Properties of semi-elementary imsets as sums of elementary imsets

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Abstract. We study properties of semi-elementary imsets and elementary imsets introduced by Studený [10]. The rules of the semi-graphoid axiom (decomposition, weak union and contraction) for conditional independence statements can be translated into a simple identity among three semi-elementary imsets. By recursively applying the identity, any semi-elementary imset can be written as a sum of elementary imsets, which we call a representation of the semi-elementary imset. A semi-elementary imset has many representations. We study properties of the set of possible representations of a semi-elementary imset and prove that all representations are connected by relations among four elementary imsets.

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1. Introduction

The method of imsets introduced by Studený [10] provides a very powerful algebraic method for studying conditional independence statements which hold under a probability distribution. In this paper we prove some facts on semi-elementary imsets when they are represented as non-negative integer combinations of elementary imsets. In particular we prove that all representations of a semi-elementary imset are connected by relations among four elementary imsets.

Let N denote a finite set of random variables and let A, B, C denote disjoint subsets of N. The union $A \cup B$ of two sets A, B is abbreviated as AB. As usual $A \perp\!\!\!\perp B | C$ denotes that the random variables in A are conditionally independent of those in B given the

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variables in C. Three rules of the semi-graphoid axiom, i) decomposition, ii) weak union and iii) contraction, can be summarized in the following single equivalence

$$X \perp\!\!\!\perp Y_1 Y_2 | Z \quad \Leftrightarrow \quad X \perp\!\!\!\perp Y_1 | Z \text{ and } X \perp\!\!\!\perp Y_2 | Y_1 Z \tag{1}$$

for any disjoint subsets X, Y_1, Y_2, Z of N.

Let $\mathcal{P}(N)$ denote the power set of N. For each triplet of disjoint subsets of N, denoted by $\langle A, B \,|\, C \rangle$, Studený [10] defined the *semi-elementary imset* $u_{\langle A, B \,|\, C \rangle} : \mathcal{P}(N) \mapsto \mathbb{Z}$ by

$$u_{\langle A, B \,|\, C \rangle}(S) = \begin{cases} 1 & \text{if } S = ABC \text{ or } S = C \\ -1 & \text{if } S = AC \text{ or } S = BC \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In terms of semi-elementary imsets, (1) is written as an identity

$$u_{\langle X, Y_1 Y_2 \,|\, Z \rangle} = u_{\langle X, Y_1 \,|\, Z \rangle} + u_{\langle X, Y_2 \,|\, Y_1 Z \rangle} \tag{3}$$

among three semi-elementary imsets. This relation can be visualized in the form of the diagram of Figure 1. In Figure 1, each edge point corresponds to an element of $\mathcal{P}(N)$, and a diamond represents a semi-elementary imset which takes 1 at the top and bottom of the diamond and takes -1 at the left and right. This diagram is a sub-diagram of Hasse diagram. From Figure 1, it is easily seen that the values of $u_{\langle X, Y_1 \,|\, Z \rangle} + u_{\langle X, Y_2 \,|\, Y_1 Z \rangle}$ at $Y_1 Z$ and $XY_1 Z$ cancel out, become 1 at $XY_1 Y_2 Z$ and Z, become -1 at XZ and $Y_1 Z Y_2$, and hence (3) holds. Identity (3) is very convenient, because it can be regarded as an identity among three $2^{|N|}$-dimensional integer vectors.
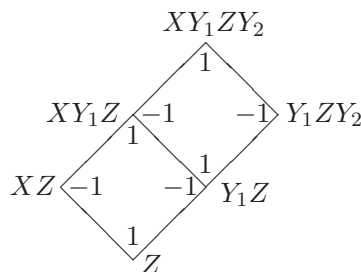

Figure 1: Sum of two semi-elementary imsets

When A and B are singletons, written with lower-case letters as a and b, $u_{\langle a, b \,|\, C \rangle}$ is called an *elementary* imset. The following fact which has been stated in Proposition 4.2 of Studený [10] plays a central role in this paper.

Lemma 1. *(Proposition 4.2 of Studený [10]) Every semi-elementary imset can be written as a non-negative integer sum of elementary imsets.*

Proof. Let $u_{\langle A, B \,|\, C \rangle}$ be any semi-elementary imset. Note that $u_{\langle B, A \,|\, C \rangle} = u_{\langle A, B \,|\, C \rangle}$ from the definition of (2). If A or B is not singleton, then the semi-elementary imset

can be split into two semi-elementary imsets by applying (3). By recursively applying (3), any semi-elementary imset $u_{\langle A,B\,|\,C\rangle}$ can be written as a non-negative integer sum of elementary imsets $u_{\langle a,b\,|\,\Gamma\rangle}$, where $a \in A$, $b \in B$ and $C \subseteq \Gamma \subseteq ABC$.

We call a sum of elementary imsets which is equal to $u_{\langle A,B\,|\,C\rangle}$ a *representation* of $u_{\langle A,B\,|\,C\rangle}$ by elementary imsets. Depending on the order of applications of (3) to various subsets of A and B, there are many representations of a semi-elementary imset $u_{\langle A,B\,|\,C\rangle}$. Furthermore, as we show in this paper, there are representations which can not be obtained by recursive application of (3). See Figure 3 below. For a given $\langle A, B\,|\,C\rangle$, let $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ denote the set of all possible representations of $u_{\langle A,B\,|\,C\rangle}$. We call $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ the $u_{\langle A,B\,|\,C\rangle}$-*fiber*. In this paper we study the properties of this fiber.

Let X, Y_1, Y_2 be singletons a, b_1, b_2 in (3) and let $C \subseteq Z = \Gamma \subseteq ABC$. Then by changing the roles of b_1 and b_2, we obtain the following relation:

$$u_{\langle a,b_1\,|\,\Gamma\rangle} + u_{\langle a,b_2\,|\,b_1\Gamma\rangle} = u_{\langle a,b_2\,|\,\Gamma\rangle} + u_{\langle a,b_1\,|\,b_2\Gamma\rangle}. \tag{4}$$

In the similar way, we obtain the following relation:

$$u_{\langle a_1,b\,|\,\Gamma\rangle} + u_{\langle a_2,b\,|\,a_1\Gamma\rangle} = u_{\langle a_2,b\,|\,\Gamma\rangle} + u_{\langle a_1,b\,|\,a_2\Gamma\rangle}. \tag{5}$$

We call these relation *two-by-two basic relations*. In this paper by a relation we mean an equality between two non-negative combinations of elementary imsets. We call the difference of two sides of a two-by-two basic relation a *two-by-two move*.

We say that two representations $\theta, \theta' \in \mathcal{F}_{\langle A,B\,|\,C\rangle}$ are adjacent if they just differ by a two-by-two basic relation, i.e. $\theta - \theta'$ is a two-by-two move. Furthermore, we say that $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ is *connected* by two-by-two basic relations if for any two representations $\theta, \theta' \in \mathcal{F}_{\langle A,B\,|\,C\rangle}$ there exists a sequence of representations $\theta = \theta_0, \theta_1, \ldots, \theta_K = \theta'$, such that θ_{k-1} and θ_k are adjacent, for $1 \le k \le K$. Now our main result is stated as follows.

Theorem 1. *Every fiber $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ is connected by two-by-two basic relations.*

From the viewpoint of toric ideals and Markov bases (e.g.[11],[2],[3]), this result is closely related to connectivity of a specific fiber by a subset of a Markov basis. See [4],[1],[12] for relevant results. Since a Markov basis for the whole configuration of elementary imsets is very complicated ([5]), it is remarkable that $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ is connected by the two-by-two basic relations.

Our result is also of interest from the viewpoint of the conditional independence inference based in imsets. Relations other than the two-by-two basic relations in a Markov basis correspond to equivalences of conditional independence statements which are not derived from the semi-graphoid axiom. The fact that $\mathcal{F}_{\langle A,B\,|\,C\rangle}$ is connected by the two-by-two basic relations suggest that we have to explore other fibers for finding equivalences of conditional independence statements which are not derived from the semi-graphoid axiom.

The organization of the paper is as follows. In Section 2 we set up our notation and summarize basic facts on imsets. In Section 3 we state our results, including a sketch of

the proof of Theorem 1. In Section 4 we show some numerical and computational results on the fiber $\mathcal{F}_{\langle A,B \,|\, C \rangle}$, when the cardinalities of A and B are small. Long proofs of some lemmas and Theorem 1 are given in Section 5.

2. Preliminaries

In this section we set up our notation and definitions following [10] and [6]. Let N be a finite set and let $\mathcal{P}(N) = \{A \mid A \subseteq N\}$ denote its power set. An integer-valued multiset $f : \mathcal{P}(N) \to \mathbb{Z}$ is called an imset. We write the union $A \cup B$ as AB. A singleton set $\{a\}$ is simply written as a.

For a triplet $\langle A, B \,|\, C \rangle$ of disjoint subsets of N, the semi-elementary imset $u_{\langle A,B \,|\, C \rangle}$ is defined as (2). When $A = a$ and $B = b$ are singletons, $u_{\langle a,b \,|\, C \rangle}$ is called elementary. The set of all elementary imsets for N is denoted as $\mathcal{E}(N)$. If $A = \emptyset$ or $B = \emptyset$, then $u_{\langle A,B \,|\, C \rangle}$ is the zero imset. Hence we usually assume that A, B are non-empty. On the other hand C may well be an empty set.

For a given triplet $\langle A, B \,|\, C \rangle$, we consider the following set of elementary imsets:

$$\mathcal{E}_{\langle A,B \,|\, C \rangle} = \{u_{\langle a,b \,|\, \Gamma \rangle} \mid a \in A, b \in B, C \subseteq \Gamma \subseteq ABC\}. \tag{6}$$

The cardinality of $\mathcal{E}_{\langle A,B \,|\, C \rangle}$ is given by $|\mathcal{E}_{\langle A,B \,|\, C \rangle}| = |A|2^{|A|-1}|B|2^{|B|-1}$. Starting from a given $u_{\langle A,B \,|\, C \rangle}$, consider recursively applying (3). Then $u_{\langle A,B \,|\, C \rangle}$ is written as a non-negative integer combination of elementary imsets from $\mathcal{E}_{\langle A,B \,|\, C \rangle}$:

$$u_{\langle A,B \,|\, C \rangle} = \sum_{u \in \mathcal{E}_{\langle A,B \,|\, C \rangle}} \theta_u u, \quad \theta_u \in \mathbb{N} = \{0, 1, 2, \dots\}. \tag{7}$$

We call the right-hand side a representation of $u_{\langle A,B \,|\, C \rangle}$. There are many representations. We call the set of possible representations the $u_{\langle A,B \,|\, C \rangle}$-fiber and denote it by $\mathcal{F}_{\langle A,B \,|\, C \rangle}$. Even for the case $|A| = 1$, $|B| = 2$, the two-by-two basic relation in (4) shows that there are two representations of $u_{\langle a,b_1 b_2 \,|\, C \rangle}$. If $A = a$ is a singleton, it is easily seen that there are $|B|!$ different representations of $u_{\langle a,B \,|\, C \rangle}$. However for the general case $|A| \geq 2$, $|B| \geq 2$, it is not trivial to enumerate $\mathcal{F}_{\langle A,B \,|\, C \rangle}$.

