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On 2-Absorbing Primary Ideals in Commutative Semirings

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Abstract. In this paper, we define 2-absorbing and weakly 2-absorbing primary ideals in a commutative semiring S with $1 \neq 0$ which are generalization of primary ideals of commutative ring. A proper ideal I of a commutative semiring S is said to be a 2-absorbing primary (weakly 2-absorbing primary) ideal of S if $abc \in I$ $(0 \neq abc \in I)$ implies $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Some results concerning 2-absorbing primary and weakly 2-absorbing primary ideals are given. It is proved that a subtractive weakly 2-absorbing primary ideal I that is not a 2-absorbing primary ideal satisfies $\sqrt{I} = \sqrt{0}$.

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1. Introduction

The algebraic structure of semiring plays a prominent role in various branches of mathematics as well as some other branches of applied science. The concept of semiring was first introduced by H. S. Vandiver [13] in 1934 and has since then been studied by many authors. The structure of prime ideals in semiring theory have gained importance and many mathematicians have exploited its usefulness in algebraic systems over the decades. Anderson and Smith [2] introduced the notion of weakly prime ideals in commutative ring for the study of factorization in commutative rings with zero divisors. The concepts of 2-absorbing and weakly 2-absorbing ideals of commutative ring with nonzero unity have been introduced by Badawi [7] and Badawi and Darani [8] respectively which are generalizations of prime and weakly prime ideals in commutative rings. Recently, Badawi *et al.* [9] introduced the concept of 2-absorbing primary ideals in commutative rings with $1 \neq 0$ and gave some characterizations related to it.

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A commutative semiring is a commutative semigroup $(S,\cdot)$ and a commutative monoid $(S,+,0_S)$ in which 0_S is the additive identity and $0_S \cdot x = x \cdot 0_S = 0_S$ for all $x \in S$, both are connected by ring like distributivity. A nonempty subset I of a semiring S is called an ideal of S if $a,b \in I$ and $r \in S$, $a+b \in I$ and $ra, ar \in I$. An ideal I of a semiring S is called subtractive if $a, a+b \in I$, $b \in S$, then $b \in I$.

Let I be an ideal of S. Then, the radical of I is defined as

$$\mathrm{Rad}(I) = \sqrt{I} = \{a \in S : a^n \in I \text{ for some positive integer.}\}n$$

Annihilator of a semiring S is defined as $Ann(a) = \{x \in S : ax = 0\}$. Recall from [10], that a proper ideal I of a commutative semiring S is said to be a 2-absorbing (weakly 2-absorbing)ideal of S if whenever $a,b,c \in S$ and $abc \in I$ $(0 \neq abc \in I)$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It is easy to see that every 2-absorbing ideal of a semiring S is a weakly 2-absorbing ideal of S but converse need not be true. For further understanding the concept of semiring, refer [11] and the properties of a 2-absorbing and weakly 2-absorbing ideals in commutative semirings, we refer[10]. The paper is organized as follows: In section 2, we introduce the concepts of 2-absorbing primary ideal of a commutative semiring and prove some results corresponding to ring theory. In section 3, we introduce the concept of weakly 2-absorbing primary ideal of a commutative semiring and give some generalizations of [5, 6, 10] and [12] which are analogous to commutative ring theory. Throughout this paper, semiring S is considered as commutative with identity $1 \neq 0$.

2. 2-Absorbing Primary Ideals

In this section, we introduce the concept of 2-absorbing primary ideal of a commutative semiring and prove some results related to it.

Definition 1. *Let S be a commutative semiring and I be a proper ideal of S. Then I is said to be a 2-absorbing primary ideal of S if whenever $a,b,c \in S$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.*

It is easy to see that every 2-absorbing ideal of a commutative semiring S is a 2-absorbing primary ideal of S but converse need not be true. For instance, consider a semiring $S = Z^+ \cup \{0\}$ and an ideal $I = \langle 8 \rangle$ of S. Then I is a 2-absorbing primary ideal of S but it is not a 2-absorbing ideal of S, as $2.2.2 \in \langle 8 \rangle$ but $2.2 \notin \langle 8 \rangle$. Also, every primary ideal of S is a 2-absorbing primary ideal of S but converse is not true, as $\langle 10 \rangle$ is a 2-absorbing primary ideal of S but it is not a primary ideal of S.

