EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 1, 2016, 34-38
ISSN 1307-5543 – www.ejpam.com

$\mathscr{D}$-Sets Generated by a Subset of a Group

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Abstract. A subset D of a group G is a $\mathscr{D}$-set if every element of G, not in D, has its inverse in D. Let A be a non-empty subset of G. A smallest $\mathscr{D}$-set of G that contains A is called a $\mathscr{D}$-set generated by A, denoted by $\langle A\rangle$. Note that $\langle A\rangle$ may not be unique. This paper characterized sets A with unique $\langle A\rangle$ and sets whose number of generated $\mathscr{D}$-sets is equal to the index minimum.

2010 Mathematics Subject Classifications: 20D99

Key Words and Phrases: Groups, $\mathscr{D}$-sets, Index minimum

1. Introduction

Let G be a group. A subset D of G is called a $\mathscr{D}$-set if for every $x \in G\backslash D$, $x^{-1} \in D$. A smallest $\mathscr{D}$-set in G is called a *minimum $\mathscr{D}$-set*. The number of minimum $\mathscr{D}$-sets of G is called the *index minimum* of G. Please refer to [3] for the concepts that are not defined in this paper.

In [1], we proved that if $x^2 = e$, then x is an element of any $\mathscr{D}$-set. Thus, if $S = \left\{s \in G : s^2 = e\right\}$, then $S \subseteq D$ for all $\mathscr{D}$-set D.

It is mention in [2] that the relation $\sim$ defined on $G\backslash S$ given by $x \sim y$ if and only if $x = y$ or $x^{-1} = y$ is an equivalence relation, and the equivalence class containing x is $\{x, x^{-1}\}$. Thus, $G\backslash S = \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_c, a_c^{-1}\}$. If $a_i \neq a_j$ for $i \neq j$, then we call the given partition a *canonical partition* of $G\backslash S$, and c is called the $\mathscr{C}$-number of G. Clearly, $c = |G\backslash S|/2$.

Remark 1. *Let G be a finite group and D be a $\mathscr{D}$-set of G. Then $D = S \cup \{x_1, x_2, \ldots, x_c\}$, where $x_i \in \{a_i, a_i^{-1}\}$ for $i = 1, 2, \ldots, c$ and $G\backslash S = \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_c, a_c^{-1}\}$ is a canonical partition, if and only if D is a minimum $\mathscr{D}$-set.*

To see this, assume that $D = S \cup \{x_1, x_2, \ldots, x_c\}$, where $x_i \in \{a_i, a_i^{-1}\}$ for $i = 1, 2, \ldots, c$ and $G\backslash S = \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_c, a_c^{-1}\}$ is a canonical partition, and D is not a minimum $\mathscr{D}$-set. Let D' be a minimum $\mathscr{D}$-set. Then $\left|D'\right| < |D|$. Let $x \in D\backslash D'$. Since the elements of

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S must be in D', $x = x_i$ for some $i \in \{1, 2, \ldots, c\}$. Since D' is a $\mathscr{D}$-set, $x_i^{-1} \in D'$. Hence, $x_i, x_i^{-1} \in D$. This is a contradiction.

Conversely, assume that D is a minimum $\mathscr{D}$-set and $D \neq S \cup \{x_1, x_2, \ldots, x_c\}$, where $x_i \in \{a_i, a_i^{-1}\}$ for $i = 1, 2, \ldots, c$ and $G \backslash S = \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_c, a_c^{-1}\}$ is a canonical partition. Since the elements of S must be in D and $D \neq S \cup \{x_1, x_2, \ldots, x_c\}$, where $x_i \in \{a_i, a_i^{-1}\}$ for $i = 1, 2, \ldots, c$ and $G \backslash S = \{a_1, a_1^{-1}\} \cup \{a_2, a_2^{-1}\} \cup \cdots \cup \{a_c, a_c^{-1}\}$ is a canonical partition, there exist $x \in G \backslash S$ such that $\{x, x^{-1}\} \in D$ (since one of x and x^{-1} must be in D). If $\{x, x^{-1}\} \in D$, then $D \backslash \{x\}$ is a $\mathscr{D}$-set smaller than D. This is a contradiction.