Now consider writing $u_{\langle A,B \,|\, C \rangle}$ as a $2^{|N|}$-dimensional integer column vector and $u \in \mathcal{E}_{\langle A,B \,|\, C \rangle}$ into a $2^{|N|} \times |\mathcal{E}_{\langle A,B \,|\, C \rangle}|$ integer matrix $\mathcal{U}_{\langle A,B \,|\, C \rangle}$. For example $\mathcal{U}_{\langle a_1 a_2, b_1 b_2 \,|\, C \rangle}$ is written as in Table 1. We call $\mathcal{U}_{\langle A,B \,|\, C \rangle}$ the configuration for the semi-elementary imset $u_{\langle A,B \,|\, C \rangle}$. Then a representation (7) is written in a matrix form as

$$u_{\langle A,B \,|\, C \rangle} = \mathcal{U}_{\langle A,B \,|\, C \rangle} \theta,$$

where θ is the column vector of coefficients θ_u in (7). From now on we identify a non-negative integer combination of elementary imsets from $\mathcal{E}_{\langle A,B \,|\, C \rangle}$ with the vector of non-negative integer coefficients. Then $\mathcal{F}_{\langle A,B \,|\, C \rangle}$ is written as

$$\mathcal{F}_{\langle A,B \,|\, C \rangle} = \{\theta \in \mathbb{N}^{|\mathcal{E}_{\langle A,B \,|\, C \rangle}|} \mid u_{\langle A,B \,|\, C \rangle} = \mathcal{U}_{\langle A,B \,|\, C \rangle} \theta\}. \tag{8}$$

Table 1: Configuration $\mathcal{U}_{\langle a_1 a_2, b_1 b_2 \mid C\rangle}$ of elementary imsets in $\mathcal{E}_{\langle a_1 a_2, b_1 b_2 \mid C\rangle}$

	$\langle a_1, b_1 \mid a_2 b_2 C\rangle$	$\langle a_1, b_2 \mid a_2 b_1 C\rangle$	$\langle a_2, b_1 \mid a_1 b_2 C\rangle$	$\langle a_2, b_2 \mid a_1 b_1 C\rangle$	$\langle a_2, b_1 \mid b_2 C\rangle$	$\langle a_1, b_1 \mid b_2 C\rangle$	$\langle a_2, b_2 \mid b_1 C\rangle$	$\langle a_1, b_2 \mid b_1 C\rangle$	$\langle a_1, b_2 \mid a_2 C\rangle$	$\langle a_1, b_1 \mid a_2 C\rangle$	$\langle a_2, b_2 \mid a_1 C\rangle$	$\langle a_2, b_1 \mid a_1 C\rangle$	$\langle a_2, b_2 \mid C\rangle$	$\langle a_2, b_1 \mid C\rangle$	$\langle a_1, b_2 \mid C\rangle$	$\langle a_1, b_1 \mid C\rangle$
$a_1a_2b_1b_2C$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$a_2b_1b_2C$	-1	-1	0	0	1	0	1	0	0	0	0	0	0	0	0	0
$a_1b_1b_2C$	0	0	-1	-1	0	1	0	1	0	0	0	0	0	0	0	0
$a_1a_2b_2C$	-1	0	-1	0	0	0	0	0	1	0	1	0	0	0	0	0
$a_1a_2b_1C$	0	-1	0	-1	0	0	0	0	0	1	0	1	0	0	0	0
b_1b_2C	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0
a_2b_2C	1	0	0	0	-1	0	0	0	-1	0	0	0	1	0	0	0
a_2b_1C	0	1	0	0	0	0	-1	0	0	-1	0	0	0	1	0	0
a_1b_2C	0	0	1	0	0	-1	0	0	0	0	-1	0	0	0	1	0
a_1b_1C	0	0	0	1	0	0	0	-1	0	0	0	-1	0	0	0	1
a_1a_2C	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0
b_2C	0	0	0	0	1	1	0	0	0	0	0	0	-1	0	-1	0
b_1C	0	0	0	0	0	0	1	1	0	0	0	0	0	-1	0	-1
a_2C	0	0	0	0	0	0	0	0	1	1	0	0	-1	-1	0	0
a_1C	0	0	0	0	0	0	0	0	0	0	1	1	0	0	-1	-1
C	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1

In this form it is evident that $\mathcal{F}_{\langle A, B \mid C\rangle}$ is a particular fiber in the theory of Markov basis.

Note that the configuration $\mathcal{U}_{\langle A, B \mid C\rangle}$ is a subconfiguration of the set of all elementary imsets $\mathcal{E}(N)$. It is known that the elementary imsets in $\mathcal{E}_{\langle A, B \mid C\rangle}$ are the extreme rays of a face of the cone generated by all the elementary imsets. Hence the subconfiguration $\mathcal{U}_{\langle A, B \mid C\rangle}$ generates a combinatorial pure subring in the sense of Ohsugi, Herzog and Hibi ([8, 7, 9]).

From the form of $\mathcal{U}_{\langle A, B \mid C\rangle}$ in Table 1, it is evident that the structure of the fiber $\mathcal{F}_{\langle A, B \mid C\rangle}$ only depends on $|A|$ and $|B|$. Therefore, for our study of the structure of the fiber $\mathcal{F}_{\langle A, B \mid C\rangle}$, we can assume that $C = \emptyset$ and $N = AB$, without loss of generality.

For the rest of this section, and for the purpose of illustration, we write out the fiber $\mathcal{F}_{\langle a_1 a_2, b_1 b_2 \mid \emptyset\rangle}$. If we split $b_1 b_2$ first, we have $u_{\langle a_1 a_2, b_1 b_2 \mid \emptyset\rangle} = u_{\langle a_1 a_2, b_1 \mid \emptyset\rangle} + u_{\langle a_1 a_2, b_2 \mid b_1\rangle}$ or $u_{\langle a_1 a_2, b_1 b_2 \mid \emptyset\rangle} = u_{\langle a_1 a_2, b_2 \mid \emptyset\rangle} + u_{\langle a_1 a_2, b_1 \mid b_2\rangle}$. Consider the former case. Then we can split $a_1 a_2$, independently in two terms on the right-hand side. Then we have four representations:

$$\begin{aligned}
u_{\langle a_1 a_2, b_1 b_2 \mid \emptyset\rangle} &= u_{\langle a_1, b_1 \mid \emptyset\rangle} + u_{\langle a_2, b_1 \mid a_1\rangle} + u_{\langle a_1, b_2 \mid b_1\rangle} + u_{\langle a_2, b_2 \mid a_1 b_1\rangle} \qquad (9) \\
&= u_{\langle a_2, b_1 \mid \emptyset\rangle} + u_{\langle a_1, b_1 \mid a_2\rangle} + u_{\langle a_1, b_2 \mid b_1\rangle} + u_{\langle a_2, b_2 \mid a_1 b_1\rangle} \\
&= u_{\langle a_1, b_1 \mid \emptyset\rangle} + u_{\langle a_2, b_1 \mid a_1\rangle} + u_{\langle a_2, b_2 \mid b_1\rangle} + u_{\langle a_1, b_2 \mid a_2 b_1\rangle} \qquad (10) \\
&= u_{\langle a_2, b_1 \mid \emptyset\rangle} + u_{\langle a_1, b_1 \mid a_2\rangle} + u_{\langle a_2, b_2 \mid b_1\rangle} + u_{\langle a_1, b_2 \mid a_2 b_1\rangle}.
\end{aligned}$$

When we enumerate the representations, apparently we have 16 representations. However in fact there are 12 distinct representations of $\mathcal{F}_{\langle a_1 a_2, b_1 b_2 \mid \emptyset\rangle}$. If we consider symmetry with respect to the interchanges $a_1 \leftrightarrow a_2$ and $b_1 \leftrightarrow b_2$, there are three types of representations
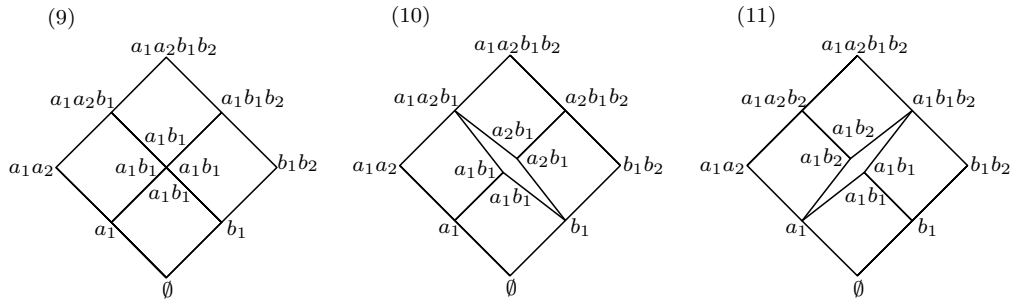
Figure 2: Three types of representations of $u_{\langle a_1 a_2, b_1 b_2 \,|\, \emptyset \rangle}$

as shown in Figure 2. In Figure 2 the whole diamond depicts a representation and each unit diamond depicts an elementary imset $u_{\langle a,b \,|\, \Gamma \rangle}$, whose four vertices represent $ab\Gamma$ (top), $a\Gamma$ (left), $b\Gamma$ (right) and Γ (bottom) . Note that two types of representations (the middle and the right) have "rifts". At a rift, the values of four elementary imsets surrounding an internal node of the whole diamond cancel in two different pairs. The precise definition of rifts will be given in Section 5. The representations of the first and second types in Figure 2 are obtained in (9) and (10). By splitting $a_1 a_2$ first, the representation of the third type in Figure 2 is given as follows:

$$u_{\langle a_1 a_2, b_1 b_2 \,|\, \emptyset \rangle} = u_{\langle a_1, b_1 \,|\, \emptyset \rangle} + u_{\langle a_2, b_2 \,|\, a_1 \rangle} + u_{\langle a_1, b_2 \,|\, b_1 \rangle} + u_{\langle a_2, b_1 \,|\, a_1 b_2 \rangle}. \tag{11}$$

The type without rifts is obtained by applying the same splitting to two intermediate terms on the right-hand side of $u_{\langle a_1 a_2, b_1 b_2 \,|\, \emptyset \rangle} = u_{\langle a_1 a_2, b_1 \,|\, \emptyset \rangle} + u_{\langle a_1 a_2, b_2 \,|\, b_1 \rangle}$ or $u_{\langle a_1 a_2, b_1 b_2 \,|\, \emptyset \rangle} = u_{\langle a_1 a_2, b_2 \,|\, \emptyset \rangle} + u_{\langle a_1 a_2, b_1 \,|\, b_2 \rangle}$. This is an important fact in proving Theorem 1.

