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Essentiality in the Category of S-acts

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Abstract. *Essentiality* is an important notion closely related to injectivity. In this paper, we study essentiality with respect to monomorphisms of acts. We give some criterion to characterize and describe essentiality explicitly.

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1. Introduction and Preliminaries

An important notion related to injectivity with respect to monomorphisms or any other class $\mathscr{M}$ of morphisms in a category $\mathscr{A}$, is essentiality. In fact, injectivity is characterized and injective hulls are defined using essentiality (see, for example, [1, 10] and [5]).

Throughout this paper S will be denoted by the semigroup with or without identity. We take $\mathscr{A}$ = **Act-S** to be the category of right acts over a semigroup S and $\mathscr{M}ono$ to be the class of all monomorphisms of right S-acts, and then, we study the notion of essentiality with respect to this class. Essentiality with respect to the subclass $\mathscr{M}$ of monomorphisms have studied and some equivalent conditions, so called "essential test lemma for essentiality", have introduced(see, [7, 9]).

We substantially improve the usual characterization of essentiality in terms of congruences, Lemma 1, of an extension B of A by giving a characterization in terms of the elements of B, Theorems 5 and 6, which are essential test lemmas. Also, similar to the case of modules, which essentiality has an expression by submodules, in Theorem 1 an equivalent condition in terms of Ress congruences is obtained for essentiality.

Although the Baer Criterion for injectivity (weak injectivity implies injectivity) is true for modules over a ring (with an identity), it is an open problem for acts over a semigroup S (with or without identity). In fact, we are not aware of any type of weak injectivity implying injectivity of S-acts, in general, other than Skornjakov-Baer Criterion, which says that injectivity with respect to subacts of cyclic acts implies injectivity with respect to all monomorphisms.

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One of the well known theorem about the injectivity says that, an S-act A is injective if and only if it has no proper essential extension(see, [8] or [4]). So essentiality play an important role in the study of the Baer problem.

In section 2 some necessity conditions on essentiality are obtained and Theorem 5, "which is the main result of this article", is in fact an essential test lemma which introduces an equivalent condition to essentiality.

Let us first recall the definition and some ingredients of the category **Act − S** of acts over a semigroup S needed in the sequel. For more information and the notions not mentioned here see, for example, [6] and [8].

Recall that, for a semigroup S, a set A is an S-*act* (or an S-*set*) if there is a, so called, *action* $\mu : A \times S \to A$ such that, denoting $\mu(a,s) := as$, $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$.

Each semigroup S can be considered as an S-act with the action given by its multiplication. Notice that, adjoining an external left identity 1 to a semigroup S an S-act $S^1 := S \cup \{1\}$ is obtained.

Also, recall that an element $a \in A$ is said to be *fixed* if $as = a$ for all $s \in S$. The S-act $A \cup \{0\}$ with a fixed adjoined to A is denoted by A^0. All fixed elements of as an S-act A is a subact of A and denoted by $Fix(A)$. A fixed element of a Semigroup S is called a left zero element. All left zero elements of a Semigroup S is a right ideal of S and denoted by $Z(S)$.

The definitions of a *homomorphism of* S-*acts* or S-*maps*, *subact* A of B, written as $A \leq B$, an *extension* of A, a *congruence* ρ on A and a *quotient* A/ρ of A are all clear. For $H \subseteq A \times A$, the *congruence generated by* H, that is the smallest congruence on A containing H, is denoted by $\rho(H)$. Let $H \subseteq A \times A$ and $\rho = \rho(H)$. Then, for $a, b \in A$, one has $a\rho b$ if and only if either $a = b$ or there exist $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in A, s_1, s_2, \ldots, s_n \in S^1$ where for $i = 1, \ldots, n, (p_i, q_i) \in H \cup H^{-1}$, such that $a = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, \ldots, q_n s_n = b$.