The following results are found in [2]. They will be used in the succeeding sections.

Theorem 1. *Let G be a finite group. If c is the $\mathscr{C}$-number of G, then $i(G) = 2^c$.*

Theorem 2. *Let G be a finite group and T be the family of all of its $\mathscr{D}$-sets. If c is the $\mathscr{C}$-number of G, then $|T| = 3^c$.*

Theorem 2 says that if $A \subseteq G$, then $i(A) \leq 3^c$.

2. $\mathscr{D}$-Sets Generated by a Subset

All groups considered here are finite groups. Let G be a group and A be a non-empty subset of G. A smallest $\mathscr{D}$-set of G that contains A is called a $\mathscr{D}$-set generated by A, denoted by $\langle A \rangle$.

For example, consider the additive group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Note that $S_{\mathbb{Z}_6} = \{0, 3\}$, and $\{\{1, 5\}, \{2, 4\}\}$ is a canonical partition of $\mathbb{Z}_6$. Thus, the $\mathscr{C}$-number of $\mathbb{Z}_6$ is 2. Hence by Theorem 2, $|T| = 3^2 = 9$. The elements of T would be

$$\begin{aligned}
D_1 &= \{0, 3, 1, 5, 2, 4\} \\
D_2 &= \{0, 3, 1, 2, 5\} \\
D_3 &= \{0, 3, 1, 4, 5\} \\
D_4 &= \{0, 3, 1, 2, 4\} \\
D_5 &= \{0, 3, 2, 4, 5\} \\
D_6 &= \{0, 3, 1, 2\} \\
D_7 &= \{0, 3, 1, 4\} \\
D_8 &= \{0, 3, 2, 5\} \\
D_9 &= \{0, 3, 4, 5\}.
\end{aligned}$$

Observe that $A = \{1, 4\}$ is a subset of D_1, D_3, D_4, and D_7 and the smallest set among these is D_7. Thus, $\langle 1, 4 \rangle = D_7 = \{0, 3, 1, 4\}$.

Note that $\langle A \rangle$ may not be unique. To see this, let $B = \{0, 3\}$. Then D_6 and D_9 are the smallest $\mathscr{D}$-sets of G containing $\{0, 3\}$. Hence, $\langle 0, 3 \rangle = D_6$ or D_9. We denote by $i(A)$ the number of distinct $\mathscr{D}$-sets of G generated by A.

Remark 2. *For any nonempty subset A of a finite group G, $\langle A \rangle$ always exist since G is itself a $\mathscr{D}$-set. Hence, $i(A) > 0$.*

Remark 3. *Let G be a group and D be a minimum $\mathscr{D}$-set of G. If $x \in D \backslash S$, then $x^{-1} \notin D$.*

To see this, suppose that $x \in D \backslash S$ and $x^{-1} \in D$. Then $D \backslash \{x\}$ is a $\mathscr{D}$-set smaller than D. This is a contradiction.

Theorem 3. *Let G be a finite group, $S = \left\{s \in G : s^2 = e\right\}$ and $A \subseteq G$. Then $A \subseteq S$ if and only if $A \subseteq D$ for all minimum $\mathscr{D}$-set D of G.*

Proof. Let G be a finite group, $S = \left\{s \in G : s^2 = e\right\}$ and $A \subseteq G$. In [2], $S \subseteq D$ for all $\mathscr{D}$-set D of G. So, if $A \subseteq S$, then $A \subseteq D$ for all $\mathscr{D}$-set D of G. In particular, $A \subseteq D$ for all minimum $\mathscr{D}$-set D of G.