3. Main results

In this section we establish some facts on the fiber of representations $\mathcal{F}_{\langle A, B \,|\, C \rangle}$. At the end of the section we give a sketch of a proof of Theorem 1. See section 5 for the full proof of Theorem 1.

First we show that a representation always involves $|A||B|$ elementary imsets. This fact is obvious if a representation is obtained by recursive application of (3). However since there are representations not obtained by recursive application of (3), we need to show this fact for all representations.

As before we consider an elementary imset $u \in \mathcal{E}_{\langle A, B \,|\, C \rangle}$ as a function from $\mathcal{P}(N)$ to $\mathbb{Z} \subset \mathbb{R}$ or a $2^{|N|}$-dimensional integer vector. We define the inner product of two functions f, g from $\mathcal{P}(N)$ to $\mathbb{R}$ by the standard inner product of $2^{|N|}$-dimensional vectors:

$$\langle f, g \rangle = \sum_{S \in \mathcal{P}(N)} f(S) g(S) = f(\emptyset) g(\emptyset) + \cdots + f(N) g(N).$$

For $u = u_{\langle a,b \,|\, \Gamma \rangle} \in \mathcal{E}_{\langle A, B \,|\, C \rangle}$, let $s = |A \cap \Gamma|, t = |B \cap \Gamma|$. Note that $C \subset \Gamma \subset ABC$ in

the definition of $\mathcal{E}_{\langle A,B \,|\, C\rangle}$ in (6). Then $0 \leq s \leq |A| - 1$, $0 \leq t \leq |B| - 1$. Let

$$\mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle} = \{u \in \mathcal{E}_{\langle A,B \,|\, C\rangle} \mid s = |A \cap \Gamma|, t = |B \cap \Gamma|\}.$$

Classifying the terms in a representation, we can rewrite (7) as

$$u_{\langle A,B \,|\, C\rangle} = \sum_{s=0}^{|A|-1} \sum_{t=0}^{|B|-1} \sum_{u \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}} \theta_u u.$$

We now prove that a representation is a sum of $|A||B|$ terms and each term is taken just once from $\mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}$.

Proposition 1. *Let* $\theta : u_{\langle A,B \,|\, C\rangle} = \sum_{u \in \mathcal{E}_{\langle A,B \,|\, C\rangle}} \theta_u u$ *be a representation. Then for each* (s,t), $0 \leq s \leq |A| - 1$, $0 \leq t \leq |B| - 1$, *there is one* $u \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}$ *such that* $\theta_u = 1$ *and* $\theta_{u'} = 0$ *for other* $u' \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}$, $u' \neq u$.

Proof. For $0 \leq s \leq |A| - 1$ and for $S \subseteq A$ let $g_s : \mathcal{P}(A) \to \mathbb{Z}$ be defined as

$$g_s(S) = 1_{\{|S|>s\}} = \begin{cases} 1 & \text{if } |S| > s \\ 0 & \text{otherwise.} \end{cases}$$

Let $h_t : \mathcal{P}(B) \to \mathbb{Z}$ be similarly defined. Let $f(S) = g_s(A \cap S)h_t(B \cap S)$. We consider the inner product of f with $u = u_{\langle a,b \,|\, \Gamma\rangle} \in \mathcal{E}_{\langle A,B \,|\, C\rangle}$. Write $A' = A \cap \Gamma$, $B' = B \cap \Gamma$. It is easily seen that the inner product of f with $u_{\langle a,b \,|\, \Gamma\rangle}$ is given as

$$\langle f, u_{\langle a,b \,|\, \Gamma\rangle}\rangle = (1_{\{|aA'|>s\}} - 1_{\{|A'|>s\}})(1_{\{|bB'|>t\}} - 1_{\{|B'|>t\}}).$$

Hence

$$\langle f, u\rangle = \begin{cases} 1 & \text{if } u \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle} \\ 0 & \text{otherwise .} \end{cases}$$

On the other hand $\langle u_{\langle A,B \,|\, C\rangle}, f\rangle = (1-0)(1-0) = 1$. Hence we have

$$1 = \sum_{u \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}} \theta_u.$$

Since θ_u are non-negative integers we have the proposition. $\square$

From this proposition, given a representation $\theta : u_{\langle A,B \,|\, C\rangle} = \sum \theta_u u$, for each $0 \leq s \leq |A| - 1$, $0 \leq t \leq |B| - 1$, there exists $u(s,t) = u_\theta(s,t) \in \mathcal{E}^{s,t}_{\langle A,B \,|\, C\rangle}$ such that

$$u_{\langle A,B \,|\, C\rangle} = \sum_{s=0}^{|A|-1} \sum_{t=0}^{|B|-1} u_\theta(s,t). \tag{12}$$

We often omit the subscript θ in $u_\theta(s,t)$ and simply write $u(s,t)$ for notational simplicity.

Next we give a definition of σ-decomposability. Here σ stands for "semi-graphoid". Simply put, a representation (12) is said to be σ-decomposable if it is obtained by recursive application of (3). A precise definition of σ-decomposability is given after we first give another definition. It is analogous to the definition of decomposable graphs. We first define separability of a representation.

Definition 1. *Let $1 \le s_0 \le |A| - 1$. Then we say that a representation (12) is $(s_0, *)$-separable if there is some $A_0 \subseteq A$ such that $|A_0| = s_0$ and*

$$u_{\langle A_0, B \,|\, C\rangle} = \sum_{s=0}^{s_0-1} \sum_{t=0}^{|B|-1} u(s,t), \quad u_{\langle A\setminus A_0, B \,|\, A_0 C\rangle} = \sum_{s=s_0}^{|A|-1} \sum_{t=0}^{|B|-1} u(s,t). \tag{13}$$

Similarly, for $1 \le t_0 \le |B| - 1$, we say that a representation (12) is $(*, t_0)$-separable if there is some $B_0 \subseteq B$ such that $|B_0| = t_0$ and

$$u_{\langle A, B_0 \,|\, C\rangle} = \sum_{s=0}^{|A|-1} \sum_{t=0}^{t_0-1} u(s,t), \quad u_{\langle A, B\setminus B_0 \,|\, B_0 C\rangle} = \sum_{s=0}^{|A|-1} \sum_{t=t_0}^{|B|-1} u(s,t). \tag{14}$$

For example, a representation

$$u_{\langle a_1 a_2, b_1 b_2 \,|\, \emptyset\rangle} = u(0,0) + u(1,0) + u(0,1) + u(1,1)$$
$$= u_{\langle a_1, b_1 \,|\, \emptyset\rangle} + u_{\langle a_2, b_1 \,|\, a_1\rangle} + u_{\langle a_2, b_2 \,|\, b_1\rangle} + u_{\langle a_1, b_2 \,|\, a_2 b_1\rangle}$$

is $(*, 1)$-separable with $B_0 = b_1$, but not $(1, *)$-separable.

Now σ-decomposability is recursively defined as follows.

Definition 2. *We say that a representation (12) of $u_{\langle A, B \,|\, C\rangle}$ is σ-decomposable if at least one of the following conditions is satisfied:*

(i) $|A| = |B| = 1$.

(ii) *There is some $1 \le s_0 \le |A| - 1$ such that (12) is $(s_0, *)$-separable and the two representations in (13) are respectively σ-decomposable.*

(iii) *There is some $1 \le t_0 \le |B| - 1$ such that (12) is $(*, t_0)$-separable and the two representations in (14) are respectively σ-decomposable.*

Note that a representation is $(s_0, *)$-separable for some $1 \le s_0 \le |A| - 1$ if and only if the representation can be split into two semi-elementary imsets which can be obtained by one application of (3). Therefore, a representation is σ-decomposable if and only if it is obtained by recursive application of (3). If a representation is not σ-decomposable, we call the representation σ-*indecomposable*. The following proposition will be proved in Section 5.

Proposition 2. *All of the representations of $u_{\langle A,B\,|\,C\rangle}$ by elementary imsets in $\mathcal{E}_{\langle A,B\,|\,C\rangle}$ are σ-decomposable if and only if $|A| \leq 2$ or $|B| \leq 2$.*

We now consider the case of $u_{\langle A,B\,|\,C\rangle}$ where $|A| \geq 3$ and $|B| \geq 3$. In this case, we can construct counter-examples, i.e., representations of $u_{\langle A,B\,|\,C\rangle}$ by elementary imsets which are σ-indecomposable. Now, let us consider the following representation of $u_{\langle a_1 a_2 a_3, b_1 b_2 b_3\,|\,\emptyset\rangle}$:

$$
\begin{aligned}
u_{\langle a_1 a_2 a_3, b_1 b_2 b_3\,|\,\emptyset\rangle} \;=\;& u_{\langle a_1, b_1\,|\,\emptyset\rangle} + u_{\langle a_2, b_1\,|\,a_1\rangle} + u_{\langle a_3, b_3\,|\,a_1 a_2\rangle} \\
&+ u_{\langle a_2, b_2\,|\,b_1\rangle} + u_{\langle a_1, b_3\,|\,a_2 b_1\rangle} + u_{\langle a_3, b_1\,|\,a_1 a_2 b_3\rangle} \\
&+ u_{\langle a_2, b_3\,|\,b_1 b_2\rangle} + u_{\langle a_3, b_2\,|\,a_2 b_1 b_3\rangle} + u_{\langle a_1, b_2\,|\,a_2 a_3 b_1 b_3\rangle}.
\end{aligned}
\tag{15}
$$

See Figure 3. From the definition of σ-decomposability and Figure 3, it is clear that the representation of (15) is σ-indecomposable. Furthermore, from Proposition 2, the counter-example given by (15) is the smallest representation which is σ-indecomposable. The counter-example (15) can be extended to $|A| > 3$ or $|B| > 3$ as in Figure 5 below.
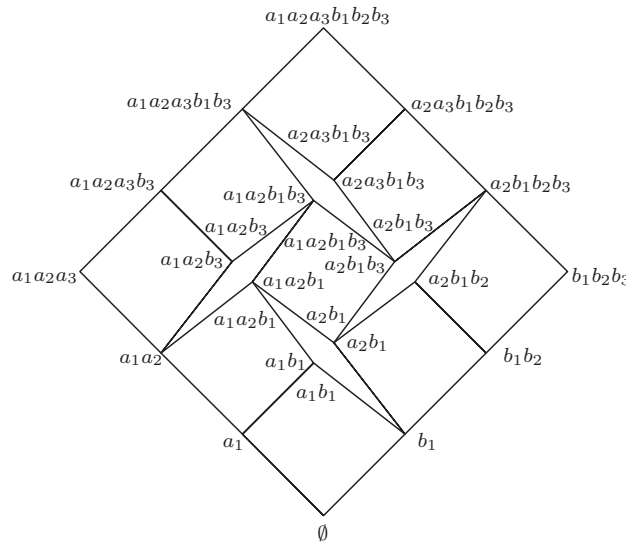

Figure 3: An counter-example which is not σ-decomposable

In the rest of this section, we give only a sketch of a proof of Theorem 1. We will give the complete proof in Section 5.

