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Controllability of Mild Solutions for Evolution Equations with Infinite State-Dependent Delay

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Abstract. We consider in this paper the controllability of mild solutions defined on the semi-infinite positive real interval for two classes of first order partial functional and neutral functional evolution equations with infinite state-dependent delay using a nonlinear alternative due to Avramescu for sum of compact and contraction operators in Fréchet spaces, combined with the semigroup theory.

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1. Introduction

Controllability of mild solutions is given in this paper over the semi-infinite real interval $J := [0, +\infty)$ for two classes of first order partial and neutral functional evolution equations with infinite state-dependent delay in a real separable Banach space $(E, |\cdot|)$.

In Section 3, we study the following evolution equation

$$\begin{aligned} y'(t) =& A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \text{ a.e. } t \in J, \\ y_0 =& \phi \in \mathscr{B} \end{aligned} \tag{1}$$

and in Section 4, we study the following neutral evolution equation

$$\begin{aligned} \frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] =& A(t)y(t) + Cu(t) + f(t, y_{\rho(t,y_t)}), \text{ a.e. } t \in J, \\ y_0 =& \phi \in \mathscr{B}, \end{aligned} \tag{2}$$

where $\mathscr{B}$ is an abstract *phase space* to be specified later, $f, g : J \times \mathscr{B} \to E$, $\rho : J \times \mathscr{B} \to \mathbb{R}$ and $\phi \in \mathscr{B}$ are given functions, the control function $u(\cdot)$ is given in $L^2(\mathbb{R}_+; E)$, the Banach space

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of admissible control function with E is a real separable Banach space with the norm $|\cdot|$, C is a bounded linear operator from E into E and $\{A(t)\}_{0\le t<+\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an evolution system of operators $\{U(t,s)\}_{(t,s)\in J\times J}$ for $s\le t$.

For any continuous function y and any $t\le 0$, we denote by y_t the element of $\mathscr{B}$ defined by $y_t(\theta)=y(t+\theta)$ for $\theta\le 0$: Here $y_t(\cdot)$ represents the history of the state from time $t\le 0$ up to the present time t.

Finally in Section 5, we illustrate by examples the previous abstract theory obtained.

Controllability problem of linear and nonlinear systems represented by ODEs in finite dimensional space has been extensively studied. Several authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators (see [13, 28]) and developed more results in [26, 30, 34]. Carmichael and Quinn [12] have shown that the controllability problem can be converted into a fixed point problem. Then, interesting controllability results are given for neutral problems with impulses by Balachandran *et al.* in [4] and for integrodifferential equations by Machado *et al.* in [27] and for inclusions by Gunasekar *et al.* in [18, 19, 31]. Recently Baghli *et al.* have studied many classes of functional evolution equations and inclusions in [6, 7] and proposed some controllability results in [1] and [8] when the delay is finite and infinite.

However, complicated situations in which the delay depends on the unknown functions have been proposed in modelling in recent years. These equations are frequently called equations with state-dependent delay. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay; see for instance the books [21, 24, 32], and the papers [14, 20]. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Hernandez *et al.* [22] and Li *et al.* [25]. Very recently, Baghli *et al.* considered when the solution is depending in the delay for evolution equations in [9], for multivalued problems in [10] and for perturbed evolution equations in [3].

Our main purpose in this paper is to extend the controllability results obtained by Baghli *et al.* in [1] and [8] when $\rho(t,y_t)=t$ to the control problems (1) and (2) with infinite *state-dependent delay* as in [9]. We provide sufficient conditions for the existence of mild solutions using the nonlinear alternative of Avramescu [5] due to Burton and Kirk [11] for contractions maps in Fréchet spaces, combined with semigroup theory [2, 29].

2. Preliminaries

We introduce notations, definitions and theorems which are used in this paper.

Let $C(\mathbb{R}_+;E)$ be the space of continuous functions from $\mathbb{R}_+$ into E and $B(E)$ be the space of all bounded linearoperators from E into E, with the usual supremum norm
$\|N\|_{B(E)}=\sup\{\,|N(y)|\;:\;|y|=1\,\}$ for all $N\in B(E)$.

A measurable function $y:\mathbb{R}_+\to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable (See the Bochner integral properties in Yosida [33]).

Let $L^1(\mathbb{R}_+, E)$ be the Banach space of measurable functions $y : \mathbb{R}_+ \to E$ which are Bochner integrable normed by $\|y\|_{L^1} = \int_0^{+\infty} |y(t)|\, dt$.

In this paper, we will employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato in [20] and follow the terminology used by Hino *et al.* in [23] (More details and some examples of phase spaces could be found in [23]). Thus, $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms :

(A1) If $y : (-\infty, b) \to E$, $b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathscr{B}$, then for every $t \in [0, b)$ the following conditions hold :

- (i) $y_t \in \mathscr{B}$;
- (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathscr{B}}$;
- (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathscr{B}} \leq K(t) \sup_{s\in[0,t]} |y(s)| + M(t)\|y_0\|_{\mathscr{B}}.$$

(A2) For the function y in (A_1), y_t is a $\mathscr{B}-$valued continuous function on $[0, b]$.

(A3) The space $\mathscr{B}$ is complete.

Denote $K_b = \sup_{t\in[0,b]} K(t)$ and $M_b = \sup_{t\in[0,b]} M(t)$.

Definition 1. *A function $f : J \times \mathscr{B} \to E$ is said to be an L^1-Carathéodory function if it satisfies:*

(i) for each $t \in J$ the function $f(t, \cdot) : \mathscr{B} \to E$ is continuous;

(ii) for each $y \in \mathscr{B}$ the function $f(\cdot, y) : J \to E$ is measurable;

(iii) for every positive integer k there exists $h_k \in L^1(J; \mathbb{R}^+)$ such that

$$|f(t, y)| \leq h_k(t) \text{ for all } \|y\|_{\mathscr{B}} \leq k \text{ and almost every } t \in J$$

In what follows, for the family $\{A(t), t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [2]).

(P1) The domain $D(A(t))$ is independent of t and is dense in E.