2. Essentiality of Acts

Here, some characterizations and some properties of essentiality are given. Many of results of this section are similar to the work done in [3].

Definition 1. *A monomorphism $f : A \to B$ of S-acts is said to be essential if for each homomorphism $g : B \to C$ which gf is a monomorphism, then g is so. If f is an inclusion map, then B is said to be an essential extension of A.*

The following two theorems give the usual (external) characterizations for the essentiality (mainly in terms of congruences). More (internal) characterizations (in terms of elements) will be given later in this section.

The set of all congruences on an S-act B is denoted by $Con(B)$ and Δ is the trivial congruence(i.e. $a\Delta b$ if and only if $a = b$.)

Lemma 1. *For a monomorphism $f : A \to B$, the following are equivalent:*

(i) f is an essential monomorphism.

(ii) For every epimorphism $g : B \to C$ such that gf is a monomorphism, g itself is a monomorphism.

(iii) For every congruence ρ on B such that for the canonical epimorphism $\pi : B \to B/\rho$, πf is a monomorphism, we get $\rho = \Delta$.

(iv) For every monogenic congruence ρ on B such that for the canonical epimorphism $\pi : B \to B/\rho$, πf is a monomorphism, we get $\rho = \Delta$.

Proof. We just prove (iv) $\Rightarrow$ (i). Let $g : B \to C$ be a homomorphism with gf a monomorphism, and $g(b) = g(b')$. Then, since $\rho(b, b') \subseteq ker(g)$, we can factorize g through $B/\rho(b, b')$, and hence πf is a monomorphism, where $\pi : B \to B/\rho(b, b')$. So, by (iv), $\rho(b, b') = \Delta$, and thus $b = b'$. □

Corollary 1. *An S-act B is an essential extension of A if and only if for each congruence ρ on B, if $\rho\,|_A = \Delta$, then $\rho = \Delta$.*

Theorem 1. *An extension B of A is an essential extension if and only if for every non trivial $\theta \in Con(B)$, $\theta \cap \rho_A \neq \Delta$, where ρ_A is the Rees congruence on B.*

Proof. ($\Rightarrow$) Let $\theta \neq \Delta$ and $\theta \cap \rho_A = \Delta$. Then, considering the canonical epimorphism $\pi : B \to B/\theta$, we see that $\pi|_A$ is a monomorphism, and so by hypothesis $\theta = \Delta$ which is a contradiction.

($\Leftarrow$) Let $g : B \to C$ be a homomorphism such that $g|_A$ is a monomorphism. It is clear that $ker(g) \cap \rho_A = \Delta$. So by hypothesis, $ker(g) = \Delta$ and hence g is a monomorphism. □

Theorem 2. *The monomorphisms $f : A \to B$ and $g : B \to C$ are essential monomorphisms if and only if gf is so.*

Proof. We just prove the case that if gf is an essential monomorphism, then f is so. Let $h : B \to D$ be a homomorphism such that hf is a monomorphism. Then there exists an extension $\bar{h} : C \to E(D)$ of h to the injective hull of D, and since gf is essential, $\bar{h}$ is a monomorphism, and hence so is h. □

Proposition 1. *Let A and C be subacts of B such that $|C| \geq 2$ and B is an essential extension of A. Then $|C \cap A| \geq 2$. In particular, $B \setminus A$ does not have two fixed elements.*

Proof. Let $|C \cap A| \leq 1$. It is clear that $\pi|_A : A \to B/\rho_C$, in which ρ_C is a Ress congruence on C, is a monomorphism. Hence, π is a monomorphism and so $|C| = 1$, which is a contradiction. □

Corollary 2. *If $a_0 \in A$ and $b_0 \in B \setminus A$ are fixed elements, then B is not an essential extension of A.*

For a subact A of an S-act B and $b \in B$, we use the notation $I_b = \{s \in S \mid bs \in A\}$.