First we note that every representation of $u_{\langle A,B\,|\,C\rangle}$ has its rift pattern. Intuitively, a rift pattern is a rectangle consisting of $|A||B|$ cells with some rifts, where each cell corresponds to an elementary imset $u(s,t)$ in (12). For example, (15) has a rift pattern with nine cells and four rifts (of length 2) as shown in Figure 3. A precise definition of rifts and rift patterns will be given in Definition 3.

We label elements of A and B as $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$. As discussed in the last paragraph of Section 2, representations without rifts are obtained by

applying the same splitting rules of (3) to the related intermediate terms recursively. In particular, we can obtain the following representation without rifts:

$$u_{\langle A,B \,|\, C\rangle} = \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} u_{\langle a_i,b_j \,|\, a_1...a_{i-1}b_1...b_{j-1}C\rangle}. \tag{16}$$

We call the above representation the *standard representation* of $u_{\langle A,B \,|\, C\rangle}$. As we will show in Proposition 3, every other representations without rifts are obtained by permuting a_i's and b_j's independently in the standard representation. The converse is also true, i.e., any representations obtained by permuting a_i's and b_j's independently in the standard representation have no rifts. By the following proposition, it suffices to show that we can eliminate all of the rifts in the representations by applying two-by-two basic relations.

Proposition 3. *Any representations of $u_{\langle A,B \,|\, C\rangle}$ without rifts are obtained by applying two-by-two basic relations to the standard representation in (16).*

Proof. First, note that any permutation is constructed from adjacent transpositions $a_i \leftrightarrow a_{i+1}$ and $b_j \leftrightarrow b_{j+1}$. Therefore, we only need to show that any adjacent transposition is constructed by two-by-two basic relations. For example, suppose that we want to interchange $a_1 \leftrightarrow a_2$ in the standard representation in (16). Then, this adjacent transposition is realized by applying $|B|$ two-by-two basic relations:

$$
\begin{aligned}
u_{\langle a_1,b_1 \,|\, C\rangle} + u_{\langle a_2,b_1 \,|\, a_1 C\rangle} &= u_{\langle a_2,b_1 \,|\, C\rangle} + u_{\langle a_1,b_1 \,|\, a_2 C\rangle}, \\
u_{\langle a_1,b_2 \,|\, b_1 C\rangle} + u_{\langle a_2,b_2 \,|\, a_1 b_1 C\rangle} &= u_{\langle a_2,b_2 \,|\, b_1 C\rangle} + u_{\langle a_1,b_2 \,|\, a_2 b_1 C\rangle}, \\
&\vdots
\end{aligned}
$$

$$u_{\langle a_1,b_{|B|} \,|\, b_1...b_{|B|-1}C\rangle} + u_{\langle a_2,b_{|B|} \,|\, a_1 b_1...b_{|B|-1}C\rangle} = u_{\langle a_2,b_{|B|} \,|\, b_1...b_{|B|-1}C\rangle} + u_{\langle a_1,b_{|B|} \,|\, a_2 b_1...b_{|B|-1}C\rangle}.$$

In the same way, we can show that any adjacent transpositions $a_i \leftrightarrow a_{i+1}$ in arbitrary representations without rifts are constructed by using two-by-two basic relations. The proof for any adjacent transpositions $b_j \leftrightarrow b_{j+1}$ is also similar to the proof for a_i's. $\square$

Rather than stating the method to eliminate the rifts in general, let us work through an example to see how we can do it. Here, we use the example of (15) to explain the method to eliminate the rifts. First, by applying a two-by-two basic relation $u_{\langle a_1,b_1 \,|\, \emptyset\rangle} + u_{\langle a_2,b_1 \,|\, a_1\rangle} = u_{\langle a_2,b_1 \,|\, \emptyset\rangle} + u_{\langle a_1,b_1 \,|\, a_2\rangle}$, we can change the representation (i) in Figure 4 into (ii) in Figure 4 and eliminate the lower left rift. Note that the shaded areas in Figure 4 represent the elementary imsets to which we are applying two-by-two basic relations. In the same way, we can eliminate the lower right rift by applying $u_{\langle a_2,b_2 \,|\, b_1\rangle} + u_{\langle a_2,b_3 \,|\, b_1 b_2\rangle} = u_{\langle a_2,b_3 \,|\, b_1\rangle} + u_{\langle a_2,b_2 \,|\, b_1 b_3\rangle}$. The third step changing from (iii) to (v) in Figure 4 is rather complicated. We can eliminate the upper left rift by applying $u_{\langle a_1,b_1 \,|\, a_2\rangle} + u_{\langle a_1,b_3 \,|\, a_2 b_1\rangle} = u_{\langle a_1,b_3 \,|\, a_2\rangle} + u_{\langle a_1,b_1 \,|\, a_2 b_3\rangle}$. However, this operation also generates a lower left rift as in (4) of Figure 4. In this case, we must apply further two-by-two basic relation $u_{\langle a_2,b_1 \,|\, \emptyset\rangle} + u_{\langle a_2,b_3 \,|\, b_1\rangle} = u_{\langle a_2,b_3 \,|\, \emptyset\rangle} + u_{\langle a_2,b_1 \,|\, b_3\rangle}$, to eliminate the rift generated by side effects of a two-by-two basic relation. The operation of the fourth step changing from (v) to (vii) in Figure 4
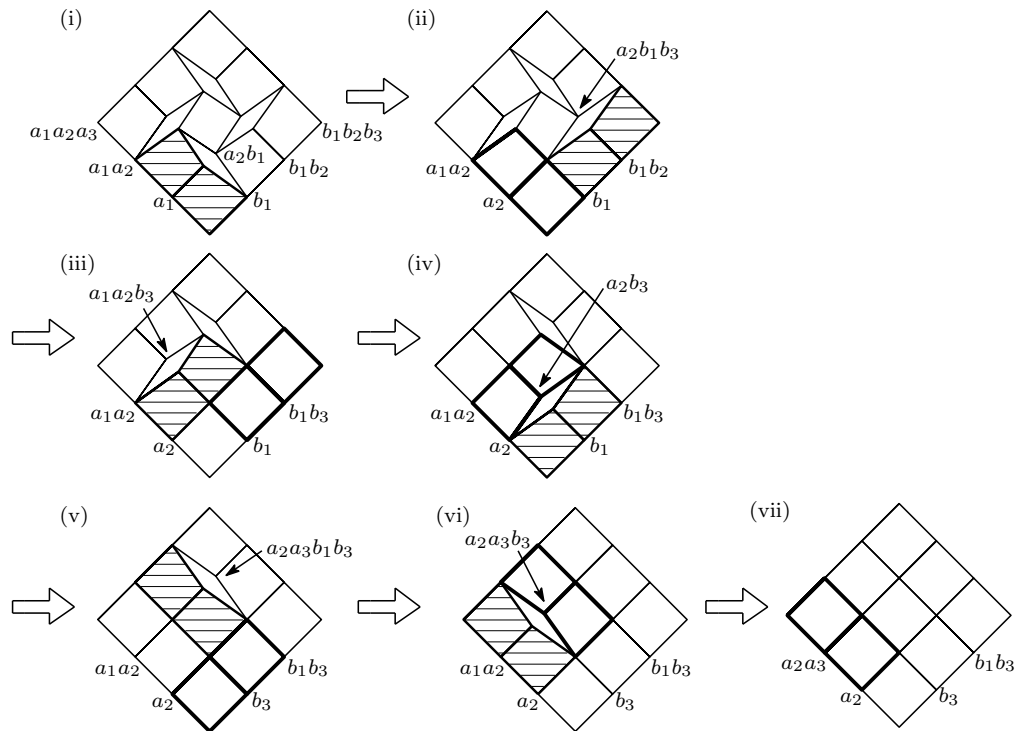
Figure 4: Elimination of rifts

is almost the same as in the third step. In this manner, we can eliminate all the rifts in the representations. Note that, in the first step, we can not apply $u_{\langle a_1,b_1\,|\,a_2\rangle} + u_{\langle a_1,b_3\,|\,a_2b_1\rangle} = u_{\langle a_1,b_3\,|\,a_2\rangle} + u_{\langle a_1,b_1\,|\,a_2b_3\rangle}$ which is used in the third step to eliminate the upper left rift. This means that order of eliminations of the rifts is important. The rigorous definition of these operations and the detailed proofs about the properties are given in Section 5.

4. Some numerical and computational results

We report our numerical and computational study to enumerate all of the representations of $u_{\langle A,B\,|\,C\rangle}$. In this section, we identify two representations that one is obtained by permuting a_i's and b_j's independently in another. For example, we identify $u_{\langle a_1,b_1\,|\,\Gamma\rangle}+u_{\langle a_2,b_1\,|\,a_1\Gamma\rangle}+u_{\langle a_1,b_2\,|\,b_1\Gamma\rangle}+u_{\langle a_2,b_2\,|\,a_1b_1\Gamma\rangle}$ and $u_{\langle a_2,b_2\,|\,\Gamma\rangle}+u_{\langle a_1,b_2\,|\,a_2\Gamma\rangle}+u_{\langle a_2,b_1\,|\,b_2\Gamma\rangle}+u_{\langle a_1,b_1\,|\,a_2b_2\Gamma\rangle}$ which are representations of $u_{\langle a_1a_2,b_1b_2\,|\,\Gamma\rangle}$. Hence, we count the number of representatives of the representations under this equivalence relation. Note that if the permutation is not the identity, the two representations are different from each other since they are distinguished by, for example, $u(s,0)$'s and $u(0,t)$'s in (12). Hence each equivalence class has $|A|!|B|!$ elements. We call the set of all of the rifts of a representation a *rift pattern*. See Definition 3 for the precise definition. Note that two representations may have the same rift pattern.