Theorem 1. *Let $f : S \mapsto S'$ be a homomorphism of commutative semirings. Then, if I' is a 2-absorbing primary ideal of S', then $f^{-1}(I')$ is a 2-absorbing primary ideal of S.*

Proof. Let $abc \in f^{-1}(I')$ for some $a,b,c \in S$. Then $f(abc) \in I'$, that is, $f(a)f(b)f(c) \in I'$. Since I' is a 2-absorbing primary ideal of S', therefore $f(a)f(b) \in I'$ or $f(b)f(c) \in \sqrt{I'}$ or $f(c)f(a) \in \sqrt{I'}$. Hence, $ab \in f^{-1}(I')$ or $bc \in f^{-1}(\sqrt{I'})$ or $ca \in f^{-1}(\sqrt{I'})$. Since $f^{-1}(\sqrt{I'}) \subseteq \sqrt{f^{-1}(I')}$, we have $f^{-1}(I')$ is a 2-absorbing primary ideal of S. □

Theorem 2. *If I is a 2-absorbing primary ideal of a semiring S, then $\sqrt{I}$ is a 2-absorbing ideal of S.*

Proof. Let $abc \in \sqrt{I}$ for some $a, b, c \in S$. Suppose that $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Since $abc \in \sqrt{I}$, then there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in I$. This gives $a^n b^n \in I$, since I is a 2-absorbing primary ideal of S and $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Hence, $ab \in \sqrt{I}$. Thus, $\sqrt{I}$ is a 2-absorbing ideal of S. □

Corollary 1. *Let I be an ideal of a semiring S. Then the following statements are equivalent:*

(1) I is a 2-absorbing primary ideal of S.

(2) $\sqrt{I}$ is a 2-absorbing ideal of S and if $abc \in I$ with $bc \notin \sqrt{I}$ and $ca \notin \sqrt{I}$ then $ab \in I$.

Definition 2. *Let I be a 2-absorbing primary ideal of a semiring S. Then by above theorem $P = \sqrt{I}$ is a 2-absorbing ideal of S. In this case, I is said to be a $P - 2$-absorbing primary ideal of S.*

Theorem 3. *Let $I_1, I_2, \ldots, I_n$ be $P - 2$-absorbing primary ideals of S, where P is a 2-absorbing ideal of S. Then $I = \bigcap_{i=i}^{n} I_i$ is a $P - 2-$absorbing primary ideal of S.*

Proof. Proof is similar to [9, Theorem 2.16]. □

Theorem 4. *Let S be a semiring. Suppose that I_1 is a P_1-primary ideal of S for some prime ideal P_1 of S, and I_2 is a P_2-primary ideal of S for some prime ideal P_2 of S. Then the following statements hold:*

(1) $I_1 I_2$ is a 2-absorbing primary ideal of S.

(2) $I_1 \cap I_2$ is a 2-absorbing primary ideal of S.

Proof. Proof is similar to [9, Theorem 2.4]. □

Theorem 5. *Let I be a 2-absorbing primary ideal of S such that $\sqrt{I} = P$ is a prime ideal of S. Then $(I : x)$ is a 2-absorbing primary ideal of S with $\sqrt{(I : x)} = P$ for all $x \in S \setminus \sqrt{I}$, where $(I : x) = \{r \in S : xr \in I\}$.*

Proof. Let $x \in S \setminus \sqrt{I}$ and $a \in (I : x)$. Then $ax \in I \subseteq \sqrt{I}$, gives $a \in \sqrt{I}$, since $x \notin \sqrt{I}$ and $\sqrt{I}$ is prime. Hence, $a \in \sqrt{I}$, gives $I \subseteq (I : x) \subseteq \sqrt{I} = P$, which implies that $P = \sqrt{I} \subseteq \sqrt{(I : x)} \subseteq \sqrt{I} = P$. Thus, we have $\sqrt{(I : x)} = P$. Now, let $a, b, c \in S$ be such that $abc \in (I : x)$. Then $abcx \in I$, implies that either $abc \in I$ or $ax \in \sqrt{I}$ or $bcx \in \sqrt{I}$. If $ax \in \sqrt{I}$ or $bcx \in \sqrt{I}$, we get $ac \in \sqrt{(I : x)}$ or $bc \in \sqrt{(I : x)}$, since $\sqrt{(I : x)} = \sqrt{I}$ and $x \notin \sqrt{I}$. Next, if $abc \in I$, we have either $ab \in I$ or $bc \in \sqrt{I}$ or $ca \in \sqrt{I}$, since I is a 2-absorbing primary ideal of S. Thus, $ab \in (I : x)$ or $bc \in \sqrt{(I : x)}$ or $ca \in \sqrt{(I : x)}$. Therefore $(I : x)$ is a 2-absorbing primary ideal of S. □