Corollary 3. *Let A have at least one fixed element and B be an essential extension of A. Then:*

(i) $Fix(B) \subseteq A$.

(ii) For every $b \in B$, $I_b \neq \emptyset$.

Corollary 4. *If S has at least one left zero element and S is an essential extension of a right ideal I, then* $Z(S) \subseteq I$. *If S is a left zero semigroup, then* $I = S$.

Lemma 2. *If A has no fixed element, then* A^0 *is an essential extension of A.*

Proof. Let $g : A^0 \to B$ be a homomorphism such that $g|_A$ is a monomorphism. Then g itself is one-one. In fact, if $g(a) = g(0)$ for some $a \in A$, then for every $s \in S, g(as) = g(a)s = g(0)s = g(0s) = g(0) = g(a)$ and so $as = a$. This means that a is a fixed element, which is a contradiction. Thus g is an injection. □

Lemma 3. *Pushouts do not necessarily transfer essential monomorphisms.*

Proof. Let A have no fixed element. By Lemma 2, the inclusion $\tau : A \to A^0$ is an essential extension. Consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\tau} & A^0 \\ \tau \downarrow & & \downarrow q \\ A^0 & \xrightarrow{p} & A^{z_1,z_2} \end{array}$$

where z_1, z_2 are two fixed elements adjoint to A and $p(a) = q(a) = a$ $(a \in A)$ and $p(0) = z_1$, $q(0) = z_2$. By [2, Theorem 3.2(1)], p and q are monomorphisms. Define a homomorphism $h : A^{z_1,z_2} \to A^0$ by $h(a) = a$ and $h(z_1) = h(z_2) = 0$. Then $hp = id_{A^0}$. But, h is not one-one, and hence p is not an essential extension. □

Recall that a directed system of S-acts and S-maps is a family $(B_\alpha)_{\alpha\in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha\leq\beta\in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta\in I})$ in **Act-S** is given as $\underrightarrow{lim}_\alpha B_\alpha = \coprod_\alpha B_\alpha/\rho$ where the congruence ρ is given by $b_\alpha \rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$ in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ is the natural S-map.

Theorem 3. *Any direct limit of essential monomorphisms is an essential monomorphism.*

Proof. Let $f : A \to \underrightarrow{lim}_\alpha B_\alpha$ be a direct limit in **Act-S** of essential monomorphisms $f_\alpha : A \to B_\alpha$, $\alpha \in I$, and directed S-maps $g_{\alpha\beta} : B_\alpha \to B_\beta$ $(\alpha \leq \beta)$, and consider $g_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ as before. To show that $f : A \to \underrightarrow{lim} B_\alpha$ is essential, let hf, for $h : \underrightarrow{lim}_\alpha B_\alpha \to C$, be a monomorphism. Then, for every $\alpha \in I$, $hg_\alpha f_\alpha \in \mathcal{M}$. Since each f_α is essential, each hg_α is a monomorphism. Now if $h([b_\alpha]) = h([b_\beta])$, then $hg_\gamma g_{\alpha\gamma}(b_\alpha) = hg_\gamma g_{\beta\gamma}(b_\beta)$, for some $\gamma \geq \alpha, \beta$. Thus $g_{\alpha\gamma}(b_\alpha) = g_{\beta\gamma}(b_\beta)$ which means that $b_\alpha \rho b_\beta$, and hence h is a monomorphism and f is essential. □

Definition 2. *The category $\mathscr{A}$ is called essentially bounded, if every $A \in \mathscr{A}$ has only a set of essential extensions.*

Proposition 2. *The category* **Act-S** *is essentially bounded.*

Proof. Any essential extension B of A can be clearly embedded into the injective hull $E(A)$ of A. So, we get the result. □

The following theorem is an other form of [4, Theorem 8].