First, let us consider the number of rift patterns. As will be explained in Section 5, rifts occur at $(|A| - 1)(|B| - 1)$ points in $\mathcal{P}(N)$ and rifts never intersect one another, i.e., each point in $\mathcal{P}(N)$ has only three patterns: no rift, slash rift and backslash rift. The precise definitions of slash rift and backslash rift are given in Section 5 (see Figure 8 below). Therefore, the number of rift patterns is given by $3^{(|A|-1)(|B|-1)}$. The list of all the rift patterns are also given by this enumeration.

For each rift pattern, we can enumerate all the representations which are consistent with the rift pattern by making an exhaustive investigation. If we only want to count the number of representations, then we can compute it as follows. First, note that the diversity of representations is given by rifts, and, for each rift, the degree of freedom is decided only by the "length of the rift". Intuitively, in Figure 8 in Section 5, the length of the rift is given by $t_U - t_L$ for a slash rift and $s_U - s_L$ for a backslash rift. The following lemma is useful for counting the degree of freedom.

Proposition 4. *For any rift with length l, we can compute the degree of freedom $d(l)$ by the following recurrence formula:*

$$d(l) = l! - \sum_{k=1}^{l-1} (l-k)! \, d(k), \quad l \geq 2, \tag{17}$$

where $d(1) = 1$.

We will give the proof of the above proposition in Section 5. Since the diversity increases in a multiplicative fashion independently for each rift, the number of all of the representations for a rift pattern is obtained by taking the products of the degree of freedoms for all of the rifts. Hence, by summing the number of representations for all of the rift patterns, we can count the number of representations.

The number of σ-indecomposable representations is calculated by applying the recursive algorithm given in Definition 2 to each rift pattern.

If $|A| = 1$ or $|B| = 1$, then there is no rift. Therefore, the numbers of rift patterns and representations of $u_{\langle A,B \,|\, C \rangle}$ for $|A| = 1$ or $|B| = 1$ are both 1. Furthermore, from Proposition 2, the number of σ-indecomposable representations of $u_{\langle A,B \,|\, C \rangle}$ for $|A| \leq 2$ or $|B| \leq 2$ is zero.

Next, let us consider the case for $|A| = 2$. In this case, we have the following lemma.

Proposition 5. *Let $r_2(|B|)$ denote the number of representatives of the representations of $u_{\langle A,B \,|\, C \rangle}$ for $|A| = 2$. Then, we have the following recurrence formula:*

$$r_2(m) = m! + \sum_{k=1}^{m-1} k! \, r_2(m-k), \quad m \geq 2, \tag{18}$$

where $r_2(1) = 1$.

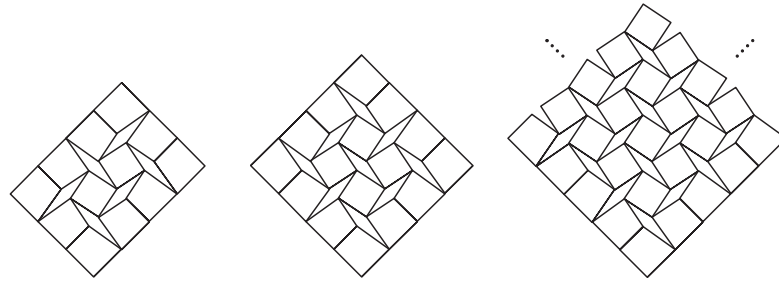We will give the proof of the above proposition in Section 5.

Figure 5: σ-indecomposable rift patterns

For the case of $2 \leq |A|, |B| \leq 5$, we give the numbers of rift patterns, representations, σ-indecomposable rift patterns, σ-indecomposable representations of $u_{\langle A,B \mid C \rangle}$ in Table 2. We also give some examples of σ-indecomposable rift patterns in Figure 5. The left-hand side of Figure 5 is an example of σ-indecomposable rift pattern for $|A| = 3, |B| = 4$. This can be seen as an extension of the counter-example of Figure 3 for $|A| = 3, |B| = 3$. The example in the middle of Figure 5 is an example of σ-indecomposable rift pattern for $|A| = 4, |B| = 4$ and also can be seen as an extension of the counter-example of Figure 3. It is easily seen that we can construct an example of σ-indecomposable rift pattern in general case by extending the counter-example of Figure 3 as in the right-hand side of Figure 5.

Table 2: Number of rift patterns, representations, σ-indecomposable rift patterns and σ-indecomposable representations for each $|A|$ and $|B|$

$|A|$	$|B|$	rift patterns	representations	σ-indecomposable rift patterns	σ-indecomposable representations
2	2	3	3	0	0
2	3	9	11	0	0
2	4	27	47	0	0
2	5	81	231	0	0
3	3	81	161	2	2
3	4	729	2971	40	96
3	5	6561	69281	562	3582
4	4	19683	241291	2436	19996
4	5	531441	25897047	102576	3420918
5	5	43046721	12606896129	12833474	2714509138

5. Proofs

Let us fix an arbitrary representation θ and let

$$\mathcal{E}^\theta_{\langle A,B \mid C \rangle} = \{u \in \mathcal{E}_{\langle A,B \mid C \rangle} \mid 0 \leq \exists s \leq |A|-1, 0 \leq \exists t \leq |B|-1 \ s.t. \ u = u_\theta(s,t) \in \mathcal{E}^{s,t}_{\langle A,B \mid C \rangle}\}.$$

$$q_\uparrow(u) = ab\Gamma$$

$$q_\leftarrow(u) = a\Gamma \quad \langle u = u_{\langle a,b\,|\,\Gamma\rangle}\rangle \quad q_\rightarrow(u) = b\Gamma$$
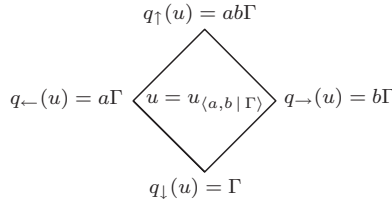
$$q_\downarrow(u) = \Gamma$$

Figure 6: Four functions from $\mathcal{E}_{\langle A,B\,|\,C\rangle}$ to $\mathcal{P}(N)$

Then there exist a_{st}, b_{st} and $\Gamma_{st} \in \mathcal{P}(N)$ for every $u_\theta(s,t) \in \mathcal{E}^\theta_{\langle A,B\,|\,C\rangle}$ such that $u_\theta(s,t) = u_{\langle a_{st}, b_{st}\,|\,\Gamma_{st}\rangle}$. We define four functions from $\mathcal{E}^\theta_{\langle A,B\,|\,C\rangle}$ to $\mathcal{P}(N)$ as $q^\theta_\downarrow(u_\theta(s,t)) = \Gamma_{st}$, $q^\theta_\leftarrow(u_\theta(s,t)) = a_{st}\Gamma_{st}$, $q^\theta_\rightarrow(u_\theta(s,t)) = b_{st}\Gamma_{st}$ and $q^\theta_\uparrow(u_\theta(s,t)) = a_{st}b_{st}\Gamma_{st}$. See Figure 6. In Figure 6, a diamond represents a semi-elementary imset $u_{\langle a,b\,|\,\Gamma\rangle}$ and four edge points at the bottom, left, right and top represent elements of $\mathcal{P}(N)$ which correspond to $q^\theta_\downarrow(u_{\langle a,b\,|\,\Gamma\rangle}) = \Gamma$, $q^\theta_\leftarrow(u_{\langle a,b\,|\,\Gamma\rangle}) = a\Gamma$, $q^\theta_\rightarrow(u_{\langle a,b\,|\,\Gamma\rangle}) = b\Gamma$ and $q^\theta_\uparrow(u_{\langle a,b\,|\,\Gamma\rangle}) = ab\Gamma$ respectively. We often omit the superscript θ in $q^\theta_\downarrow, q^\theta_\leftarrow, q^\theta_\rightarrow, q^\theta_\uparrow$ and simply write $q_\downarrow, q_\leftarrow, q_\rightarrow, q_\uparrow$ for notational simplicity. Then we have the following lemma.

Lemma 2. *For every $v \in \mathcal{E}^\theta_{\langle A,B\,|\,C\rangle}$, the following equations hold.*

$$|A \cap q_\downarrow(v)| + 1 = |A \cap q_\leftarrow(v)| = |A \cap q_\rightarrow(v)| + 1 = |A \cap q_\uparrow(v)|,$$

$$|B \cap q_\downarrow(v)| + 1 = |B \cap q_\leftarrow(v)| + 1 = |B \cap q_\rightarrow(v)| = |B \cap q_\uparrow(v)|$$

Proof. This follows immediately from the definitions of $q_\downarrow, q_\leftarrow, q_\rightarrow$ and $q_\uparrow$. □

Let $\mathcal{Q} = \cup_{v \in \mathcal{E}^\theta_{\langle A,B\,|\,C\rangle}} \{q_\downarrow(v), q_\leftarrow(v), q_\rightarrow(v), q_\uparrow(v)\}$. Then we have the following lemma.

Lemma 3. *For any $Q \in \mathcal{Q}$, let $s = |A \cap Q|$ and $t = |B \cap Q|$. Then each of $q^{-1}_\downarrow(Q)$, $q^{-1}_\leftarrow(Q)$, $q^{-1}_\rightarrow(Q)$ and $q^{-1}_\uparrow(Q)$ has at most one element: $q^{-1}_\downarrow(Q) = \emptyset$ or $\{u(s,t)\}$, $q^{-1}_\leftarrow(Q) = \emptyset$ or $\{u(s-1,t)\}$, $q^{-1}_\rightarrow(Q) = \emptyset$ or $\{u(s,t-1)\}$ and $q^{-1}_\uparrow(Q) = \emptyset$ or $\{u(s-1,t-1)\}$. Especially, $q^{-1}_\downarrow(q_\downarrow(v)) = \{v\}$, $q^{-1}_\leftarrow(q_\leftarrow(v)) = \{v\}$, $q^{-1}_\rightarrow(q_\rightarrow(v)) = \{v\}$ and $q^{-1}_\uparrow(q_\uparrow(v)) = \{v\}$ hold.*

Proof. This follows immediately from Proposition 1. □

For the four "vertices" of the representation, we have the following lemma.