(P2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$ and there is a constant M independent of λ and t such that

$$\|R(t, A(t))\| \leq M(1 + |\lambda|)^{-1}, \text{ for } Re\lambda \leq 0.$$

(P3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t - \tau|^{\alpha}, \text{ for } t, \theta, \tau \in J.$$

Lemma 1 ([2]). *Under assumptions (P1)-(P3), the Cauchy problem*

$$y'(t) - A(t)y(t) = 0, t \in J \text{ and } y(0) = y_0,$$

has a unique evolution system $U(t,s)$, $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \le s \le t < +\infty\}$ satisfying the following properties :

(i) $U(t,t) = I$ *where I is the identity operator in E,*

(ii) $U(t,s)U(s,\tau) = U(t,\tau)$ *for* $0 \le \tau \le s \le t < +\infty$,

(iii) $U(t,s) \in B(E)$ *the space of bounded linear operators on E where for every $(t,s) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\ y$ is continuous.*

For more details on evolution systems and their properties, see [2, 15, 16, 29].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies: $\|x\|_1 \le \|x\|_2 \le \|x\|_3 \le \ldots$ for every $x \in X$. Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that $\|y\|_n \le \overline{M}_n$ for all $y \in Y$. To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

We give now the definition of the appropriate concept of contraction in X then we state the corresponding nonlinear alternative result.

Definition 2. *[17] A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that: $\|f(x) - f(y)\|_n \le k_n\ \|x - y\|_n$ for all $x, y \in X$.*

Theorem 1 (Avramescu's Nonlinear Alternative [5]). *Let X be a Fréchet space and let $A, B : X \to X$ be two operators satisfying:*

(i) A is a compact operator,

(ii) B is a contraction.

Then either one of the following statements holds:

(C1) The operator $A + B$ has a fixed point;

(C2) The set $\left\{x \in X, x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0,1)$.

3. Semilinear Evolution Equations

Before stating and proving our first main result, we define firstly the corresponding mild solution then we define the concept of controllability for that problem and finally we expose the properties of state-dependent delay.

Definition 3. *We say that the function $y : \mathbb{R} \to E$ is a mild solution of (1) if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies for each $t \geq 0$ the following integral equation*

$$y(t) = U(t,0)\phi(0) + \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)\, f(s, y_{\rho(s,y_s)})ds. \tag{3}$$

Definition 4. *The evolution problem (1) is said to be controllable if for every initial function $\phi \in \mathscr{B}$, $y^* \in E$ and for some $n \in \mathbb{N}$, there is some control $u \in L^2([0,n];E)$ such that the mild solution $y(\cdot)$ of (1) satisfies the terminal condition $y(n) = y^*$.*

Set $\mathscr{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathscr{B}, \rho(s,\phi) \leq 0\}$. We always assume that $\rho : J \times \mathscr{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathscr{R}(\rho^-)$ into $\mathscr{B}$ and there exists a continuous and bounded function $L^\phi : \mathscr{R}(\rho^-) \to (0,+\infty)$ such that

$$\|\phi_t\|_{\mathscr{B}} \leq L^\phi(t)\|\phi\|_{\mathscr{B}} \text{ for every } t \in \mathscr{R}(\rho^-).$$

Remark 1. *Continuous and bounded functions verified frequently the condition (H_ϕ), for more details, see for instance [23].*

Lemma 2 ([22]). *If $y : (-\infty, b] \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathscr{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathscr{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, max\{0,s\}]\},\ s \in \mathscr{R}(\rho^-) \cup J$$

where $L^\phi = \sup_{t \in \mathscr{R}(\rho^-)} L^\phi(t)$.

Proposition 1. *From (H_ϕ), (A1) and Lemma 2, for all $t \in [0,n]$ and $n \in \mathbb{N}$ we have*

$$\|y_{\rho(t,y_t)}\|_{\mathscr{B}} \leq K_n|y(t)| + (M_n + L^\phi)\|\phi\|_{\mathscr{B}}.$$

We will need to introduce the following hypothesis which are assumed thereafter :

(H0) $U(t,s)$ is compact for $t - s > 0$.

(H1) There exists a constant $\widehat{M} \geq 1$ such that $\|U(t,s)\|_{B(E)} \leq \widehat{M}$ for every $(t,s) \in \Delta$.

(H2) There exists a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,+\infty)$ and such that:

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathscr{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathscr{B}.$$

(H3) For all $R > 0$, there exists $l_R \in L^1_{loc}(J;\mathbb{R}_+)$ such that:

$$|f(t,u)-f(t,v)| \leq l_R(t)\ \|u-v\|_{\mathscr{B}}$$

for all $u, v \in \mathscr{B}$ with $\|u\|_{\mathscr{B}} \leq R$ and $\|v\|_{\mathscr{B}} \leq R$.

(H4) For each $n \in \mathbb{N}$, the linear operator $W : L^2([0,n];E) \to E$ is defined by

$$Wu = \int_0^n U(n,s)Cu(s)ds,$$

has a pseudo invertible operator $\tilde{W}^{-1}$ which takes values n $L^2([0,n];E)/\ker W$ and there exists positive constants $\widetilde{M}$ and $\widetilde{M}_1$ such that: $\|C\| \leq \widetilde{M}$ and $\|\tilde{W}^{-1}\| \leq \widetilde{M}_1$.

For the construction of $\tilde{W}^{-1}$ see the paper of Carmichael *et al.* [12].

Consider the following space

$$B_{+\infty} = \left\{y : \mathbb{R} \to E : y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathscr{B}\right\}$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by:

$$\|y\|_n := \sup_{t\in[0,n]} e^{-\tau\ L_n^*(t)}\ |y(t)|$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)\ ds$, $\bar{l}_n(t) = K_n\widehat{M}l_n(t)$ and l_n is the function from (H3).

Then $B_{+\infty}$ is a Fréchet space with those family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

Theorem 2. *Assume that* (H_ϕ) *and (H0)-(H4) hold and moreover for each* $n \in \mathbb{N}$*, there exists a constant* $M_\star^n > 0$ *such that*

$$\frac{M_\star^n}{\alpha_n + K_n\widehat{M}(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1)\psi(M_\star^n)\ \|p\|_{L^1}} > 1, \tag{4}$$

with $\alpha_n = K_n\widehat{M}\widetilde{M}\widetilde{M}_1 n|y^*| + \left[M_n + L^\phi + K_n\widehat{M}H\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\right)\right]\|\phi\|_{\mathscr{B}}$. *Then the evolution problem* (1) *is controllable on* $\mathbb{R}$.

Proof. We transform the problem (1) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by:

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \leq 0; \\ U(t,0)\phi(0) + \int_0^t U(t,s)\ C\ u_y(s)ds + \int_0^t U(t,s)\ f(s, y_{\rho(s,y_s)})ds & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1).