Theorem 4. **Act-S** *fulfills Banaschewski's condition, that is, for every monomorphism $f : A \to B$ there exists a homomorphism $g : B \to C$ such that gf is an essential monomorphism.*

For an S-act A and $a \in A$ we denote the homomorphism $f : \mathbf{S} \to A$, given by $f(s) = as$ for all $s \in S$, by λ_a.

Lemma 4. *Let B be an essential extension of A and $C^p(A) = \{b \in B \mid \exists a \in A, \lambda_b = \lambda_a\}$. Then $C^p(A) = A$.*

Proof. Since B is an essential extension of A, $C^p(A)$ is an essential extension of A too. Let $C^p(A) \neq A$ and $b \in C^p(A) \setminus A$. By the Axiom of choice, choose and fix an element $a_b \in A$ such that $\lambda_b = \lambda_{a_b}$. Consider the homomorphism $g : C^p(A) \to A$ defined by

$$g(b) = \begin{cases} b, & \text{if } b \in A \\ a_b, & \text{if } b \notin A \end{cases}.$$

It is clear that $g\mid_A = id_A$ and hence g is an isomorphism. So $b = a_b$ which is a contradiction. Thus $C^p(A) = A$. □

Proposition 3. *Suppose that B is an essential extension of A and $b \in B \setminus A, b' \in B$ such that $I_b = I_{b'}$. If for each $s \in I_b = I_{b'}$, $bs = b's$, then $b = b'$.*

Proof. If $b' \in A$, then $I_b = I_{b'} = S$ and $b \in C^p(A)$. By Lemma 4, $b \in A$, which is impossible. So $b' \notin A$. Consider the canonical epimorphism $\pi : B \to B/\rho(b, b')$. Let $a, a' \in A$ and $a\rho(b, b')a'$. then $a = a'$ or there exist $p_i, q_i \in \{b, b'\}$, $s_i \in S^1$ and $n \in \mathbb{N}$ such that $a = p_1s_1$, $q_1s_1 = p_2s_2, q_2s_2 = p_3s_3, \ldots, q_ns_n = a'$. If $s_1 = 1$, then $a = p_1s_1 = p_1 \in \{b, b'\}$, which is impossible. Let for $i \geq 2$, $s_i = 1$ and for each $j < i, s_j \neq 1$. So $s_1 \in I_{p_1} = I_{q_1}$ and $a = p_1s_1 = q_1s_1 = p_2s_2$, which implies $s_2 \in I_{p_2} = I_{q_2}$ and $p_2s_2 = q_2s_2 = p_3s_3$. By continuing with this way, we have

$$a = p_1s_1 = q_1s_1 = p_2s_2 = q_2s_2 = p_3s_3 = \ldots = q_{i-1}s_{i-1} = p_is_i = p_i \in \{b, b'\}.$$

So b or b' belongs to A, which is a contradiction. Thus for each $1 \leq i \leq n, s_i \neq 1$, which deduced that $a = p_1s_1 = q_1s_1 = p_2s_2 = q_2s_2 = p_3s_3 = \ldots = a'$. So $\pi\mid_A$ is a monomorphism. By the hypothesize π is a monomorphism and hence $b = b'$. □

An S-act A is called s-dense in an extension B, if for each $b \in B$, $bS \subseteq A$. Also B is said to be an s-dense essential extension of A, if B is an s-dense extension as well as essential extension of A. The following lemma is an essential test lemma for s-dense essentiality.

Lemma 5. *An s-dense extension $\tau : A \to B$ is s-dense essential if and only if for each $b \in B \setminus A$, $b' \in B$, if $\lambda_b = \lambda_{b'}$, then $b = b'$.*

Proof. ($\Rightarrow$) Since A is s-dense in B, $I_b = I_{b'} = S$. Now we are done by using Proposition 3.