Theorem 6. *If I is a 2-absorbing primary ideal of a semiring S, then the following holds:*

(1) $(\sqrt{I} : x)$ *is a 2-absorbing ideal of S for all $x \in S \setminus \sqrt{I}$.*

(2) $(\sqrt{I} : x) = (\sqrt{I} : x^2)$ *for all $x \in S \setminus \sqrt{I}$.*

Proof. *(1)* Let $a, b, c \in S$ be such that $abc \in (\sqrt{I} : x)$. Then $abcx \in \sqrt{I}$. Since $\sqrt{I}$ is a 2-absorbing ideal of S therefore $ab \in \sqrt{I}$ or $bcx \in \sqrt{I}$ or $cax \in \sqrt{I}$, that is, $ab \in (\sqrt{I} : x)$ or $bc \in (\sqrt{I} : x)$ or $ca \in (\sqrt{I} : x)$. Hence $(\sqrt{I} : x)$ is a 2-absorbing ideal of S.

(2) It is clear that $(\sqrt{I} : x) \subseteq (\sqrt{I} : x^2)$. Let $y \in (\sqrt{I} : x^2)$. Then $x^2y \in \sqrt{I}$. Since $\sqrt{I}$ is 2-absorbing ideal of S, therefore we have either $x^2 \in \sqrt{I}$ or $xy \in \sqrt{I}$. If $xy \in \sqrt{I}$, then $y \in (\sqrt{I} : x)$ and we are done. If $x^2 \in \sqrt{I}$, then $x \in \sqrt{I}$, a contradiction. Hence, $(\sqrt{I} : x) = (\sqrt{I} : x^2)$. □

Let S be a semiring and A be the set of all multiplicatively cancellable elements of S (so $1 \in S$). For further understanding of the structure of the semiring of fractions S_A of S with respect to A, refer [3].

Theorem 7. *Let I be a 2-absorbing primary ideal of a semiring S and A be the multiplicatively cancellable subset of S. Then IS_A is a 2-absorbing primary ideal of S_A.*

Proof. Let $a/s, b/t, c/r \in S_A$, where $a, b, c \in S$ and $s, t, r \in A$ be such that $abc/str \in IS_A$ but $bc/tr \notin \sqrt{IS_A}$ and $ca/rs \notin \sqrt{IS_A}$. Then there exist $p \in I$ and $z \in A$ such that $abcz = strp \in I$ but $bcz \notin I$ and $caz \notin I$ since if $bcz \in I$ and $caz \in I$, we get $bc/tr \in \sqrt{IS_A}$ and $ca/rs \in \sqrt{IS_A}$, which leads to a contradiction. Since $abcz \in I$ and I is a 2-absorbing primary ideal of S, we have $ab \in I$, implies $ab/st \in IS_A$. Hence, IS_A is a 2-absorbing primary ideal of S_A. □

Lemma 1. *Let I be a 2-absorbing primary ideal of S. Suppose that I and $\sqrt{I}$ be subtractive ideals of S and $abJ \subseteq I$ for some $a, b \in S$ and an ideal J of S. If $ab \notin I$, then either $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.*

Proof. Suppose that $aJ \nsubseteq \sqrt{I}$ and $bJ \nsubseteq \sqrt{I}$. Therefore, there are some $x, y \in J$ such that $ax \notin \sqrt{I}$ and $by \notin \sqrt{I}$. Since $abx \in I$ and $ab \notin I$ and $ax \notin \sqrt{I}$, we have $bx \in \sqrt{I}$. Since $aby \in I$ and $ab \notin I$ and $by \notin \sqrt{I}$, we have $ay \in \sqrt{I}$. Now, since $ab(x + y) \in I$ and $ab \notin I$, we have $a(x + y) \in \sqrt{I}$ or $b(x + y) \in \sqrt{I}$, since I is a 2-absorbing primary ideal of S. If $a(x + y) \in \sqrt{I}$ and $ay \in \sqrt{I}$, then $ax \in \sqrt{I}$, since $\sqrt{I}$ is subtractive, which is a contradiction. Similarly, if $b(x + y) \in \sqrt{I}$ and $bx \in \sqrt{I}$, we get $by \in \sqrt{I}$, a contradiction. Hence, either $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$. □