Using assumption (H4), for arbitrary function $y(\cdot)$, we define the control

$$u_y(t) = \tilde{W}^{-1}\left[y^* - U(n,0)\ \phi(0) - \int_0^n U(n,\tau)\ f(\tau, y_{\rho(\tau,y_\tau)})\ d\tau\right](t).$$

Applying (H2), we get

$$|u_y(t)| \leq \widetilde{M}_1\left[|y^*| + \widehat{M}H\|\phi\|_{\mathscr{B}} + \widehat{M}\int_0^n p(\tau)\,\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathscr{B}})d\tau\right]. \tag{5}$$

We shall show that using this control the operator N has a fixed point $y(\cdot)$. Then $y(\cdot)$ is a mild solution of the evolution system (1).

For $\phi \in \mathscr{B}$, we will define the function $x(\cdot) : \mathbb{R} \to E$ by $x(t) = \phi(t)$ for $t \leq 0$ and $x(t) = U(t,0)\phi(0)$ for $t \in J$. Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by $\overline{z}(t) = 0$ for $t \leq 0$ and $\overline{z}(t) = z(t)$ for $t \in J$.

If $y(\cdot)$ satisfies (3), we can decompose it as $y(t) = z(t) + x(t)$, $t \geq 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies for $t \in J$

$$z(t) = \int_0^t U(t,s)\; C\; u_{z+x}(s)ds + \int_0^t U(t,s)\; f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds.$$

Let $B^0_{+\infty} = \left\{z \in B_{+\infty} : z_0 = 0 \in \mathscr{B}\right\}$. For any $z \in B^0_{+\infty}$ we have $\|z\|_{+\infty} = \sup_{s\geq 0}|z(s)|$.

Thus $(B^0_{+\infty}, \|\cdot\|_{+\infty})$ is a Banach space. We define the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by $F(z)(t) = \int_0^t U(t,s)\; C\; u_{z+x}(s)ds$ and $G(z)(t) = \int_0^t U(t,s)\; f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds$.

Obviously the operator N has a fixed point is equivalent to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in several steps. First we show that F is continuous and compact.

Step 1: F is continuous. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_n \to z$ in $B^0_{+\infty}$. By (H1), (H4) and (5), we get for every $t \in [0, n]$

$$\begin{aligned}|F(z_n)(t) - F(z)(t)| \leq& \widehat{M}\widetilde{M}\int_0^t |u_{z_n+x}(s) - u_{z+x}(s)|ds\\ \leq& \widehat{M}^2\widetilde{M}\widetilde{M}_1\int_0^t\int_0^n |f(\tau, z_{n\rho(\tau,z_{n\tau}+x_\tau)} + x_{\rho(\tau,z_{n\tau}+x_\tau)})\\ &- f(\tau, z_{\rho(\tau,z_\tau+x_\tau)} + x_{\rho(\tau,z_\tau+x_\tau)})|d\tau ds\\ \leq& \widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n |f(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|ds.\end{aligned}$$

Since f is continuous, we obtain by the Lebesgue Dominated Convergence theorem

$$|F(z_n)(t) - F(z)(t)| \to 0 \text{ as } n \to +\infty.$$

Thus F is continuous.

Step 2: F maps bounded sets of $B^0_{+\infty}$ into bounded sets. For any $d > 0$, there exists a positive constant ℓ such that for each $z \in B_d = \{z \in B^0_{+\infty} : \|z\|_n \leq d\}$ one has $\|F(z)\|_n \leq \ell$. Let $z \in B_d$. By (H1), (H2) and (5), we have for each $t \in [0, n]$

$$|F(z)(t)| \leq \widehat{M}\widetilde{M}\int_0^t \widetilde{M}_1\left[|\widehat{y}| + \widehat{M}H\|\phi\|_{\mathscr{B}}\right.$$

$$+\widehat{M}\int_0^n p(\tau)\ \psi(\|z_{\rho(\tau,z_\tau+x_\tau)}+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathscr{B}})\ d\tau\Bigg]ds$$
$$\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}+\widehat{M}\int_0^n p(s)\ \psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}})ds\right].$$

Using Proposition 1, we get

$$\begin{aligned}\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} &\leq K_n|z(s)|+(M_n+\mathscr{L}^\phi)\|z_0\|_{\mathscr{B}}+K_n|x(s)|+(M_n+\mathscr{L}^\phi)\|x_0\|_{\mathscr{B}}\\ &\leq K_n|z(s)|+K_n\|U(s,0)\|_{B(E)}|\phi(0)|+(M_n+\mathscr{L}^\phi)\|\phi\|_{\mathscr{B}}\\ &\leq K_n|z(s)|+(M_n+\mathscr{L}^\phi+K_n\widehat{M}H)\|\phi\|_{\mathscr{B}}.\end{aligned}$$

Set $c_n := (M_n+\mathscr{L}^\phi+K_n\widehat{M}H)\|\phi\|_{\mathscr{B}}$, then we obtain

$$\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} \leq K_n|z(s)|+c_n. \tag{6}$$

Since $z\in B_d$, then we have for $\delta_n := K_n d + c_n$

$$\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}} \leq K_n|z(s)|+c_n \leq \delta_n. \tag{7}$$

Using the nondecreasing character of ψ, we get for each $t\in[0,n]$

$$|F(z)(t)| \leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}+\widehat{M}\psi(\delta_n)\|p\|_{L^1}\right] := \varrho.$$

Thus there exists a positive number ϱ such that $\|F(z)\|_n \leq \varrho$. Hence $F(B_d)\subset B_\varrho$.

Step 3: F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1,\tau_2\in J$ with $\tau_2>\tau_1$ and $z\in B_d$. Then

$$\begin{aligned}|F(z)(\tau_2)-F(z)(\tau_1)| \leq &\int_0^{\tau_1} \|U(\tau_2,s)-U(\tau_1,s)\|_{B(E)}\ \|C\||u_{z+x}(s)|\ ds\\ &+\int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)}\ \|C\||u_{z+x}(s)|\ ds.\end{aligned}$$

By the inequalities (5) and (6) and using the nondecreasing character of ψ, we get

$$|u_{z+x}(t)| \leq \widetilde{M}_1\left[|y^*|+\widehat{M}H\|\phi\|_{\mathscr{B}}+\widehat{M}\ \psi(\delta_n)\ \|p\|_{L^1}\right] := \omega. \tag{8}$$

Then

$$\begin{aligned}|F(z)(\tau_2)-F(z)(\tau_1)| \leq &\|C\|_{B(E)}\ \omega\int_0^{\tau_1} \|U(\tau_2,s)-U(\tau_1,s)\|_{B(E)}\ ds\\ &+\|C\|_{B(E)}\ \omega\int_{\tau_1}^{\tau_2} \|U(\tau_2,s)\|_{B(E)}\ ds.\end{aligned}$$