Conversely, assume that $A \subseteq D$ for all minimum $\mathscr{D}$-set D of G and $A \nsubseteq S$. Let $x \in A \backslash S$ and D be a minimum $\mathscr{D}$-set containing A. Since $A \subseteq D$ and $x \in A \backslash S$, $x \in D \backslash S$. Hence by Remark 3, $x^{-1} \notin D$. Note that $D_1 = D \backslash \{x\} \cup \{x^{-1}\}$ is a minimum $\mathscr{D}$-set that do not contain A. This is a contradiction. □

Corollary 1. *Let G be a finite group, $S = \left\{s \in G : s^2 = e\right\}$. If $A \subseteq S$, then $i(A) = 2^c$.*

Proof. This follows from Theorem 3 and Theorem 1. □

The next result characterizes sets A with unique generated $\mathscr{D}$-set.

Theorem 4. *Let G be a finite group and $A \subseteq G$. Then, $i(A) = 1$ if and only if $A \subseteq D$ and $A \supseteq (D \backslash S)$ for some $\mathscr{D}$-set D of G.*

Proof. Let G be a finite group and $A \subseteq G$. Suppose that $i(A) = 1$, and, $A \nsubseteq D$ or $A \quad (D \backslash S)$ for all $\mathscr{D}$-set D of G. If $A \nsubseteq D$ for all $\mathscr{D}$-set D of G, then $i(A) = 0$. This is a contradiction (by Remark 2). So we assume that $A \subseteq D$. Let D_1 be a smallest $\mathscr{D}$-set containing A. If $A \quad (D \backslash S)$ for all $\mathscr{D}$-set D of G, then $A \cup S$ is not a $\mathscr{D}$-set, that $A \cup S$ is properly contained in D_1. Let $x \in D_1 \backslash (A \cup S)$. Then $D_1 \backslash \{x\} \cup \{x^{-1}\}$ is a $\mathscr{D}$-set containing A with $\left|D_1\right| \leq |D|$. This is a contradiction.

Conversely, assume that $A \subseteq D$, $A \supseteq (D \backslash S)$ for some $\mathscr{D}$-set D of G, and $i(A) > 1$. If $A \subseteq D$ and $A \supseteq (D \backslash S)$ for some $\mathscr{D}$-set D of G, then $D = A \cup S$ is a smallest $\mathscr{D}$-set containing A. Since $i(A) > 1$, let D_1 be another smallest $\mathscr{D}$-set containing A. Since $A \supseteq (D \backslash S)$, $D_1 \supseteq (D \backslash S)$. If $D \neq D_1$ and $D_1 \supseteq (D \backslash S)$, D is a proper subset of D_1 (since D_1 must contain S). Hence $|D| \neq \left|D_1\right|$. This is a contradiction. □

Theorem 5. *Let G be a finite group, $S = \left\{s \in G : s^2 = e\right\}$, and $A \subseteq G$. If $x_i \neq x_j^{-1}$ for all $x_i, x_j \in A \backslash S$, then A is a subset of a minimum $\mathscr{D}$-set. Hence, $i(A) \leq 2^c$.*

Proof. Let G be a finite group, $S = \left\{s \in G : s^2 = e\right\}$, and $A \subseteq G$. From Remark 1, if D is a minimum $\mathscr{D}$-set, then $D = S \cup \left\{a_1, a_2, \ldots, a_c\right\}$ where $a_i \in \left\{x_i, x_i^{-1}\right\}$ for $i = 1, 2, \ldots, c$ where $\left\{\{x_i, x_i^{-1}\} : i = 1, 2, \ldots, c\right\}$ is a partition of $G \backslash S$ in the sense of Remark 1. Thus, if $x_i \neq x_j^{-1}$ for all $x_i, x_j \in A \backslash S$, Then A is a subset of a minimum $\mathscr{D}$-set. □

We recall the disjoint union of sets. Let X and Y be sets. The *disjoint union* of X and Y, denoted by $X \dot{\cup} Y$, is found by combining the elements of X and Y, treating all elements to be distinct. Thus, $\left|X \dot{\cup} Y\right| = |X| + |Y|$.