Theorem 8. *Let I be a proper subtractive ideal of S and suppose that $\sqrt{I}$ is a subtractive ideal of S. Then I is a 2-absorbing primary ideal of S if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_3I_1 \subseteq \sqrt{I}$.*

Proof. Proof is similar to the proof of [9, Theorem 2.19] □

Definition 3 ([1, Definition (4)]). *An ideal I of a semiring S is called a Q-ideal (partitioning ideal) if there exists a subset Q of S such that*

(1) $S = \cup\{q + I : q \in Q\}$

(2) If $q_1, q_2 \in Q$*, then* $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a semiring S. Then $S/I_Q = \{q + I : q \in Q\}$ forms a semiring under the following addition '$\oplus$' and multiplication '$\odot$', $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is unique such that $q_1 q_2 + I \subseteq q_4 + I$. This semiring S/I_Q is called the quotient semiring of S and denoted by $(S/I_Q, \oplus, \odot)$ or S/I_Q. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of S/I_Q. Clearly, if S is commutative then S/I_Q is commutative.

Theorem 9. *Let S be a semiring, I a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. Then P is a* 2*-absorbing primary ideal of S if and only if $P/I_{Q\cap P}$ is a* 2*-absorbing primary ideal of S/I_Q.*

Proof. Let P be a 2-absorbing primary ideal of S. Suppose that $q_1 + I, q_2 + I, q_3 + I \in S/I_Q$ are such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{Q\cap P}$ where $q_4 \in Q \cap P$ is a unique element such that $q_1 q_2 q_3 + I \subseteq q_4 + I \in P/I_{Q\cap P}$. So $q_1 q_2 q_3 = q_4 + i$ for some $i \in I$. Since P is a 2-absorbing primary ideal of S and $q_1 q_2 q_3 \in P$, therefore $q_1 q_2 \in P$ or $(q_2 q_3)^m \in P$ or $(q_3 q_1)^n \in P$ for some positive integers m, n. Consider the case $q_1 q_2 \in P$. If $(q_1 + I) \odot (q_2 + I) = i_1 + I$ where $i_1 \in Q$ is a unique element such that $q_1 q_2 + I \subseteq i_1 + I$. So $i_1 + f = q_1 q_2 + e$ for some $e, f \in I$. Since P is subtractive and $I \subseteq P$, we have $i_1 \in P$, therefore $i_1 \in Q \cap P$. Thus, $P/I_{Q\cap P}$ is a 2-absorbing primary ideal of S/I_Q. Next, if $q_2^m q_3^m \in P$ for some positive integer m. Let $(q_2^m + I) \odot (q_3^m + I) = i_2 + I$ where $i_2 \in Q$ is a unique element such that $q_2^m q_3^m + I \subseteq i_2 + I$. So, $i_2 + f_1 = q_2^m q_3^m + e_1$ for some $f_1, e_1 \in I$. Since P is subtractive and $I \subseteq P$, we have $i_2 \in P$, therefore $i_2 \in Q \cap P$. This gives,

$$(q_2 + I)^m \odot (q_3 + I)^m = (q_2^m + I) \odot (q_3^m + I) = q_2^m q_3^m + I \subseteq i_2 + I$$

where $i_2 \in Q \cap P$. Hence, $P/I_{Q\cap P}$ is a 2-absorbing primary ideal of S/I_Q. Similarly, if $(q_3 q_1)^n \in P$ for some positive integer n, we get $P/I_{Q\cap P}$ is a 2-absorbing primary ideal of S/I_Q.