Noting that $|F(z)(\tau_2)-F(z)(\tau_1)|$ tends to zero as $\tau_2-\tau_1\to 0$ independently of $z\in B_d$. The right-hand side of the above inequality tends to zero as $\tau_2-\tau_1\to 0$. Since $U(t,s)$ is a strongly continuous operator and the compactness of $U(t,s)$ for $t>s$ implies the continuity in the uniform operator topology (see [2, 29]). As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t\in J$ be fixed and let ϵ be such that $0<\epsilon<t$. For $z\in B_d$ we define

$$F_\epsilon(z)(t)=U(t,t-\epsilon)\int_0^{t-\epsilon} U(t-\epsilon,s)\ C\ u_{z+x}(s)\ ds.$$

Since $U(t,s)$ is a compact operator, the set $Z_\epsilon(t)=\{F_\epsilon(z)(t):\ z\in B_d\}$ is pre-compact in E for every ϵ sufficiently small, $0<\epsilon<t$. Moreover using (8), we have

$$\begin{aligned}|F(z)(t)-F_\epsilon(z)(t)| &\leq \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ \|C\|\ |u_{z+x}(s)|\ ds\\ &\leq \|C\|_{B(E)}\ \omega\ \int_{t-\epsilon}^{t} \|U(t,s)\|_{B(E)}\ ds.\end{aligned}$$

Therefore there are precompact sets arbitrary close to the set $\{F(z)(t):z\in B_d\}$. Hence the set $\{F(z)(t):z\in B_d\}$ is precompact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.

Step 4: We shall show now that the operator G is a contraction. Indeed, consider $z,\overline{z}\in B^0_{+\infty}$. By (H1), (H3) and (7), we get for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$\begin{aligned}|G(z)(t)-G(\overline{z})(t)| &\leq \int_0^t \widehat{M}\ l_n(s)\ \|z_{\rho(s,z_s+x_s)}-\overline{z}_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}ds\\ &\leq \int_0^t \widehat{M}\ K_n\ l_n(s)\ |z(s)-\overline{z}(s)|ds\\ &\leq \int_0^t \left[\overline{l}_n(s)\ e^{\tau L^*_n(s)}\right]\left[e^{-\tau L^*_n(s)}\ |z(s)-\overline{z}(s)|\right]ds\\ &\leq \int_0^t \left[\frac{e^{\tau\ L^*_n(s)}}{\tau}\right]'\ ds\ \|z-\overline{z}\|_n\\ &\leq \frac{1}{\tau}\ e^{\tau\ L^*_n(t)}\ \|z-\overline{z}\|_n.\end{aligned}$$

Therefore,

$$\|G(z)-G(\overline{z})\|_n\leq\frac{1}{\tau}\ \|z-\overline{z}\|_n.$$

So, the operator G is a contraction for all $n\in\mathbb{N}$.

Step 5: To apply Theorem 1, we must check *(C2)*: i.e. it remains to show that the following set is bounded $\mathscr{E}=\left\{z\in B^0_{+\infty}:\ z=\lambda\ F(z)+\lambda\ G\left(\frac{z}{\lambda}\right)\text{ for some }0<\lambda<1\right\}$.

Let $z\in\mathscr{E}$. By (5), we have for each $t\in[0,n]$

$$\frac{|z(t)|}{\lambda}\leq\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}\right]$$

$$+\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\ \psi\left(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}\right)ds$$
$$+\widehat{M}\int_0^t p(s)\ \psi\left(\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathscr{B}}\right)ds.$$

Using the first inequality in (6), we get

$$\begin{aligned}\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathscr{B}} &\leq \frac{K_n|z(s)|}{\lambda}+\frac{M_n+\mathscr{L}^\phi}{\lambda}\|z_0\|_{\mathscr{B}}\\ &\quad + K_n|x(s)|+\left(M_n+\mathscr{L}^\phi\right)\|x_0\|_{\mathscr{B}}\\ &\leq \frac{K_n|z(s)|}{\lambda}+K_n\|U(s,0)\|_{B(E)}|\phi(0)|+\left(M_n+\mathscr{L}^\phi\right)\|\phi\|_{\mathscr{B}}\\ &\leq \frac{K_n|z(s)|}{\lambda}+\left(K_n\widehat{M}H+M_n+\mathscr{L}^\phi\right)\|\phi\|_{\mathscr{B}}.\end{aligned}$$

Then, we get

$$\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathscr{B}} \leq \frac{K_n|z(s)|}{\lambda}+c_n. \tag{9}$$

By the previous inequality and the nondecreasing character of ψ, we obtain

$$\begin{aligned}\frac{|z(t)|}{\lambda} &\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}\right]+\widehat{M}^2\widetilde{M}\widetilde{M}_1 n\int_0^n p(s)\ \psi(K_n|z(s)|+c_n)ds\\ &\quad+\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n|z(s)|}{\lambda}+c_n\right)ds.\end{aligned}$$

Consider the function $\widetilde{u}(t):=\sup_{\theta\in[0,t]}|z(\theta)|$. Then by the nondecreasing character of ψ, we get for $\lambda<1$ and for $t\in[0,n]$

$$\frac{\widetilde{u}(t)}{\lambda}\leq \widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}\right]+\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\right)\int_0^n p(s)\ \psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds.$$

We consider the function μ defined by $\mu(t)=\sup_{s\in[0,t]}\frac{K_n\widetilde{u}(s)}{\lambda}+c_n$ for $t\in J$. Let $t^\star\in[0,t]$ be such that $\mu(t)=\frac{K_n u(t^\star)}{\lambda}+c_n$. If $t^\star\in[0,n]$, by the previous inequality and the nondecreasing character of ψ, we have for $\alpha_n:=c_n+K_n\widehat{M}\widetilde{M}\widetilde{M}_1 n\left[|\widehat{y}|+\widehat{M}H\|\phi\|_{\mathscr{B}}\right]$

$$\mu(t)\leq\alpha_n+K_n\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\right)\int_0^n p(s)\ \psi(\mu(s))\ ds.$$

Consequently,

$$\frac{\|z\|_n}{\alpha_n+K_n\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1 n+1\right)\psi(\|z\|_n)\|p\|_{L^1}}\leq 1.$$

Then by the condition (4), there exists a constant $M_\star^n$ such that $\mu(t)\leq M_\star^n$. Since $\|z\|_n\leq\mu(t)$, we have $\|z\|_n\leq M_\star^n$. This shows that the set $\mathscr{E}$ is bounded, i.e. the statement

(C2) in Theorem 1 does not hold. Then the Avramescu's nonlinear alternative [5] implies that *(C1)* holds: i.e. the operator $F + G$ has a fixed-point $z^\star$. Then, there exists at least $y^\star(t) = z^\star(t) + x(t)$, $t \in \mathbb{R}$ which is a fixed point of the operator N, which is a mild solution of the problem (1). Thus the evolution system (1) is controllable on $\mathbb{R}$. Then, the proof is complete. □

4. Semilinear Neutral Evolution Equations

Before stating and proving our second main result, we define firstly the corresponding mild solution then we define the concept of controllability for that problem.