Theorem 6. *Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, $Q=\left\{x\in A\backslash S: x^{-1}\in A\right\}$, and $A\subseteq G$. Then $i(A)=2^{c-n}$, where $n=|A|-|A\cap S|-\frac{|Q|}{2}$.*

Proof. Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, $Q=\{x\in A\backslash S: x^{-1}\in A\}$ and $A\subseteq G$. Consider $P=A\backslash(S\cup Q)$. Note that if $x\in P$, then $x^{-1}\in G\backslash A$. Thus,

$$\begin{aligned} A &= (A\cap S)\ \dot\cup\, Q\, \dot\cup\, P \\ &= (A\cap S)\ \dot\cup\ \left\{x_1,x_1^{-1},x_2,x_2^{-1},\ldots,x_k,x_k^{-1}\right\}\ \dot\cup\ \left\{x_{k+1},x_{k+2},\ldots,x_n\right\}. \end{aligned} \tag{1}$$

It can be shown that if D is a smallest $\mathscr{D}$-set containing A, then D is of the form

$$\begin{aligned} D &= S\,\dot\cup\, Q\,\dot\cup\, P\,\dot\cup\ \left\{x_{n+1},x_{n+2},\ldots,x_c\right\} \\ &= S\,\dot\cup\ \left\{x_1,x_1^{-1},x_2,x_2^{-1},\ldots,x_k,x_k^{-1},x_{k+1},x_{k+2},\ldots,x_n\right\}\ \dot\cup\ \left\{x_{n+1},x_{n+2},\ldots,x_c\right\}. \end{aligned} \tag{2}$$

By this, the number of ways to choose a smallest $\mathscr{D}$-set containing A is $2\cdot 2\cdot\dots\cdot 2=2^{c-n}$, where $n=\frac{|Q|}{2}+(n-k)$. Since $|A|=|A\cap S|+|Q|+(n-k)$, $n=|A|-|A\cap S|-\frac{|Q|}{2}$. □

3. $\mathscr{D}$-Sets Generated by a Subgroup

The following are consequences of Theorem 6.

Corollary 2. *Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. Then $i(A)=2^{c-n}$, where $n=|H\backslash S|/2$.*

Proof. Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. If $H\leq G$, then $x,x^{-1}\in H$ for all $x\in H$. Let $Q=\{x\in A\backslash S: x^{-1}\in A\}$. Then $Q=H\backslash S$, that is, $|Q|=|H\backslash S|$. Thus, by Theorem 6, $i(H)=2^{c-\left(|H|-|H\cap S|-\frac{|H\backslash S|}{2}\right)}=2^{c-\left(|H\backslash S|-\frac{|H\backslash S|}{2}\right)}=2^{c-\left(\frac{|H\backslash S|}{2}\right)}$. □

Corollary 3. *Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. If $S\subseteq H$, then $i(H)=2^{c-\left(\frac{|H|-|S|}{2}\right)}$.*

Proof. Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. If $S\subseteq H$, then $|H\backslash S|=|H|-|S|$. Thus, by Corollary 2, $i(H)=2^{c-\left(\frac{|H|-|S|}{2}\right)}$. □

Corollary 4. *Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. If $H\cong\mathbb{Z}_p$ where p is an odd number, then $i(H)=2^{c-\left(\frac{|H|-1}{2}\right)}$.*

Proof. Let G be a finite group, $S=\left\{s\in G: s^2=e\right\}$, and $H\leq G$. If $H\cong\mathbb{Z}_p$ where p is an odd number, then $S=\{e\}$. Thus, by Corollary 3, $i(H)=2^{c-\left(\frac{|H|-1}{2}\right)}$. □

ACKNOWLEDGEMENTS The authors would like to thank Cebu Normal University, Cebu City and Negros Oriental State University, Dumaguete City for allowing the authors to use some of their facilities and resources in the conduct of this research.

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