Conversely, if $P/I_{Q\cap P}$ is a 2-absorbing primary ideal of S/I_Q. Let $abc \in P$ for some $a, b, c \in S$. Since I is a Q-ideal of S therefore there exist $q_1, q_2, q_3, q_4 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$, $c \in q_3 + I$. Now, $abc \in (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$. So, $abc = q_4 + i_3 \in P$ for some $i_3 \in I$. Since P is a subtractive ideal of S and $I \subseteq P$, we have $q_4 \in P$. So,

$$(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I \in P/I_{Q\cap P},$$

which gives $(q_1+I)\odot(q_2+I) \in P/I_{Q\cap P}$ or $(q_2^r+I)\odot(q_3^r+I) \in P/I_{Q\cap P}$ or $(q_3^t+I)\odot(q_1^t+I) \in P/I_{Q\cap P}$ for some positive integers r, t, since $P/I_{Q\cap P}$ is a 2-absorbing primary ideal of S/I_Q. If $(q_1 + I) \odot (q_2+I) \in P/I_{Q\cap P}$, then there exists $q_5 \in Q \cap P$ such that $ab \in (q_1+I)\odot(q_2+I) = q_5+I$. This gives, $ab = q_5 + i_4$ for some $i_4 \in I$. This implies $ab \in P$. Thus P is a 2-absorbing primary ideal of S. If $(q_2^r+I)\odot(q_3^r+I) \in P/I_{Q\cap P}$, then there exists $q_6 \in Q \cap P$ such that $b^r c^r \in (q_2^r+I)\odot(q_3^r+I) = q_6+I$. This gives, $b^r c^r = q_6+i_5$ for some $i_5 \in I$. This implies, $(bc)^r \in P$. Therefore, $bc \in \sqrt{P}$. Similarly, we can prove that $ca \in \sqrt{P}$. Hence, P is a 2-absorbing primary ideal of S. □

3. Weakly 2-Absorbing Primary Ideals

In this section, we introduce the notion of weakly 2-absorbing primary ideal of a commutative semiring and prove some results related to it.

Definition 4. *Let S be a commutative semiring and I be a proper ideal of S. Then I is said to be a weakly 2-absorbing primary ideal of S if whenever $a, b, c \in S$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.*

It is clear that every 2-absorbing primary ideal of S is a weakly 2-absorbing primary ideal of S but converse is not true, as $\langle 0 \rangle$ is a weakly 2-absorbing primary ideal of S but not a 2-absorbing primary ideal of S. Consider the set $S = Z_{16} = \{0, 1, 2, \ldots, 15\}$. Then S forms a semiring under addition and multiplication modulo 16. If we take the set $I = \{0, 8\}$. Then it is easy to check that I is a weakly 2-absorbing primary ideal of S but it not a weakly 2-absorbing ideal of S because $0 \neq 2.2.2 \in I$ but $2.2 \notin I$. For any ideal, the following implications hold:

$$\begin{array}{ccccc}
\text{Prime} & \Rightarrow & \text{2-absorbing} & \Rightarrow & \text{Weakly 2-absorbing} \\
\text{ideal} & \not\Leftarrow & \text{ideal} & \not\Leftarrow & \text{ideal} \\
\not\Uparrow\Downarrow & & \not\Uparrow\Downarrow & & \not\Uparrow\Downarrow \\
\text{Primary} & \Rightarrow & \text{2-absorbing primary} & \Rightarrow & \text{Weakly 2-absorbing primary} \\
\text{ideal} & \not\Leftarrow & \text{ideal} & \not\Leftarrow & \text{ideal}
\end{array}$$

Lemma 2 ([6, Lemma 2.5]). *Let I be a subtractive ideal of a semiring S and let $a \in I$ and $a + b \in \sqrt{I}$. Then $b \in \sqrt{I}$.*

Proof. Let $a \in I$ and $a + b \in \sqrt{I}$. Then, we can assume that there exists a positive integer m such that $(a + b)^m = c + b^m \in I$, where $c \in I$ (as $a \in I$). This gives $b^m \in I$ since I is subtractive. Hence $b \in \sqrt{I}$. □

Theorem 10. *Let S be a semiring and I be a subtractive weakly 2-absorbing primary ideal that is not a 2-absorbing primary ideal of S. Then $\sqrt{I} = \sqrt{0}$.*