($\Leftarrow$) Consider $A \overset{\tau}{\to} B \overset{g}{\to} C$, which τ is inclusion map and $g\tau$ is a monomorphism. Let $g(b) = g(b')$ such that $b \in B \setminus A$. For each $s \in S$ we have $g(bs) = g(b's)$ and $\{bs, b's\} \in A$. Since $g\mid_A$ is one to one, $\lambda_b = \lambda_{b'}$ and hence $b = b'$. □

Lemma 6. *Let A have a fixed element and B be a proper essential extension of A. Then for every $b \in B$ and every nonempty right ideal J of S, $I_b \cap J \neq \emptyset$.*

Proof. For $b \in A$, $I_b = S$ and the result is obvious. For $b \notin A$, let J be a nonempty right ideal of S and $I_b \cap J = \emptyset$. By Corollary 3, $I_b \neq \emptyset$ and $Fix(B) \subseteq A$. It is clear that $B' = \{bs | s \in J\}$ is a subact of B and $B' \cap A = \emptyset$. By Proposition 1, $|B'| = 1$ and hence for every $s \in J$, $bs = b_0$ for some $b_0 \in B \setminus A$. Consider $s_0 \in J$. Then for every $t \in S$, $b_0 t = (bs_0)t = b(s_0 t) = b_0$. So $b_0 \in Fix(B) \subseteq A$, which is impossible. □

The following two theorems are the main results of this article, which is in fact a kind of essential test lemma. In these theorems we give an (internal) characterization for essential monomorphisms (in terms of elements rather than congruences).

Theorem 5. *(Essential Test Lemma 1) An S-act B is an essential extension of A if and only if for every $x \in B$ and $y \in B \setminus A$ whenever the following two conditions hold then $x = y$:*

(i) For each $s \in S$ with $s \in I_x \cap I_y$, we have $xs = ys$.

(ii) If $I_1 = I_x \setminus I_y$ and $I_2 = I_y \setminus I_x$, then $ker\lambda_y \mid_{I_1} \subseteq ker\lambda_x$ and $ker\lambda_x \mid_{I_2} \subseteq ker\lambda_y$.

Proof. ($\Leftarrow$) Let $g : B \to C$ be a homomorphism with $g|_A$ a monomorphism, and $g(x) = g(y)$ for $x, y \in B$. Then, clearly conditions (i) and (ii) hold for $x \in B$, $y \in B \setminus A$. Thus $x = y$, and so B is an essential extension of A.

($\Rightarrow$) Let B be an essential extension of A, $x \in B$, and $y \in B \setminus A$. Let the conditions (i) and (ii) hold. At first we show that $I_x = I_y$. On the contrary, let $I_x \neq I_y$. In this case, there exists $t \in S$ such that $a = xt \in A$ and $b = yt \notin A$ (or $xt \notin A, yt \in A$). So, by (i), we have

($*$) For every $s \in I_b$, $as = xts = yts = bs$.

Now, consider the congruence $\rho = \rho(a, b)$ on B and $(a_1, a_2) \in \rho$ with $a_1, a_2 \in A$. Then $a_1 = a_2$ or there exist $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ in B with $\{p_i, q_i\} = \{a, b\}$ and $s_1, s_2, \ldots, s_n \in S^1$ such that $a_1 = p_1 s_1, q_1 s_1 = p_2 s_2, \ldots, q_n s_n = a_2$. We prove, by induction on n, that $a_1 = a_2$,. If $n = 1$, then $a_1 = p_1 s_1$, $a_2 = q_1 s_1$ (where $s_1 \neq 1$, since otherwise $p_1 = a$ and hence $a_2 = q_1 = b$ which is a contradiction). But, one of p_1 or q_1 is b, so $bs_1 \in A$ and hence using ($*$) $a_1 = p_1 s_1 = q_1 s_1 = a_2$. Now, let the result be true when the path connecting a_1 to a_2 has length less than n. Assume we have the above path of length $n \geq 1$. Then:

If $q_1 s_1 \in A$ then $s_1 \neq 1$ (because otherwise $q_1 = a$ and $p_1 = b$ which contradicts $a_1 = p_1 s_1$). Also $bs_1 \in A$ because it is one of $p_1 s_1$ or $q_1 s_1$. Thus $a_1 = p_1 s_1 = q_1 s_1$. This means $p_2 s_2 = a_1$ and so we get a path with length $n-1$ which connects a_1 to a_2. Then, by induction hypothesis, $a_1 = a_2$.