Definition 5. *We say that the function $y(\cdot) : \mathbb{R} \to E$ is a mild solution of* (2) *if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$\begin{aligned} y(t) =& U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \int_0^t U(t,s)A(s)g(s, y_{\rho(s,y_s)})ds \\ &+ \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds, \end{aligned} \tag{10}$$

for each $t \geq 0$.

Definition 6. *The neutral evolution problem* (2) *is said to be controllable if for every initial function $\phi \in \mathscr{B}$, $y^* \in E$ and $n \in \mathbb{N}$, there is some control $u \in L^2([0,n]; E)$ such that the mild solution $y(\cdot)$ of* (2) *satisfies $y(n) = y^*$.*

We consider the function $\rho : J \times \mathscr{B} \longrightarrow \mathbb{R}$ satisfies the hypothesis (H_ϕ) and the Lemma 1. We assume here that the hypotheses (H0)-(H4) hold and we will need the following assumptions:

(H5) There exists a constant $\overline{M}_0 > 0$ such that $\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0$ for all $t \in J$.

(H6) There exists a constant $0 < L < \frac{1}{\overline{M}_0 K_n}$, such that
$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathscr{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathscr{B}.$$

(H7) There exists a constant $L_* > 0$ such that
$$|A(s)\ g(s,\phi) - A(\overline{s})\ g(\overline{s},\overline{\phi})| \leq L_*\ (|s - \overline{s}| + \|\phi - \overline{\phi}\|_{\mathscr{B}})$$
for all $s, \overline{s} \in J$ and $\phi, \overline{\phi} \in \mathscr{B}$.

(H8) The function g is completely continuous and for each bounded sub-set $Q \subset \mathscr{B}$, the mapping $\{t \longrightarrow g(t, x_{\rho(s,y_s)})\}$ is equicontinous in $C(J,E)$.

Theorem 3. *Suppose that hypotheses (H0)-(H8) are satisfied and moreover*

$$\frac{M^{\star\star}}{\gamma_n + \frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\left[M^{\star\star}+\psi(M^{\star\star})\right]\ \psi(\|z\|_n)\|\zeta\|_{L^1}} > 1, \tag{11}$$

where $\zeta(t) = \max(L; p(t))$ *and* $\gamma_n = (M_n + \mathscr{L}^{\phi} + K_n\widehat{M}H)\|\phi\|_{\mathscr{B}} + \frac{K_n\beta_n}{1-\overline{M}_0LK_n}$ *with*

$$\begin{aligned}\beta_n =& \left[(\widehat{M}+1)\overline{M}_0L + \widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right] + \widehat{M}\widetilde{M}\widetilde{M}_1n\left(1+K_n\overline{M}_0L\right)|\widehat{y}| \\ &+ \left[\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\overline{M}_0L\left[\widehat{M}+M_n+\mathscr{L}^{\phi}\right] + \widehat{M}H\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+\overline{M}_0LK_n\right)\right]\|\phi\|_{\mathscr{B}}.\end{aligned}$$

Then the neutral evolution problem (2) *is controllable on* $\mathbb{R}$.

Proof. Consider the operator $\widetilde{N}: B_{+\infty} \to B_{+\infty}$ defined by:

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t) & \text{if } t \le 0; \\ U(t,0)\,[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \int_0^t U(t,s)A(s)g(s, y_{\rho(s,y_s)})ds & \\ + \int_0^t U(t,s)Cu(s)ds + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})ds & \text{if } t \in J. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (2).
Using assumption (H4), for arbitrary function $y(\cdot)$, we define the control

$$\begin{aligned}u_y(t) =& \tilde{W}^{-1}\left[y^* - U(n,0)\left(\phi(0)-g(0,\phi)\right) - g(n, y_{\rho(n,y_n)})\right. \\ &\left. - \int_0^n U(n,\tau)A(\tau)g(\tau, y_{\rho(\tau,y_\tau)})d\tau - \int_0^n U(n,\tau)f(\tau, y_{\rho(\tau,y_\tau)})d\tau\right](t).\end{aligned}$$

Noting that by (H1), (H2), (H4), (H5) and (H7) we get

$$\begin{aligned}|u_y(t)| \le& \widetilde{M}_1\left[|y^*| + \widehat{M}\left(H+\overline{M}_0L\right)\|\phi\|_{\mathscr{B}} + \left(\widehat{M}+1\right)\overline{M}_0L + \widehat{M}Ln\right] \\ &+ \widetilde{M}_1\overline{M}_0L\|y_{\rho(n,y_n)}\|_{\mathscr{B}} + \widetilde{M}_1\widehat{M}L\int_0^n \|y_{\rho(\tau,y_\tau)}\|_{\mathscr{B}}d\tau \\ &+ \widetilde{M}_1\widehat{M}\int_0^n p(\tau)\psi(\|y_{\rho(\tau,y_\tau)}\|_{\mathscr{B}})d\tau.\end{aligned} \tag{12}$$

Using this control the operator $\widetilde{N}$ has a fixed point $y(\cdot)$. Then $y(\cdot)$ is a mild solution of the neutral evolution system (2).

For $\phi \in \mathscr{B}$, we will define the function $x(\cdot): \mathbb{R} \to E$ by $x(t) = \phi(t)$ for $t \le 0$ and $x(t) = U(t,0)\,\phi(0)$ for $t \in J$. Then $x_0 = \phi$. For each function $z \in B_{+\infty}$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by $\overline{z}(t) = 0$ for $t \le 0$ and $\overline{z}(t) = z(t)$ for $t \in J$.

If $y(\cdot)$ satisfies (10), we decompose it as $y(t) = z(t) + x(t)$, $t \ge 0$, which implies $y_t = z_t + x_t$, for every $t \in J$ and the function $z(\cdot)$ satisfies $z_0 = 0$ and for $t \in J$, we get

$$z(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - U(t,0)g(0,\phi$$

$$+\int_0^t U(t,s)A(s)g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds+\int_0^t U(t,s)Cu_{z+x}(s)ds$$
$$+\int_0^t U(t,s)f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds.$$

Let us define the operators $\widetilde{F},G:B^0_{+\infty}\longrightarrow B^0_{+\infty}$ by

$$\begin{aligned}\widetilde{F}(z)(t)=&g(t,z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)})-U(t,0)g(0,\phi)\\&+\int_0^t U(t,s)A(s)g(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds+\int_0^t U(t,s)Cu_{z+x}(s)ds\end{aligned}$$

and $G(z)(t)=\int_0^t U(t,s)f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})ds$.