Proof. We first prove that $I^3 = 0$. Suppose that $I^3 \neq 0$. Then, we prove that I is a 2-absorbing primary ideal of S. Let $abc \in I$ for some $a, b, c \in S$. Suppose that $abc \neq 0$, then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ since I is a weakly 2-absorbing primary ideal of S. So, assume that $abc = 0$. If $abI \neq 0$, then there exists an element a' in I such that $aba' \neq 0$, which implies $0 \neq aba' = ab(c + a') \in I$. Since I is a weakly 2-absorbing primary ideal of S, therefore either $ab \in I$ or $b(c + a') \in \sqrt{I}$ or $a(c + a') \in \sqrt{I}$. By Lemma 2, we have $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. So, we assume that $abI = 0$. Similarly, we can assume that $aIc = 0$ and $Ibc = 0$. Now, let $aI^2 \neq 0$. Then there exist $i_1, i_2 \in I$ such that $ai_1i_2 \neq 0$. Since $abI = aIc = Ibc = 0$, we have $0 \neq a(b + i_1)(c + i_2) = ai_1i_2 \in I$. Therefore, either $a(b + i_1) \in I$ or $a(c + i_2) \in \sqrt{I}$ or $(b + i_1)(c + i_2) \in \sqrt{I}$. Hence, we have either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. So, we can assume that $aI^2 = 0$. Likewise, we can assume that $bI^2 = 0$ and $cI^2 = 0$. Since $I^3 \neq 0$, there exist $p, q, r \in I$ such that $pqr \neq 0$. Again, $(a+p)(b+q)(c+r) = pqr \in I$, so either $(a+p)(b+q) \in I$ or $(b+q)(c+r) \in \sqrt{I}$ or $(a+p)(c+r) \in \sqrt{I}$, that is, $ab+aq+pb+pq \in I$ or $bc+br+qc+qr \in \sqrt{I}$

or $ac + ar + pc + pr \in \sqrt{I}$. Hence, either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. This implies that I is a 2-absorbing primary ideal of S, which is a contradiction. Therefore, $I^3 = 0$. Clearly, $\sqrt{0} \subseteq \sqrt{I}$. As $I^3 = 0$, we get $I \subseteq \sqrt{0}$. This concludes that $\sqrt{I} \subseteq \sqrt{0}$. Thus, $\sqrt{I} = \sqrt{0}$. □

Theorem 11. *Let S be a semiring and $\{I_i\}_{i\in\Delta}$ be a family of subtractive weakly 2-absorbing primary ideals of S that are not 2-absorbing primary ideals of S. Then $I = \bigcap_{i\in\Delta} I_i$ is a weakly 2-absorbing primary ideal of S.*

Proof. Let $\{I_i\}_{i\in\Delta}$ be a family of weakly 2-absorbing primary ideals of S that are not 2-absorbing primary ideals of S. Therefore, by Theorem 10, we have $\sqrt{I_i} = \sqrt{0}$ for all $i \in \Delta$. This gives $\bigcap_{i\in\Delta} \sqrt{I_i} = \sqrt{0}$. Thus we have $\sqrt{I} = \sqrt{0}$, since $\bigcap_{i\in\Delta} \sqrt{I_i} = \sqrt{I}$. Next, let $a, b, c \in S$ be such that $0 \neq abc \in I$ but $ab \notin I$. Then there exists $i \in \Delta$ such that $ab \notin I_i$ and $0 \neq abc \in I_i$. This gives $bc \in \sqrt{I_i}$ or $ac \in \sqrt{I_i}$ since I_i is a weakly 2-absorbing primary ideal of S and $ab \notin I_i$. Thus, either $bc \in \sqrt{I_i} = \sqrt{0} = \sqrt{I}$ or $ca \in \sqrt{I_i} = \sqrt{0} = \sqrt{I}$. Hence I is a weakly 2-absorbing primary ideal of S. □

Definition 5 ([4, Definition 1(i)]). *A proper ideal I of a semiring S is said to be a strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$.*

Proposition 1. *Let S and S' be semirings, $f : S \mapsto S'$ be an epimorphism such that $f(0) = 0$ and I be a subtractive strong ideal of S. Then the following holds:*

(1) If I is a weakly 2-absorbing primary ideal of S such that $kerf \subseteq I$, then $f(I)$ is a weakly 2-absorbing primary ideal of S'.

(2) If I is a 2-absorbing primary ideal of S such that $kerf \subseteq I$, then $f(I)$ is a 2-absorbing primary ideal of S'.

