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On Degree Sum Energy of a Graph

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Abstract. The degree sum energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the degree sum matrix of G. In this paper, we obtain some lower bounds for the degree sum energy of a graph G.

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1. Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex v, denoted by d_i. Specially, $\Delta = \Delta(G)$ and $\delta = \delta(G)$ are called the maximum and minimum degree of vertices of G respectively. G is said to be r-regular if $\delta(G) = \Delta(G) = r$ for some positive integer r. For any integer $x, \lfloor x \rfloor$ is the positive integer less than or equal to x. For undefined terminologies we refer the reader to [5].

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced almost 30 years ago [6] and having a clear connection to chemical problems, has in newer times attracted much attention of mathematicians and mathematical chemists [3, 7–9, 13–15].

Motivated by work on maximum degree energy [1], Ramane *et al.* [12] introduced the concept of degree sum energy, which is defined as follow:

Definition 1. *Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of $v_i, i = 1, 2, \ldots, n$. Then $DS(G) = [d_{ij}]$ is called the degree sum matrix of a graph G, where*

$$d_{ij} = \begin{cases} d_i + d_j & \text{if } i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

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The characteristic polynomial of $DS(G)$ is denoted by $f_n(G,\lambda) := det(\lambda I - DS(G))$. Since $DS(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The maximum degree energy of G is then defined as

$$E_{DS}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In this paper, we are interested in to obtain some new lower bounds for the degree sum energy of a graph G.

2. Results

For the sake of completeness, we mention below some results which are important throughout the paper.

Lemma 1 ([12]). *Since $trace(DS(G)) = 0$, the eigenvalues of $DS(G)$ satisfied the following relations*

(1) $\sum_{i=1}^{n} \lambda_i = 0$

(2) $\sum_{i=1}^{n} \lambda_i^2 = 2\mathscr{R}$, *where* $\mathscr{R} = \sum_{1\leq i<j\leq n} (d_i + d_j)^2$

Lemma 2 ([12]). *If G is any graph with n vertices, then* $\sqrt{2\mathscr{R}} \leq E_{DS}(G)$.

Theorem 1 ([11]). *Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2 \tag{1}$$

where $M_1 = max_{1\leq i\leq n}(a_i)$*;* $M_2 = max_{1\leq i\leq n}(b_i)$*;* $m_1 = min_{1\leq i\leq n}(a_i)$ *and* $m_2 = min_{1\leq i\leq n}(b_i)$

Theorem 2 ([10]). *Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2 \tag{2}$$

where M_i and m_i are defined similarly to Theorem 1.

Theorem 3 ([2]). *Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then*

$$|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i| \leq \alpha(n)(A-a)(B-b) \tag{3}$$

where a, b, A and B are real constants, that for each i, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor\left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Theorem 4 ([4]). *Let a_i and $b_i, 1 \leq i \leq n$ are nonnegative real numbers, then*

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r+R)\Big(\sum_{i=1}^{n} a_i b_i\Big) \tag{4}$$

where r and R are real constants, so that for each i, $1 \leq i \leq n$, holds, $ra_i \leq b_i \leq Ra_i$.

3. Bounds for the Degree Sum Energy of Graphs

Theorem 5. *Let G be a graph of order n and size m, then*

$$E_{DS}(G) \geq \sqrt{2\mathscr{R}n - \frac{n^2}{4}(\lambda_1 - \lambda_n)^2} \tag{5}$$

where λ_1 and λ_n are maximum and minimum of the absolute value of λ_i's.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 2 implies

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2 \leq \frac{n^2}{4}\big(\lambda_1 - \lambda_n\big)^2$$

$$2\mathscr{R}n - (E_{DS}(G))^2 \leq \frac{n^2}{4}\big(\lambda_1 - \lambda_n\big)^2$$

$$E_{DS}(G) \geq \sqrt{2\mathscr{R}n - \frac{n^2}{4}(\lambda_1 - \lambda_n)^2},$$

as asserted. □

Theorem 6. *Suppose zero is not an eigenvalue of $DS(G)$. Then*

$$E_{DS}(G) \geq \frac{2\sqrt{\lambda_1 \lambda_n}\sqrt{2\mathscr{R}n}}{\lambda_1 + \lambda_n}. \tag{6}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of λ_i's.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $a_i = |\lambda_i|$ and $b_i = 1$, which by Theorem 1 implies

$$\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\Big(\sqrt{\frac{\lambda_n}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_n}}\Big)^2 \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2$$

$$2\mathscr{R}n \leq \frac{1}{4}\Big(\frac{(\lambda_1 + \lambda_n)^2}{\lambda_1 \lambda_n}\Big)(E_{DS}(G))^2$$

$$E_{DS}(G) \geq \frac{2\sqrt{\lambda_1 \lambda_n}\sqrt{2\mathscr{R}n}}{\lambda_1 + \lambda_n},$$

as desired. □

Theorem 7. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $DS(G)$. Then*

$$E_{DS}(G) \geq \sqrt{2\mathscr{R}n - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \tag{7}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $a_i = |\lambda_i| = b_i$, $a = |\lambda_n| = b$ and $A = |\lambda_1| = b$, which by Theorem 3 implies

$$|n\sum_{i=1}^{n} |\lambda_i|^2 - \left(\sum_{i=1}^{n} |\lambda_i|\right)^2| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2 \tag{8}$$

Since, $E_{DS}(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2\mathscr{R}$, the above inequality becomes

$$2\mathscr{R}n - E_{DS}(G)^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

and a simple calculation gives us the required result. □

Corollary 1. *Since $\alpha(n) \leq \frac{n^2}{4}$, then according to (7), we have*

$$\begin{aligned} E_{DS}(G) &\geq \sqrt{2\mathscr{R}n - \alpha(n)(|\lambda_1| - |\lambda_n|)^2} \\ &\geq \sqrt{2\mathscr{R}n - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}. \end{aligned}$$

This means that inequality (7) is stronger of inequality (5).

Theorem 8. *Let G be a graph of order n and size m. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $DS(G)$. Then*

$$E_{DS}(G) \geq \frac{|\lambda_1||\lambda_n|n + 2\mathscr{R}}{|\lambda_1| + |\lambda_n|} \tag{9}$$

where λ_1 and λ_n are minimum and maximum of the absolute value of $\lambda_i's$.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $b_i = |\lambda_i|$, $a_i = 1, r = |\lambda_n|$ and $R = |\lambda_1|$, which by Theorem 4 implies

$$\sum_{i=n}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|)\sum_{i=1}^{n} |\lambda_i|. \tag{10}$$

Since, $E_{DS}(G) = \sum_{i=1}^{n} |\lambda_i|$, $\sum_{i=1}^{n} |\lambda_i|^2 = 2\mathscr{R}$, from (10), inequality (9) directly follows from Theorem 4. □

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