Obviously the operator $\widetilde{N}$ has a fixed point is equivalent to $\widetilde{F}+G$ has one, so it turns to prove that $\widetilde{F}+G$ has a fixed point.

We can show as in Section 3 that the operator $\widetilde{F}$ is continuous and compact and the operator G is a contraction. For applying Avramescu's nonlinear alternative, we must check *(C2)* in Theorem 1: i.e. it remains to show that the following set

$$\widetilde{\mathscr{E}}=\left\{z\in B^0_{+\infty}:\ z=\lambda\widetilde{F}(z)+\lambda G\left(\frac{z}{\lambda}\right)\text{ for some }0<\lambda<1\right\}$$

is bounded.

Let $z\in\widetilde{\mathscr{E}}$. Then, using (H1)-(H6) and (12), we have for each $t\in[0,n]$

$$\begin{aligned}\frac{|z(t)|}{\lambda}\leq&\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right]+\widehat{M}\widetilde{M}\widetilde{M}_1n|\widehat{y}|\\&+\widehat{M}\left[\overline{M}_0L\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)+\widehat{M}\widetilde{M}\widetilde{M}_1nH\right]\|\phi\|_{\mathscr{B}}\\&+\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0Ln\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathscr{B}}\\&+\overline{M}_0L\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathscr{B}}+\widehat{M}L\int_0^t\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathscr{B}}\,ds\\&+\widehat{M}^2\widetilde{M}\widetilde{M}_1Ln\int_0^n\|\left(z_{\rho(\tau,z_\tau+x_\tau)}\right)+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathscr{B}}\,d\tau\\&+\widehat{M}^2\widetilde{M}\widetilde{M}_1n\int_0^n p(\tau)\,\psi(\|\left(z_{\rho(\tau,z_\tau+x_\tau)}\right)+x_{\rho(\tau,z_\tau+x_\tau)}\|_{\mathscr{B}})\,d\tau\\&+\widehat{M}\int_0^t p(s)\,\psi\left(\left\|\frac{z_{\rho(s,\frac{z_\tau}{\lambda}+x_s)}}{\lambda}+x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\right\|_{\mathscr{B}}\right)ds.\end{aligned}$$

By Proposition 1, we obtain $\|z_{\rho(n,z_n+x_n)}+x_{\rho(n,z_n+x_n)}\|_{\mathscr{B}}\leq K_n|\widehat{y}|+(M_n+\mathscr{L}^\phi)\|\phi\|_{\mathscr{B}}$. Using the inequalities (6) and (9), we have

$$\frac{|z(t)|}{\lambda}\leq\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right]+\widehat{M}\widetilde{M}\widetilde{M}_1n|\widehat{y}|$$

$$
\begin{aligned}
&+\widehat{M}\left[\overline{M}_0L\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)+\widehat{M}\widetilde{M}\widetilde{M}_1nH\right]\|\phi\|_{\mathscr{B}}\\
&+\widehat{M}\widetilde{M}\widetilde{M}_1\overline{M}_0Ln\left(K_n|\widehat{y}|+(M_n+\mathscr{L}^{\phi})\|\phi\|_{\mathscr{B}}\right)\\
&+\overline{M}_0L\left(K_n|z(t)|+(M_n+\mathscr{L}^{\phi}+K_n\widehat{M}H)\|\phi\|_{\mathscr{B}}\right)\\
&+\widehat{M}L\int_0^t\left(K_n|z(s)|+c_n\right)ds+\widehat{M}^2\widetilde{M}\widetilde{M}_1Ln\int_0^n\left(K_n|z(\tau)|+c_n\right)d\tau\\
&+\widehat{M}^2\widetilde{M}\widetilde{M}_1n\int_0^n p(\tau)\,\psi\left(K_n|z(\tau)|+c_n\right)d\tau\\
&+\widehat{M}\int_0^t p(s)\,\psi\left(\frac{K_n|z(s)|}{\lambda}+c_n\right)ds.
\end{aligned}
$$

We consider the function $\widetilde{u}(t):=\sup_{\theta\in[0,t]}|z(\theta)|$ then by the nondecreasing character of ψ, we obtain for

$$
\begin{aligned}
\beta_n:=&\left[(\widehat{M}+1)\overline{M}_0L+\widehat{M}Ln\right]\left[\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right]\\
&+\widehat{M}\widetilde{M}\widetilde{M}_1n\left(1+K_n\overline{M}_0L\right)|\widehat{y}|\\
&+\left[\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\overline{M}_0L\left[\widehat{M}+M_n+\mathscr{L}^{\phi}\right]+\widehat{M}H\left(\widehat{M}\widetilde{M}\widetilde{M}_1n\overline{M}_0LK_n\right)\right]\|\phi\|_{\mathscr{B}}
\end{aligned}
$$

and for $\lambda<1$,

$$
\begin{aligned}
\frac{\widetilde{u}(t)}{\lambda}\left(1-\overline{M}_0LK_n\right)\leq&\beta_n+\widehat{M}L\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\int_0^n\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds\\
&+\widehat{M}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\int_0^n p(s)\,\psi\left(\frac{K_n\widetilde{u}(s)}{\lambda}+c_n\right)ds.
\end{aligned}
$$

We consider the function μ defined by $\mu(t)=\sup_{s\in[0,t]}\frac{K_n\widetilde{u}(s)}{\lambda}+c_n$ for $t\in J$. Let $t^{\star}\in[0,t]$ be such that $\mu(t)=\frac{K_nu(t^{\star})}{\lambda}+c_n$. By the previous inequality, we have for $\gamma_n:=c_n+\frac{K_n\beta_n}{1-\overline{M}_0LK_n}$, then we obtain for $t\in[0,n]$

$$
\mu(t)\leq\gamma_n+\frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\int_0^n\left[L\mu(s)+p(s)\,\psi(\mu(s))\right]ds.
$$

Set $\zeta(t):=\max(L;p(t))$ for $t\in[0,n]$. Consequently, we get

$$
\frac{\|z\|_n}{\gamma_n+\frac{K_n\widehat{M}}{1-\overline{M}_0LK_n}\left(\widehat{M}\widetilde{M}\widetilde{M}_1n+1\right)\left[\|z\|_n+\psi(\|z\|_n)\right]\;\psi(\|z\|_n)\|\zeta\|_{L^1}}\leq 1.
$$

Then by the condition (11), there exists a constant $M^n_{\star\star}$ such that $\mu(t)\leq M^n_{\star\star}$. Since $\|z\|_n\leq\mu(t)$, we have $\|z\|_n\leq M^n_{\star\star}$. This shows that the set $\widetilde{\mathscr{E}}$ is bounded, i.e. the statement *(C2)* in Theorem 1 does not hold. Then the nonlinear alternative due to Avramescu [5] implies that *(C1)* holds: i.e. the operator $\widetilde{F}+G$ has a fixed-point $z^{\star\star}$. Then, there exists at least $y^{\star\star}(t)=z^{\star\star}(t)+x(t)$, $t\in\mathbb{R}$ which is a fixed point of the operator $\widetilde{N}$, which is a mild solution of the problem (2). Thus the neutral evolution system (2) is controllable on $\mathbb{R}$. Then, the proof is complete. □

5. Examples

To illustrate the previous results, we give in this section two examples.