Proof. *(1)* Let $a, b, c \in S'$ be such that $0 \neq abc \in f(I)$. Then there exists an element $m \in I$ such that $0 \neq abc = f(m)$. Since f is an epimorphism, therefore there exist $p, q, r \in S$ such that $f(p) = a, f(q) = b, f(r) = c$. Also, since I is a strong ideal of S and $m \in I$, therefore there exists $n \in I$ such that $m + n = 0$. This implies $f(n + m) = 0$, that is, $f(pqr + n) = 0$, implies $pqr + n \in kerf \subseteq I$. So, $0 \neq pqr \in I$ (as I is a subtractive ideal of S) because if $pqr = 0$, then $f(m) = 0$, a contradiction. Since I is a weakly 2-absorbing primary ideal of S, therefore either $pq \in I$ or $qr \in \sqrt{I}$ or $rp \in \sqrt{I}$. Thus $ab \in f(I)$ or $bc \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $ac \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Hence, $f(I)$ is a weakly 2-absorbing primary ideal of S'.

(2) It follows from *(1)*. □

Proposition 2. *Let $a, x \in S$. Then the following holds:*

(1) suppose Sx be a subtractive ideal S and if $Ann(x) \subseteq Sx$. Then Sx is a 2-absorbing primary ideal of S if and only if Sx is a weakly 2-absorbing primary ideal of S.

(2) suppose aI be a subtractive ideal S and if $Ann(a) \subseteq aI$. Then aI is a 2-absorbing primary ideal of S if and only if it is a weakly 2-absorbing primary ideal of S.

Proof. *(1)* Let Sx be a weakly 2-absorbing primary ideal of S and $r, s, t \in S$ with $rst \in Sx$. If $rst \neq 0$, then $rs \in Sx$ or $rt \in \sqrt{Sx}$ or $st \in \sqrt{Sx}$, which implies Sx is a 2-absorbing primary ideal of S. So we assume that $rst = 0$. Evidently, $rs(x+t) \in Sx$. If $rs(x+t) \neq 0$, we have $rs \in Sx$ or $r(x+t) \in \sqrt{Sx}$ or $s(x+t) \in \sqrt{Sx}$, as Sx is a weakly 2-absorbing primary ideal of S. By Lemma 2, we have either $rs \in Sx$ or $rt \in \sqrt{Sx}$ or $st \in \sqrt{Sx}$. Therefore, we have $rs(x+t) = 0$ implies $rsx = 0$ and so $rs \in Ann(x) \subseteq Sx$ and thus $rs \in Sx$. Hence Sx is a 2-absorbing primary ideal of S.

(2) Let aI be a weakly 2-absorbing primary ideal and $r, s, t \in S$ such that $rst \in aI$. If $rst \neq 0$ then $rs \in aI$ or $rt \in \sqrt{aI}$ or $st \in \sqrt{aI}$, which implies aI is a 2-absorbing primary ideal of S. So, we assume $rst = 0$. Clearly, $r(s+a)t = rst + rat \in aI$. If $r(s+a)t \neq 0$, then $r(s+a) \in aI$ or $rt \in \sqrt{aI}$ or $(s+a)t \in \sqrt{aI}$. By Lemma 2, we get either $rs \in aI$ or $rt \in \sqrt{aI}$ or $st \in \sqrt{aI}$. So, we assume that $r(s+a)t = 0$ implies $rat = 0$, as $rst = 0$. Hence $rt \in Ann(a) \subseteq aI$. Thus $rt \in aI$ and hence aI is a 2-absorbing primary ideal of S. □

Consider $S = S_1 \times S_2$ where each S_i, $i = 1, 2$ is a commutative semiring with unity and $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$ for all $a_1, b_1 \in S_1$ and $a_2, b_2 \in S_2$.

Proposition 3. *Let I be a proper ideal of a semiring S_1. Then the following statements are equivalent:*

(1) I is a 2-absorbing primary ideal of S_1.

(2) $I \times S_2$ is a 2-absorbing primary ideal of $S = S_1 \times S_2$.

(3) $I \times S_2$ is a weakly 2-absorbing primary ideal of $S = S_1 \times S_2$.

Proof. (1) ⇒ (2) Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in S$ be such that $(a_1, a_2)(b_1, b_2)(c_1, c_2) \in I \times S_2$. Then $(a_1b_1c_1, a_2b_2c_2) \in I \times S_2$ implies $a_1b_1c_1 \in I$. This gives either $a_1b_1 \in I$ or $(b_1c_1)^m \in I$ or $(a_1c_1)^n \in I$ for some positive integers m, n, since I is a 2-absorbing primary ideal of S_1. If $a_1b_1 \in I$, then $(a_1, a_2)(b_1, b_2) \in I \times S_2$. If $b_1^m c_1^m \in I$ for some positive integer m, then $(b_1^m, b_2^m)(c_1^m, c_2^m) \in I \times S_2$, that is, $(b_1^m c_1^m, b_2^m c_2^m) \in I \times S_2$. Similarly, we can prove the case when $(a_1c_1)^n \in I$ for some positive integer n. Hence, $I \times S_2$ is a 2-absorbing primary ideal of S.