If $q_1s_1 \notin A$ then $p_2 = q_1 = b$ and $q_2 = p_1 = a$. Hence, $q_2s_2 \in A$. So

$$yts_1 = bs_1 = q_1s_1 = p_2s_2 = bs_2 = yts_2$$

where $\{xts_1, xts_2\} \subseteq A$ and $yts_2 = yts_1 \notin A$. Thus, by (ii), we get $xts_1 = xts_2$ and so $a_1 = p_1s_1 = as_1 = as_2 = q_2s_2$. This gives $p_3s_3 = a_1$ and hence we get a path with a lower length than n which yields $a_1 = a_2$, by induction hypothesis. Therefore, $a_1 = a_2$. So ρ is identity on A. Thus it is identity on B and hence $a = b$, which is a contradiction. So, $I_x = I_y$. Now using Proposition 3 deduced the result. □

Theorem 6. *(Essential Test Lemma 2) An S-act B is an essential extension of A if and only if the following hold:*

(i) for every $b, b' \in B$, *if* $\rho(b, b') \cap A \times A = \Delta$, *then* $\lambda_b = \lambda_{b'}$.

(ii) for every $b, b' \in B \setminus A$, *if* $\lambda_b = \lambda_{b'}$, *then* $b = b'$.

(iii) $C_B^p(A) = A$.

Proof. (⇒) To prove (i), let $b, b' \in B$ and $\rho(b, b') \cap A \times A = \Delta$. So for the canonical map $\pi : B \to B/\rho(b, b')$, $\pi|_A$ and so π is a monomorphism. Therefore $b = b'$ and hence $\lambda_b = \lambda_{b'}$.

To prove (ii), let $b, b' \in B \setminus A$ with $\lambda_b = \lambda_{b'}$. For the canonical map $\pi : B \to B/\rho(b, b')$, $\pi|_A$ is a monomorphism, indeed, let $a\rho(b, b')a'(a, a' \in A)$. So there are $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in \{b, b'\}$ and $s_1, s_2, \ldots, s_n \in S^1$ such that $a = p_1s_1, q_1s_1 = p_2s_2, \ldots, q_ns_n = a'$. Since $p_1s_1 = a \in A$, $s_1 \neq 1$ and hence $p_2s_2 = q_1s_1 = p_1s_1 = a$ which implies $s_2 \neq 1$. By continue to this process, for each $1 \leq i \leq n$, $s_i \neq 1$, which deduced $a = a'$. By essentiality π is a monomorphism and thus $\rho(b, b') = \Delta$ and $b = b'$.

To prove (iii), let $b \in C_B^p(A)$. So there exists $a \in A$ such that for each $s \in S$, $as = bs$. Similarly, to prove (ii), for the canonical map $\pi : B \to B/\rho(a, b)$, $\pi|_A$ is a monomorphism. Thus by essentiality, π is a monomorphism and hence $a = b$.

(⇐) By Lemma 1, it is enough to show that for every monogenic congruence $\rho = \rho(b, b')$ on B such that for the canonical epimorphism $\pi : B \to B/\rho$, $\pi|_A$ is a monomorphism, we get $b = b'$. Since $\pi|_A$ is a monomorphism, by (i), $\lambda_b = \lambda_{b'}$ and if $\{b, b'\} \subseteq A$, then $b = b'$. In the case where $b, b' \in B \setminus A$, by (ii), $b = b'$. At last condition (iii) shows that the case where $b \in A$, $b' \notin A$ may not occur. □

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