Example 1

Consider the partial functional differential equation

$$\begin{aligned}
\frac{\partial z}{\partial t}(t,\xi) = & \frac{\partial^2 z(t,\xi)}{\partial \xi^2} + d(\xi)u(t) + a_0(t,\xi)z(t,\xi) \\
& + \int_{-\infty}^{0} a_1(s-t)z\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|z(t,\theta)|^2 d\theta\right),\xi\right]ds, \\
& \text{for } t \geq 0,\ \xi \in [0,\pi], \\
& z(t,0) = z(t,\pi) = 0, \text{ for } t \geq 0, \\
& z(\theta,\xi) = z_0(\theta,\xi), \text{ for } -\infty < \theta \leq 0,\ \xi \in [0,\pi],
\end{aligned} \tag{13}$$

where $a : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t; $a_0 : \mathbb{R}^+ \times [0,\pi] \to \mathbb{R}$; $a_1 : \mathbb{R}^- \to \mathbb{R}$; $a_2 : [0,\pi] \to \mathbb{R}$; $\rho_i : \mathbb{R}^+ \to \mathbb{R}$ for $i = 1,2$; $z_0 : \mathbb{R}_- \times [0,\pi] \to \mathbb{R}$ and $d : \mathbb{R}_+ \to E$ are continuous functions. $u(\cdot) : \mathbb{R}_+ \to E$ is a given control.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with $D(A) := \{w \in E \ : w'' \in E,\ w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) = \frac{\sin(n\xi)\sqrt{2}}{\sqrt{\pi}}$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E and $T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x, z_n)z_n$ for $x \in E$ and $t \geq 0$. It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$. On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(\cdot)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}$, $\xi \in [0,\pi]$, it follows that the system $u'(t) = A(t)u(t)$ $t \geq s$, and $u(s) = x \in E$, has an associated evolution family given by $U(t,s)x(\xi) = \left[T(t-s)exp\left(\int_s^t a_0(\tau,\xi)d\tau\right)x\right](\xi)$.

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$||U(t,s)|| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in \Delta.$$

Let $\mathscr{B} = BUC(\mathbb{R}_-;E)$ the space of bounded uniformly continuous functions defined from $\mathbb{R}_-$ to E endowed with the uniform norm $\|\phi\| = \sup_{\theta\in\mathbb{R}_-} |\phi(\theta)|$.

Theorem 4. *Let $\phi \in \mathscr{B}$. Assume that the condition (H_ϕ) holds and the functions $d : \mathbb{R}_+ \to E$, $\rho_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2$, $a_1 : \mathbb{R}^- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$ are continuous. Then the evolution system* (13) *is controllable on* $(-\infty,+\infty)$.

Proof. From the assumptions, we have that $f(t,\psi)(\xi)=\int_{-\infty}^{0}a_1(s)\psi(s,\xi)ds$ and $\rho(s,\psi)=s-\rho_1(s)\rho_2\left(\int_0^{\pi}a_2(\theta)|\psi(0,\xi)|^2d\theta\right)$, are well defined functions and let $C\in L(\mathbb{R};E)$ be defined as: $Cu(t)(\xi)=d(\xi)u(t)$, $t\geq 0$, $\xi\geq 0$, $u\in\mathbb{R}$, $d(\xi)\in E$, which permit to transform system (13) into the abstract system (1). Moreover, the function f is a bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 2. Thus, the conclusion of our theorem hold. □

From Remark 1, we have the following result.

Corollary 1. *Let $\phi\in\mathscr{B}$ be continuous and bounded. Then the evolution problem* (13) *is controllable on* $\mathbb{R}$.

Example 2

Consider the semilinear neutral evolution equation

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,\xi)-\int_{-\infty}^{0}a_3(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds\right]\\
&=\frac{\partial^2u(t,\xi)}{\partial\xi^2}+a_0(t,\xi)u(t,\xi)\\
&\quad+\int_{-\infty}^{0}a_1(s-t)u\left(s-\rho_1(t)\rho_2\left(\int_0^{\pi}a_2(\theta)|u(t,\theta)|^2d\theta\right),\xi\right)ds,\\
&\text{for } t\geq 0,\ \xi\in[0,\pi],\\
&v(t,0)=v(t,\pi)=0,\ \text{for } t\geq 0,\\
&v(\theta,\xi)=v_0(\theta,\xi),\ \text{for } -\infty<\theta\leq 0,\ \xi\in[0,\pi],
\end{aligned}\tag{14}$$

where $a_3:\mathbb{R}^-\to\mathbb{R}$ is a continuous function and a, a_i for $i=0,1,2$, ρ_i for $i=1,2$, z_0, d and $u(\cdot)$ are defined as in (13).

Theorem 5. *Let $\mathscr{B}=BUC(\mathbb{R}_-;E)$ and $\phi\in\mathscr{B}$. Assume that the condition (H_ϕ) holds and the functions $d:\mathbb{R}_+\to E$, $\rho_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$, $a_1,a_3:\mathbb{R}^-\to\mathbb{R}$ and $a_2:[0,\pi]\to\mathbb{R}$ are continuous. Then the evolution system* (14) *is controllable on* $(-\infty,+\infty)$.