(2) ⇒ (3) It is obvious.

(3) ⇒ (1) Let $abc \in I$ for some $a, b, c \in S_1$. Then for each $0 \neq r \in S_2$, we have $(0,0) \neq (a,1)(b,1)(c,r) \in I \times S_2$. This gives $(a,1)(b,1) \in I \times S_2$ or $(b^m, 1)(c^m, r^m) \in I \times S_2$ or $(c^n, r^n)(a^n, 1) \in I \times S_2$, since $I \times S_2$ is a weakly 2-absorbing primary ideal of S. That is, either $ab \in I$ or $b^m c^m \in I$ or $a^n c^n \in I$ for some positive integers m, n. This shows that I is a 2-absorbing primary ideal of S_1. □

Theorem 12. *Let (S, M) be a local semiring with $M^3 = 0$. Then every proper subtractive ideal of S is a weakly 2-absorbing primary ideal of S.*

Proof. Proof is analogous to the proof of [10, Theorem 2.8]. □

Theorem 13. *Let S be a semiring, I a Q-ideal of S and P a subtractive ideal of S such that $I \subseteq P$. Then*

(1) if P is a weakly 2-absorbing primary ideal of S, then $P/I_{(Q\cap P)}$ is a weakly 2-absorbing primary ideal of $S/I_{(Q)}$.

(2) if I and $P/I_{(Q\cap P)}$ are weakly 2-absorbing primary ideals of S and $S/I_{(Q)}$ respectively, then P is a weakly 2-absorbing primary ideal of S.

Proof. (1) If $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \neq 0$ in S/I_Q then $q_1q_2q_3 \neq 0$ in S, then the proof follows from Theorem 9.

(2) Let $a, b, c \in S$ be such that $0 \neq abc \in P$. If $abc \in I$, then either $ab \in I \subseteq P$ or $bc \in I \subseteq \sqrt{I} \subseteq \sqrt{P}$ or $ca \in I \subseteq \sqrt{I} \subseteq \sqrt{P}$, since I is a weakly 2-absorbing primary ideal of S. So, assume that $abc \notin I$. Then there are elements $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$, $c \in q_3 + I$. Therefore, for some $i_1, i_2, i_3 \in I$, $a = q_1 + i_1$, $b = q_2 + i_2$, $c = q_3 + i_3$. As $abc = q_1q_2q_3 + q_1q_2i_3 + q_1q_3i_2 + q_1i_2i_3 + q_2q_3i_1 + q_2i_1i_3 + q_3i_1i_2 + i_1i_2i_3 \in P$ and since P is subtractive, we have $q_1q_2q_3 \in P$. Consider, $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ where q_4 is the unique element such that $q_1q_2q_3 + I \subseteq q_4 + I$. Since P is subtractive, we have $q_4 \in P \cap Q$, hence $q_1q_2q_3 + I \subseteq q_4 + I \in P/I_{Q\cap P}$, that is, $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I_{Q\cap P}$. Let $q \in Q$ be the unique element such that $q + I$ is the zero element in S/I_Q. If $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = 0_{S/I_Q} = q + I$, then there exit $r, s \in I$ such that $q_1q_2q_3 + r = q + s \in I$. Therefore, $q_1q_2q_3 \in I$, since I is a Q-ideal of S, it is subtractive by [Cor. 8.23, 11]. This gives $abc \in I$, a contradiction. Hence, $0_{S/I_Q} \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in P/I_{Q\cap P}$. This gives either $(q_1 + I) \odot (q_2 + I) \in P/I_{Q\cap P}$ or $(q_2^l + I) \odot (q_3^l + I) \in P/I_{Q\cap P}$ or $(q_3^t + I) \odot (q_1^t + I) \in P/I_{Q\cap P}$ for some positive integers l, t since $P/I_{Q\cap P}$ is a weakly 2-absorbing primary ideal of S/I_Q. Thus, either $ab \in P$ or $(bc)^l \in P$ or $(ca)^t \in P$ for some positive integers l, t. Hence, P is a weakly 2-absorbing primary ideal of S. □

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