Lemma 4. *For the four points $C, AC, BC, ABC \in \mathcal{P}(N)$, the following four properties hold.*

1. $q^{-1}_\downarrow(C) = \{u(0,0)\}$, $q^{-1}_\leftarrow(C) = q^{-1}_\rightarrow(C) = q^{-1}_\uparrow(C) = \emptyset$

2. $q^{-1}_\leftarrow(AC) = \{u(|A|-1, 0)\}$, $q^{-1}_\downarrow(AC) = q^{-1}_\rightarrow(AC) = q^{-1}_\uparrow(AC) = \emptyset$

3. $q^{-1}_\rightarrow(BC) = \{u(0, |B|-1)\}$, $q^{-1}_\downarrow(BC) = q^{-1}_\leftarrow(BC) = q^{-1}_\uparrow(BC) = \emptyset$

4. $q_\uparrow^{-1}(ABC) = \{u(|A| - 1, |B| - 1)\}$, $q_\downarrow^{-1}(ABC) = q_\leftarrow^{-1}(ABC) = q_\rightarrow^{-1}(ABC) = \emptyset$

Proof. We give the proof for the property in 2. The proofs for 1, 3 and 4 are almost the same as in 2.

Let us consider $q_\downarrow^{-1}(AC)$. Assume that there exists $v \in \mathcal{E}_{\langle A,B \mid C\rangle}^\theta$ such that $v \in q_\downarrow^{-1}(AC)$. Then, from Lemma 2 and $|A \cap q_\downarrow(v)| = |A|$, we have $|A \cap q_\leftarrow(v)| = |A| + 1$. This is a contradiction. Therefore such $v \in \mathcal{E}_{\langle A,B \mid C\rangle}^\theta$ does not exist.

Next, from $B \cap AC = \emptyset$, we have $q_\rightarrow^{-1}(AC) = q_\uparrow^{-1}(AC) = \emptyset$. Furthermore, because $u_{\langle A,B \mid C\rangle}$ takes -1 at AC and $q_\downarrow^{-1}(AC) = \emptyset$, $q_\leftarrow^{-1}(AC)$ must have at least one element. Hence, from Lemma 3, we have $q_\leftarrow^{-1}(AC) = \{u(|A| - 1, 0)\}$. □

Let us define the four "edges" $\mathcal{Q}_{ll}, \mathcal{Q}_{lr}, \mathcal{Q}_{ul}, \mathcal{Q}_{ur}$ of the representation as follows:

$$
\begin{aligned}
\mathcal{Q}_{ll} &= \{q_\leftarrow(u(s,0)) \mid 0 \le s \le |A| - 2\}, \\
\mathcal{Q}_{lr} &= \{q_\rightarrow(u(0,t)) \mid 0 \le t \le |B| - 2\}, \\
\mathcal{Q}_{ul} &= \{q_\leftarrow(u(|A| - 1),t)) \mid 0 \le t \le |B| - 2\}, \\
\mathcal{Q}_{ur} &= \{q_\rightarrow(u(s,|B| - 1))) \mid 0 \le s \le |A| - 2\}.
\end{aligned}
$$

For the four edges, we have the following lemma.

Lemma 5. *For $\mathcal{Q}_{ll}, \mathcal{Q}_{lr}, \mathcal{Q}_{ul}, \mathcal{Q}_{ur}$, the following four properties hold.*

1. *For $u(s,0) \in \mathcal{Q}_{ll}$, $q_\downarrow^{-1}(q_\leftarrow(u(s,0))) = \{u(s + 1, 0)\}$, $q_\leftarrow^{-1}(q_\leftarrow(u(s,0))) = \{u(s,0)\}$, $q_\rightarrow^{-1}(q_\leftarrow(u(s,0))) = \emptyset$ and $q_\uparrow^{-1}(q_\leftarrow(u(s,0))) = \emptyset$ hold.*

2. *For $u(0,t) \in \mathcal{Q}_{lr}$, $q_\downarrow^{-1}(q_\rightarrow(u(0,t))) = \{u(0, t+1)\}$, $q_\leftarrow^{-1}(q_\rightarrow(u(0,t))) = \emptyset$, $q_\rightarrow^{-1}(q_\rightarrow(u(0,t))) = \{u(0,t)\}$, and $q_\uparrow^{-1}(q_\rightarrow(u(0,t))) = \emptyset$ hold.*

3. *For $u(s,|B| - 1) \in \mathcal{Q}_{ul}$, $q_\downarrow^{-1}(q_\uparrow(u(s,|B| - 1))) = \emptyset$, $q_\leftarrow^{-1}(q_\uparrow(u(s,|B| - 1))) = \emptyset$, $q_\rightarrow^{-1}(q_\uparrow(u(s,|B| - 1))) = \{u(s+1, |B| - 1)\}$ and $q_\uparrow^{-1}(q_\uparrow(u(s,|B| - 1))) = \{u(s, |B| - 1)\}$ hold.*

4. *For $u(|A| - 1, t) \in \mathcal{Q}_{ur}$, $q_\downarrow^{-1}(q_\uparrow(u(|A| - 1, t))) = \emptyset$, $q_\leftarrow^{-1}(q_\uparrow(u(|A| - 1, t))) = \{u(|A| - 1, t+1)\}$, $q_\rightarrow^{-1}(q_\uparrow(u(|A| - 1, t))) = \emptyset$ and $q_\uparrow^{-1}(q_\uparrow(u(|A| - 1, t))) = \{u(|A| - 1, t)\}$ hold.*

Proof. We give the proof for the property in 1. The proofs for 2, 3 and 4 are almost the same as in 1.

First, from Lemma 3, we have $q_\leftarrow^{-1}(q_\leftarrow(u(s,0))) = \{u(s,0)\}$.

Next, since $B \cap q_\leftarrow(u(s,0)) = \emptyset$, we have $q_\rightarrow^{-1}(q_\leftarrow(u(s,0))) = q_\uparrow^{-1}(q_\leftarrow(u(s,0))) = \emptyset$.

Finally, we consider $q_\downarrow^{-1}(q_\leftarrow(u(s,0)))$. Because $u_{\langle A,B \mid C\rangle}$ takes 0 at $q_\leftarrow(u(s,0))$ and $q_\uparrow^{-1}(q_\leftarrow(u(s,0))) = \emptyset$, $q_\downarrow^{-1}(q_\leftarrow(u(s,0)))$ must have at least one element to cancel out the value -1 of $q_\leftarrow^{-1}(q_\leftarrow(u(s,0)))$. Hence, from Lemma 3, we have $q_\downarrow^{-1}(q_\leftarrow(u(s,0))) = \{u(s + 1, 0)\}$. □

For the "inner points" of the representation, we have the following lemma.

Lemma 6. *Let* $\mathcal{Q}_i = \mathcal{Q} \setminus (\{C, AC, BC, ABC\} \cup \mathcal{Q}_{ll} \cup \mathcal{Q}_{lr} \cup \mathcal{Q}_{ul} \cup \mathcal{Q}_{ur})$, *and* $s = |A \cap Q|$, $t = |B \cap Q|$ *for* $Q \in \mathcal{Q}_i$. *For each* $Q \in \mathcal{Q}_i$, *one of the following five properties holds.*

1. $q_\downarrow^{-1}(Q) = \{u(s,t)\}$, $q_\leftarrow^{-1}(Q) = \{u(s-1,t)\}$, $q_\rightarrow^{-1}(Q) = \{u(s,t-1)\}$, $q_\uparrow^{-1}(Q) = \{u(s-1,t-1)\}$ *((1) in Figure 7)*

2. $q_\downarrow^{-1}(Q) = \{u(s,t)\}$, $q_\leftarrow^{-1}(Q) = \emptyset$, $q_\rightarrow^{-1}(Q) = \{u(s,t-1)\}$, $q_\uparrow^{-1}(Q) = \emptyset$ *((2) in Figure 7)*

3. $q_\downarrow^{-1}(Q) = \emptyset$, $q_\leftarrow^{-1}(Q) = \{u(s-1,t)\}$, $q_\rightarrow^{-1}(Q) = \emptyset$, $q_\uparrow^{-1}(Q) = \{u(s-1,t-1)\}$ *((3) in Figure 7)*

4. $q_\downarrow^{-1}(Q) = \{u(s,t)\}$, $q_\leftarrow^{-1}(Q) = \{u(s-1,t)\}$, $q_\rightarrow^{-1}(Q) = \emptyset$, $q_\uparrow^{-1}(Q) = \emptyset$ *((4) in Figure 7)*

5. $q_\downarrow^{-1}(Q) = \emptyset$, $q_\leftarrow^{-1}(Q) = \emptyset$, $q_\rightarrow^{-1}(Q) = \{u(s,t-1)\}$, $q_\uparrow^{-1}(Q) = \{u(s-1,t-1)\}$ *((5) in Figure 7)*

Proof. Let $Q \in \mathcal{Q}_i$. First, we show that at least one of $q_\downarrow^{-1}(Q)$ and $q_\uparrow^{-1}(Q)$ is not empty. If $q_\downarrow^{-1}(Q)$ and $q_\uparrow^{-1}(Q)$ are both empty, then $q_\leftarrow^{-1}(Q)$ and $q_\rightarrow^{-1}(Q)$ must be empty because i) the value at $Q \in \mathcal{Q}_i$ is zero, ii) $q_\downarrow^{-1}(Q)$ and $q_\uparrow^{-1}(Q)$ have the value 1 by (2) and iii) $q_\leftarrow^{-1}(Q)$ and $q_\rightarrow^{-1}(Q)$ have the value -1 by (2). This contradicts $Q \in \mathcal{Q}_i$. Then, from Lemma 3, the possible patterns are classified as follows.

(I) We consider the case where both $q_\downarrow^{-1}(Q)$ and $q_\uparrow^{-1}(Q)$ are non-empty. In this case, $q_\leftarrow^{-1}(Q)$ and $q_\rightarrow^{-1}(Q)$ must be non-empty in order to cancel out the value 2 by $q_\downarrow^{-1}(Q)$ and $q_\uparrow^{-1}(Q)$ at Q. This is the case of 1 in Lemma 6.

(II) We consider the case where $q_\downarrow^{-1}(Q)$ is non-empty and $q_\uparrow^{-1}(Q)$ is empty. In this case, one of $q_\leftarrow^{-1}(Q)$ and $q_\rightarrow^{-1}(Q)$ must be non-empty and another must be empty in order to cancel out the value 1 at Q. If $q_\leftarrow^{-1}(Q)$ is non-empty and $q_\rightarrow^{-1}(Q)$ is empty, then this is the case of 4 in Lemma 6. If $q_\leftarrow^{-1}(Q)$ is empty and $q_\rightarrow^{-1}(Q)$ is non-empty, then this is the case of 2 in Lemma 6.