Proof. From the assumptions, we have that

$$\begin{aligned}
f(t,\psi)(\xi)&=\int_{-\infty}^{0}a_1(s)\psi(s,\xi)ds,\\
g(t,\psi)(\xi)&=\int_{-\infty}^{0}a_3(s)\psi(s,\xi)ds,\\
\rho(s,\psi)&=s-\rho_1(s)\rho_2\left(\int_0^{\pi}a_2(\theta)|\psi(0,\xi)|^2d\theta\right)
\end{aligned}$$

are well defined functions and let $C \in L(\mathbb{R}; E)$ be defined as: $Cu(t)(\xi) = d(\xi)u(t)$, $t \geq 0$, $\xi \geq 0$, $u \in \mathbb{R}$, $d(\xi) \in E$, which permit to transform system (14) into the abstract system (2). Moreover, the function f is a bounded linear operator. Now, the controllability of mild solutions can be deduced from a direct application of Theorem 3. Thus, the conclusion of our theorem hold. □

From Remark 1, we have the following result.

Corollary 2. *Let $\phi \in \mathscr{B}$ be continuous and bounded. Then there exists a unique mild solution of* (14) *on* $\mathbb{R}$.

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References

[1] R P Agarwal, S Baghli, and M Benchohra. Controllability of mild solutions on semiinfinite interval for classes of semilinear functional and neutral functional evolution equations with infinite delay. *Applied Mathematics and Computation*, 60(2):253–274, 2009.

[2] N U Ahmed. *Semigroup Theory with Applications to Systems and Control*. Harlow John Wiley & Sons, Inc., New York, 1991.

[3] D Aoued and S Baghli-Bendimerad. Mild solution for perturbed evolution equations with infinite state-dependent delay. *Electronic Journal of Qualitative Theory of Differential Equations*, 2013(59):1–24, 2013.

[4] G Arthi and K Balachandran. Controllability of damped second-order neutral functional differential systems with impulses. *Taiwanese Journal of Mathematics*, 16(1):89–106, 2012.

[5] C Avramescu. Some remarks on a fixed point theorem of krasnoselskii. *Electronic Journal of Qualitative Theory of Differential Equations*, 2003(5):1–15, 2003.

[6] S Baghli and M Benchohra. Multivalued evolution equations with infinite delay in fréchet spaces. *Electronic Journal of Qualitative Theory of Differential Equations*, 33:1–24, 2008.

[7] S Baghli and M Benchohra. Uniqueness results for partial functional differential equations in fréchet spaces. *Fixed Point Theory*, 9(2):395–406, 2008.

[8] S Baghli, M Benchohra, and K Ezzinbi. Controllability results for semilinear functional and neutral functional evolution equations with infinite delay. *Surveys in Mathematics and its Applications*, 4(2):15–39, 2009.

[9] S Baghli, M Benchohra, and J J Nieto. Global uniqueness results for partial functional and neutral functional evolution equations with state-dependent delay. *Journal of Advanced Research in Differential Equations*, 2(3):35–52, 2010.

[10] S Baghli-Bendimerad. Global mild solution for functional evolution inclusions with state-dependent delay. *Journal of Advanced Research in Dynamical and Control Systems*, 5(4):1–19, 2013.

[11] T A Burton and C Kirk. A fixed point theorem of krasnoselskii type. *Mathematische Nachrichten*, 189(1):23–31, 1998.

[12] N Carmichael and M D Quinn. An approach to nonlinear control problems using the fixed point methods, degree theory and pseudo-inverses. *Numerical Functional Analysis and Optimization*, 7(2-3):197–219, 1984-1985.

[13] E N Chukwu and S M Lenhart. Controllability questions for nonlinear systems in abstract spaces. *Journal of Optimization Theory and Applications*, 68(3):437–462, 1991.

[14] C Corduneanu and V Lakshmikantham. Equations with unbounded delay. *Nonlinear Analysis: Theory, Methods & Applications.*, 4(5):831–877, 1980.

[15] K J Engel and R Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, New York, 2000.

[16] A Friedman. *Partial Differential Equations*. Holt, Rinehat and Winston, New York, 1969.

[17] M Frigon and A Granas. Résultats de type leray-schauder pour des contractions sur des espaces de fréchet. *Annals of Science Mathematics of Québec*, 22(2):161–168, 1998.

[18] T Gunasekar, F Paul Samuel, and M M Arjunan. Controllability results for impulsive neutral functional evolution integrodifferential inclusions with infinite delay. *European International Journal of Science and Technology*, 2(5):196–213, 2013.

[19] T Gunasekar, F Paul Samuel, and M M Arjunan. Controllability results for second order impulsive neutral functional integrodifferential inclusions with infinite delay. *Italian Journal of Pure and Applied Mathematics*, 31(27):319–332, 2013.

[20] J Hale and J Kato. Phase space for retarded equations with infinite delay. *Funkcialaj Ekvacioj*, 21(1):11–41, 1978.

[21] J K Hale and S M Verduyn Lunel. *Introduction to Functional Differential Equations* . Applied Mathematical Sciences 99, Springer-Verlag, New York, 1993.

[22] E Hernandez, R Sakthivel, and S Tanaka Aki. Existence results for impulsive evolution differential equations with state-dependent delay. *Electronic Journal of Differential Equations*, 28(2008):1–11, 2008.

[23] Y Hino, S Murakami, and T Naito. *Functional Differential Equations with Unbounded Delay*. Springer-Verlag, Berlin, 1991.

[24] V Kolmanovskii and A Myshkis. *Introduction to the Theory and Applications of Functional-Differential Equations*. Kluwer Academic Publishers, Dordrecht, 1999.

[25] W S Li, Y K Chang, and J J Nieto. Solvability of impulsive neutral evolution differential inclusions with state-dependent delay. *Mathematical and Computer Modelling*, 49:1920–1927, 2009.

[26] X Li and J Yong. *Optimal Control Theory for Infinite Dimensional Systems*. Birkhauser, Berlin, 1995.

[27] J A Machado, C Ravichandran, M Rivero, and J J Trujillo. Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions. *Fixed Point Theory and Applications*, 66(2015):1–16, 2015.

[28] S Nakagiri and R Yamamoto. Controllability and observability for linear retarded systems in banach space. *International Journal of Control*, 49(5):1489–1504, 1989.

[29] A Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.

[30] B Radhakrishnan and K Balachandran. Controllability of nonlinear differential evolution systems in a separable banach spaces. *Electronic Journal of Differential Equations*, 2012(138):1–13, 2012.

[31] S M Ramakzishna, T Gunasekar, G V Subramaniyan, and M Suba. Controllability of impulsive neutral functional integrodifferential inclusions with an infinite delay. *Global Journal of Pure and Applied Mathematics*, 10(6):817–834, 2014.

[32] J Wu. *Theory and Applications of Partial Functional Differential Equations*. Springer-Verlag, New York, 1996.

[33] K Yosida. *Functional Analysis*. Springer-Verlag, Berlin, 6th edition, 1980.

[34] J Zabczyk. *Mathematical Control Theory*. Birkhauser, Berlin, 1992.