(III) We consider the case where $q_\downarrow^{-1}(Q)$ is empty and $q_\uparrow^{-1}(Q)$ is non-empty. In this case, one of $q_\leftarrow^{-1}(Q)$ and $q_\rightarrow^{-1}(Q)$ must be non-empty and another must be empty in order to cancel out the value 1 at Q. If $q_\leftarrow^{-1}(Q)$ is non-empty and $q_\rightarrow^{-1}(Q)$ is empty, then this is the case of 3 in Lemma 6. If $q_\leftarrow^{-1}(Q)$ is empty and $q_\rightarrow^{-1}(Q)$ is non-empty, then this is the case of 5 in Lemma 6.

$\square$

In order to prove Proposition 2, we use the following lemma.

Lemma 7. *Let* $1 \leq s_0 \leq |A| - 1$. *Then (12) is* $(s_0, *)$-*separable if and only if* $q_\downarrow(u(s_0, t)) = q_\rightarrow(u(s_0, t-1))$ *for all* $1 \leq t \leq |B| - 1$. *Similarly, for* $1 \leq t_0 \leq |B| - 1$, *(12) is* $(*, t_0)$-*separable if and only if* $q_\downarrow(u(s, t_0)) = q_\leftarrow(u(s-1, t_0))$ *for all* $1 \leq s \leq |B| - 1$.
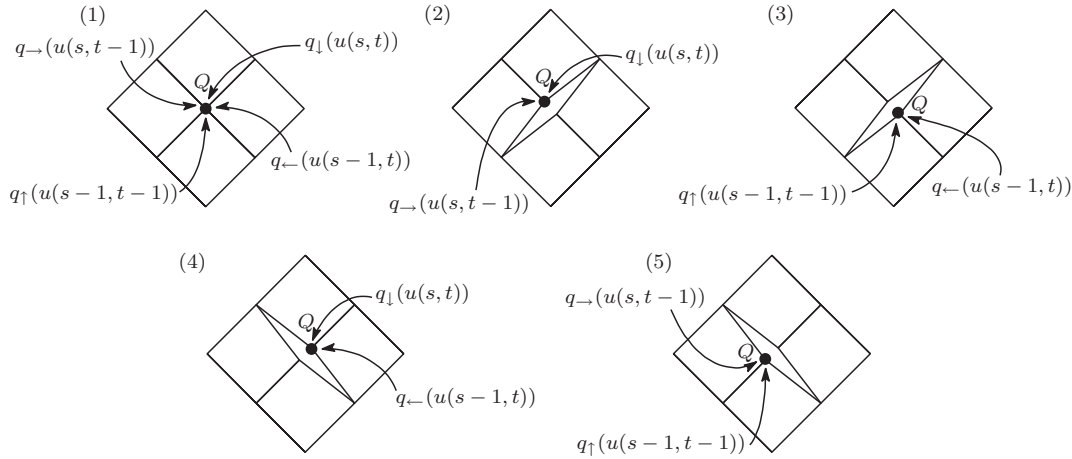
Figure 7: Five types of inner points

Proof. This follows immediately from Lemma 5 and 6.   □

Proof of Proposition 2

Proof. We prove the proposition by induction. If $|A| = |B| = 1$, then (12) is σ-decomposable from the definition. If $|A| = 1$ and $|B| \geq 2$, then (12) is $(*, t_0)$-separable for any $1 \leq t_0 \leq |B| - 1$ and therefore σ-decomposable by induction. The case $|B| = 1$ is similarly proved. Now assume $|A| = 2$ and $|B| \geq 2$. If (12) is $(1, *)$-separable, then the two representations in (14) after separation are σ-decomposable. Hence (12) is σ-decomposable. Now assume (12) is not $(1, *)$-separable. Then, by Lemma 7, $q_\downarrow(u(1, t)) \neq q_\rightarrow(u(1, t - 1))$ for any $1 \leq t \leq |B| - 1$. Therefore $q_\downarrow(u(1, t)) = q_\leftarrow(u(0, t))$ for any $1 \leq t \leq |B| - 1$. This means that, by Lemma 7 again, (12) is $(*, t)$-separable for any $1 \leq t \leq |B| - 1$. By induction, we obtain σ-decomposability of (12). The case $|A| \geq 2$ and $|B| = 2$ is similarly proved.   □

Next, we give the rigorous definition of the "rifts" of $u_{\langle A, B \mid C \rangle}$. Let $\mathcal{Q}_1$ be

$$\mathcal{Q}_1 = \{Q \in \mathcal{Q} \mid 1 \leq \exists s \leq |A| - 1, 1 \leq \exists t \leq |B| - 1, \ s.t.$$
$$Q = q_\downarrow(u(s, t)) = q_\uparrow(u(s - 1, t - 1))\}.$$

Note that $Q = q_\downarrow(u(s, t)) = q_\uparrow(u(s - 1, t - 1))$ is equivalent to $Q = q_\downarrow(u(s, t)) = q_\leftarrow(u(s - 1, t)) = q_\rightarrow(u(s, t - 1)) = q_\uparrow(u(s - 1, t - 1))$. Hence each element of $\mathcal{Q}_1$ satisfies 1 of Lemma 6. Let $\mathcal{Q}'$ be

$$\mathcal{Q}' = \{C, AC, BC, ABC\} \cup \mathcal{Q}_{ll} \cup \mathcal{Q}_{lr} \cup \mathcal{Q}_{ul} \cup \mathcal{Q}_{ur} \cup \mathcal{Q}_1.$$

Furthermore, let $\mathcal{Q}'' = \mathcal{Q} \setminus \mathcal{Q}'$. Then we have

$$\mathcal{Q}'' = \{q_\downarrow(u(s, t)), q_\uparrow(u(s - 1, t - 1)) \mid 1 \leq \exists s \leq |A| - 1, 1 \leq \exists t \leq |B| - 1, \ s.t.$$
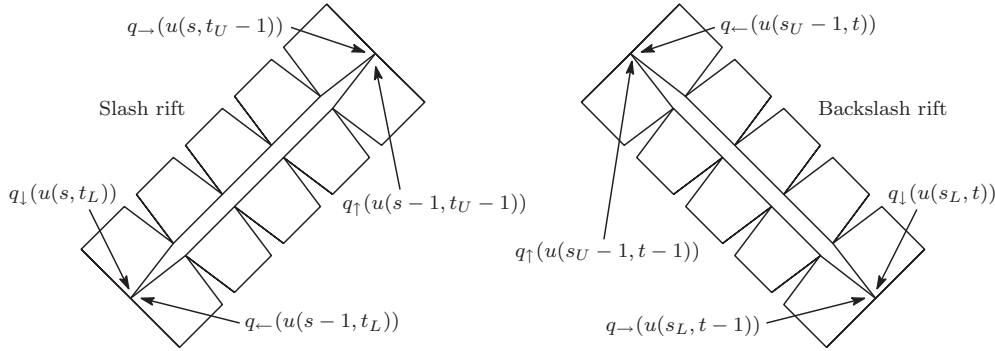
Figure 8: Definitions of slash rift and backslash rift

$$q_\downarrow(u(s,t)) \neq q_\uparrow(u(s-1,t-1))\}.$$

Each element of $\mathcal{Q}''$ satisfies one of 2, 3, 4 and 5 of Lemma 6.

Let $\tilde{s}(Q) = |A \cap Q|$ and $\tilde{t}(Q) = |B \cap Q|$ for $Q \in \mathcal{Q}$. Now we define slash rifts and backslash rifts for a representation of $u_{\langle A,B \mid C \rangle}$.

Definition 3. *For any $Q \in \mathcal{Q}''$ such that $Q = q_\downarrow(u(s,t)) \neq q_\leftarrow(u(s-1,t))$, let $t_L = t_L(Q)$ be the maximum number t' such that $q_\downarrow(u(s,t')) = q_\leftarrow(u(s-1,t'))$ and $t' \leq t$. Furthermore, let $t_U = t_U(Q)$ be the minimum number t' such that $q_\rightarrow(u(s,t'-1)) = q_\uparrow(u(s-1,t'-1))$ and $t' \geq t$. Then we define $\mathrm{R}_s(s; t_L, t_U) = \{Q \in \mathcal{Q} \mid \tilde{s}(Q) = s, t_L \leq \tilde{t}(Q) \leq t_U\}$ and call it a* slash rift. *See the left-hand side of Figure 8.*

We also define another type of rift. For any $Q \in \mathcal{Q}''$ such that $Q = q_\downarrow(u(s,t)) \neq q_\rightarrow(u(s-1,t))$, let $s_L = s_L(Q)$ be the maximum number s' such that $q_\downarrow(u(s',t)) = q_\rightarrow(u(s',t-1))$ and $s' \leq s$. Furthermore, let $s_U = s_U(Q)$ be the minimum number s' such that $q_\leftarrow(u(s'-1,t)) = q_\uparrow(u(s'-1,t-1))$ and $t' \geq t$. Then we define $\mathrm{R}_b(s_L, s_U; t) = \{Q \in \mathcal{Q} \mid s_L \leq \tilde{s}(Q) \leq s_U, \tilde{t}(Q) = t\}$ and call it a backslash rift. *See the right-hand side of Figure 8.*

A rift *is any slash rift or backslash rift, and* a rift pattern *is a set of rifts which do not intersect each other.*

Note that the existence of the maximizer and minimizer in Definition 3 is assured by Lemma 5. We denote the set of all of the slash rifts by $\mathcal{R}_s = \{\mathrm{R}_s(\tilde{s}(Q); t_L(Q), t_U(Q)) \mid Q \in \mathcal{Q}''\}$ and the set of all of the backslash rifts by $\mathcal{R}_b = \{\mathrm{R}_b(s_L(Q), s_U(Q); t(Q)) \mid Q \in \mathcal{Q}''\}$. Furthermore, note that rift occurs only at $(|A| - 1)(|B| - 1)$ pairs of $(\tilde{s}(Q), \tilde{t}(Q))$ for $Q \in \mathcal{Q}''$.

Proof of Proposition 4

Proof. The degree of freedom for any rift with length l is given by subtracting the number of cases where the rift is collapsed in the middle i.e., $q_\downarrow(Q) = q_\uparrow(Q)$ holds for a Q in the inner points of the rift, by permutations, from the number of permutations $l!$. Therefore, we obtain (17). ☐

Proof of Proposition 5

Proof. We will use induction on m. First, it is easily seen that $r_2(1) = 1$. Now we assume that (18) is true for $m-1$. If a backslash rift first appears in $q_\downarrow(u(1,k))$, i.e., there is no backslash rift in $q_\downarrow(u(1,0)), \ldots, q_\downarrow(u(1,k-1))$, then such a representation can be split in two representations: one is represented as $\sum_{i=0}^{2} \sum_{j=0}^{k-1} u(i,l)$ and another is represented as $\sum_{i=0}^{2} \sum_{j=k}^{m} u(i,l)$. In this case, since the former has $k!$ representations and the latter has $r_2(m-k)$ representations from the assumption, there are $k!r_2(m-k)$ representations. Therefore, by summing the number of representations for $k \in [1, m-1]$ and $m!$ which is the number of representations with no backslash rift, we obtain (18).   □

The following lemma implies that there exists at least one rift which can be eliminated for any representation of $u_{\langle A,B\,|\,C\rangle}$.

Proposition 6. *Assume that $\mathcal{R}_s$ or $\mathcal{R}_b$ is non-empty. Let us denote by $\mathcal{R}'_s$ the subset of all the $\mathrm{R}_s(s'; t'_L, t'_U) \in \mathcal{R}_s$ such that, for any $\mathrm{R}_s(s''; t''_L, t''_U) \in \mathcal{R}_s$, neither $s'' < s'$ nor $(t''_L, t''_U) \cap (t'_L, t'_U) \neq \emptyset$ holds where (t'_L, t'_U) means open interval. Let us denote by $\mathcal{R}'_b$ the subset of all the $\mathrm{R}_b(s'_L, s'_U; t') \in \mathcal{R}_b$ such that, for any $\mathrm{R}_b(s''_L, s''_U; t'') \in \mathcal{R}_b$, neither $t'' < t'$ nor $(s''_L, s''_U) \cap (s'_L, s'_U) \neq \emptyset$ holds. Then at least one of the following two properties holds.*

1. *There exists $\mathrm{R}_s(s; t_L, t_U) \in \mathcal{R}'_s$ such that, for any $\mathrm{R}_b(s_L, s_U; t) \in \mathcal{R}_b$, either $t \notin [t_L + 1, t_U - 1]$ or $s \leq s_L$ holds.*

2. *There exists $\mathrm{R}_b(s_L, s_U; t) \in \mathcal{R}'_b$ such that, for any $\mathrm{R}_s(s; t_L, t_U) \in \mathcal{R}_s$, either $s \notin [s_L + 1, s_U - 1]$ or $t \leq t_L$ holds.*

Proof. If $\mathcal{R}_s$ is non-empty, then $\mathrm{R}_s(\tilde{s}(Q); t_L(Q), t_U(Q))$ with the smallest s in $\mathcal{R}_s$ is clearly one of the element of $\mathcal{R}'_s$. This means that $\mathcal{R}'_s$ is non-empty if $\mathcal{R}_s$ is non-empty. The same thing holds for $\mathcal{R}_b$ and $\mathcal{R}'_b$.

If either $\mathcal{R}'_s$ or $\mathcal{R}_b$ is empty, then either 1 or 2 in the statement of Proposition 1 holds.

In the following, we consider the case where neither $\mathcal{R}'_s$ nor $\mathcal{R}_b$ is empty. We show by contradiction that either 1 or 2 holds. Suppose that neither 1 nor 2 holds.

Now we choose one element from $\mathcal{R}'_s$ and denote it by $\mathrm{R}_s(s^{(1)}; t_L^{(1)}, t_U^{(1)})$. By assumption that 1 does not hold, we can choose $\mathrm{R}_b(s_L, s_U; t) \in \mathcal{R}_b$ such that $t \in [t_L^{(1)} + 1, t_U^{(1)} - 1]$ and $s^{(1)} > s_L$ hold for $\mathrm{R}_s(s^{(1)}; t_L^{(1)}, t_U^{(1)})$.

If $\mathrm{R}_b(s_L, s_U; t) \in \mathcal{R}'_b$ holds, then we denote it by $\mathrm{R}_b(s_L^{(1)}, s_U^{(1)}; t^{(1)})$. If not, we choose $\mathrm{R}_b(s_L^{(1)}, s_U^{(1)}; t^{(1)}) \in \mathcal{R}'_b$ such that $t^{(1)} < t$ and $(s_L^{(1)}, s_U^{(1)}) \cap (s_L, s_U) \neq \emptyset$ hold. In both cases, we have $t^{(1)} \leq t_U^{(1)} - 1$.

Next, by assumption that 1 does not hold, we can choose one element from $\mathrm{R}_s(s; t_L, t_U) \in \mathcal{R}_s$ such that $s \in [s_L^{(1)} + 1, s_U^{(1)} - 1]$ and $t^{(1)} > t_L$ hold for $\mathrm{R}_b(s_L^{(1)}, s_U^{(1)}; t^{(1)})$. In this case, we have $t^{(1)} \geq t_U$. This can be shown as follows. If $t^{(1)} \geq t_U$ does not hold, i.e., $t_L < t^{(1)} < t_U$, then $Q = q_\downarrow(u(s, t^{(1)}))$ belongs to both $\mathrm{R}_s(s; t_L, t_U)$ and $\mathrm{R}_b(s_L^{(1)}, s_U^{(1)}; t^{(1)})$. This contradicts Lemma 6 and hence we have $t^{(1)} \geq t_U$.

If $R_s(s; t_L, t_U) \in \mathcal{R}'_s$ holds, then we denote it by $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)}) \in \mathcal{R}'_s$. If not, we choose $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)}) \in \mathcal{R}'_s$ such that $s^{(2)} < s$ and $(t_L^{(2)}, t_U^{(2)}) \cap (t_L, t_U) \neq \emptyset$ hold. In both cases, we have $s^{(2)} \leq s_U^{(1)} - 1$. Furthermore, by considering the relation $t_L^{(2)} \leq t_U \leq t^{(1)} \leq t_U^{(1)} - 1$ and the fact that both $R_s(s^{(1)}; t_L^{(1)}, t_U^{(1)})$ and $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)})$ are the elements of $\mathcal{R}'_s$, we have $(t_L^{(2)}, t_U^{(2)}) \cap (t_L^{(1)}, t_U^{(1)}) = \emptyset$. This means that $t_U^{(2)} \leq t_L^{(1)}$ holds. By applying much the same way as above to $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)})$, we can choose $R_b(s_L^{(2)}, s_U^{(2)}; t^{(2)}) \in \mathcal{R}'_b$ such that $s_U^{(2)} \leq s_L^{(1)}$ holds.

Therefore, for $R_s(s^{(1)}; t_L^{(1)}, t_U^{(1)})$, $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)})$ $R_b(s_L^{(1)}, s_U^{(1)}; t^{(1)})$ and $R_s(s^{(2)}; t_L^{(2)}, t_U^{(2)})$, we obtain the following inequalities:

$$s_L^{(2)} < s_U^{(2)} \leq s_L^{(1)} < s_U^{(1)}, \ t_L^{(2)} < t_U^{(2)} \leq t_L^{(1)} < t_U^{(1)} \tag{19}$$

If neither 1 nor 2 holds, we can repeat the above procedure indefinitely. However, from (19), s and t decrease in each step. This means that the same rift never appear more than once in the procedure. The procedure stops in finite times and this is contradiction. Therefore either 1 or 2 in the statement of Proposition 6 holds. □

Now we give the proof of Theorem 1.

Proof of Theorem 1

Proof. It suffices to show that any representation of $u_{\langle A,B \mid C \rangle}$ and the standard representation in (16) are connected by two-by-two basic relations. Remember, from Proposition 3 in Section 3, that the representations without rifts are connected by two-by-two basic relations. Therefore, from Proposition 6, it suffices to show that we can eliminate the rift which satisfies either 1 or 2 in the statement of Proposition 6. We prove this in the case where a rift satisfies 2 of Proposition 6. The proof for the case of 1 is similar to the proof of the case of 2 and will be omitted.

Let us consider a representation:

$$u_{\langle A,B \mid C \rangle} = \sum_{s=0}^{|A|-1} \sum_{t=0}^{|B|-1} u_\theta(s,t). \tag{20}$$

In the rest of the proof, we omit the subscript θ in $u_\theta(s,t)$. Assume that a rift in this representation satisfies 2 of Proposition 6. Then there exists $R_b(s_L, s_U; t) \in \mathcal{R}_b$ such that either $s \notin [s_L + 1, s_U - 1]$ or $t \leq t_L$ holds for any $R_s(s; t_L, t_U) \in \mathcal{R}_s$. We show that we can eliminate the backslash rift $R_b(s_L, s_U; t)$ by applying two-by-two basic relations.

Let $\Gamma' = q_\downarrow(u(s_L, 0))$, $B' = B \cap q_\rightarrow(u(s_L, t-1))$ and $A' = A \cap \{q_\leftarrow(u(s_U - 1, 0)) \setminus \Gamma'\}$. Then we obtain the following representation of $u_{\langle A',B' \mid \Gamma' \rangle}$:

$$u_{\langle A',B' \mid \Gamma' \rangle} = \sum_{s'=s_L}^{s_U-1} \sum_{t'=0}^{t-1} u(s',t'). \tag{21}$$

From the assumption, i.e., 2 of Proposition 6, it is easily seen that the representation in (21) has no rift. Let $a^{(s')} = q_\leftarrow(u(s' + s_L - 1, 0)) \setminus q_\downarrow(u(s' + s_L - 1, 0))$ for $1 \le s' \le s_U - s_L$. Note that the sequence $a^{(1)}, \ldots, a^{(s_U - s_L)}$ is ordered with respect to the upper side of the rift. In the same way, let $b^{(t')} = q_\rightarrow(u(s_L, t' - 1)) \setminus q_\downarrow(u(s_L, t' - 1))$ for $1 \le t' \le t$. Let us consider the following representation without rifts:

$$u_{\langle A', B' \mid \Gamma' \rangle} = \sum_{s'=1}^{s_U - s_L} \sum_{t'=1}^{t} u_{\langle a^{(s')}, b^{(t')} \mid a^{(1)} \ldots a^{(s'-1)} b^{(1)} \ldots b^{(t'-1)} \Gamma' \rangle}. \tag{22}$$

Remember that the terms in the representation of (21) also appear in the representation of (20). Hence, in the representation of (20), if we replace the terms used in the representation of (21) with the terms used in the representation of (22), then we can eliminate the backslash rift $\mathrm{R}_b(s_L, s_U; t)$ in the representation of (20) because of the definition of $a^{(s')}$ and $b^{(t')}$. Furthermore, from Proposition 3, this replacement is realized by applying two-by-two basic relations. Therefore we can eliminate the backslash rift $\mathrm{R}_b(s_L, s_U; t)$ by applying two-by-two basic relations.

This completes the proof of Theorem 1.   □

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