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THE NECESSARY CONDITIONS OF OPTIMALITY IN INEQUALITY TYPE FOR ONE NONSMOOTH PROBLEM

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Abstract: Necessary conditions of optimality of the first order inequality type for the problems of optimal control described by the of extreme-differential equations and quality criterion maximum type are obtained in this paper.type maximum principle of Pontryagin are obtained.

Keywords: Optimal control, nonsmooth analisus,differential equations

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Consider the following minimization problem

$$I(u)=\max_{a\in A}\int_{0}^{t_1}F(a,t,x(t),u(t))dt \qquad (1)$$

subject to

$$\dot{x}(t)=\max_{q\in Q}\ f(q,t,x(t),u(t)),\ \ x(t_0)=x_0, \qquad (2)$$

$$u(t)\in U\subset R^r, \qquad (3)$$

where $x(t)\in R^m$ is a vector-function of phase variables, t_0, x_0 are fixed, t_1 is free. $Q\subset R^s, B\in R^l$ are given compacts, m-dimensional vector-function $f(q,t,x,u)$ and $F(a,t,x,u)$ is continuons together with the first-order partial derivatives with respect to t,x on $Qx[t_0,t_1]xR^m xU$, besides

$$\min_{q\in R(t,x,u)} f_t(q,t,x,u),\quad \max_{q\in R(t,x,u)} f_t(q,t,x,u)$$

$$\min_{q\in R(t,x,u)} f_x(q,t,x,u),\quad \max_{q\in R(t,x,u)} f_x(q,t,x,u)$$

are bounded on $Qx[t_0,t_1]xR^m xU$,

$$R(t,x,u)=\{q\in Q:\max_{\bar{q}\in Q} f(\bar{q},t,x,u)=f(q,t,x,u)\ \},$$

$F(a,t,x,u)$ is continuons together with the first-order partial derivatives with respect to t,x on $Ax[t_0,t_1]xR^m xU$.

The system (2) implies that for each component maximum is taken separately. By this we mean that the set of q parameters for each row, generally speaking, differs from the corresponding set for any other row. Thus,

$$f(q,x,u)=\begin{cases} f_1(q_{11},...,q_{1s},x_1,...,x_m,u_1,...,u_r) \\ f_2(q_{21},...,q_{2s},x_1,...,x_m,u_1,...,u_r) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(q_{m11},...,q_{ms},x_1,...,x_m,u_1,...,u_r) \end{cases}$$

there, no direct connection among the maximization s of distinct rows, generally speaking, exist. In fact, here, various components are dependent on differing parameters and the maximization is derived from the set of parameters in QxQx...xQ.

Under the above assumptions to every adimisible control $u(t)$, $t\in[t_0,t_1]$ corresponds the unique solution $x(t)$ of system (2) defined on $[t_0,t_1]$.

Let $x(t)$, $t\in[t_0,t_1]$ be the solution of equation (2) corresponding to an admissible control $u(t)$.

Assume also that for every $v\geq 0$, $\tau\in[0,1]$ the following condition is satisfied

$$t_1-t_0=\int_0^1 v(\tau)d\tau$$

Analogously to the case of [1] one can show that if a measurable r-dimensional vector-function $w(\tau)$ assuming values in U are such that the equality $w(\tau)=u(t(\tau))$ is satisfied almost everywhere on

$$\Delta(v)=\{\tau\in[0,1]:v(\tau)>0\}$$

where $t(\tau)=t_0+\int_0^\tau v(s)ds$, $\tau(t)$ is the inverse to $t(\tau)$,then function $y(\tau)=x(t(\tau))$ is a solution of the following equation

$$\dot{y}(\tau)=v(\tau)\max_{q\in Q} f(q,t(\tau),\dot{y}(\tau),w(\tau)) \tag{5}$$

Conversely, if $w(\tau)$ is bounded and measurable on $\Delta(v)$ and assumes values in U and $y(\tau)$ is a solution of equation (5) corresponding to control $v(\tau)\geq 0,\ \ \tau\in[0,1]$, then $u(t)=w(\tau(t))$ is an admissible in problem (1)-(4) control and $x(t)=y(\tau(t))$ is the solution ofi equation (2) corresponding to control $u(t)$.

Let $(x_*(t),u_*(t),t_1^*)$ be an optimal solution to problem (1)-(4). Then $t_*(\tau),\ \ y_*(\tau)=x_*(t_*(\tau)),\ \ v_*(\tau)$ is a solution to the following reduced problem:

Minimize

$$I(u)=\max_{a\in A}\int_0^{t_1}F(a,t,x(t),u(t))dt \tag{6}$$

subject to

$$\dot{y}(\tau)=v(\tau)\max_{q\in Q}f(q,t(\tau),y(\tau),w(\tau)) \tag{7}$$

$$\dot{t}(\tau)=v(\tau),\qquad\qquad t(0)=t_0 \tag{8}$$

$$v(\tau)\geq 0,\qquad \tau\in[0,1] \tag{9}$$

Where

$$t_*(\tau)=t_0+\int_0^{\tau}v_*(s)ds$$

$$\nabla(v_*)=\{\tau\in[0,1]:v_*(\tau)>0\}$$

$w(\tau)$ is a given r-dimensional vector-function assuming values in U an satisfying

$$w_*(\tau)=u_*(t_*(\tau))$$

aimost every where on $\Delta(v_*)$

Again similar to [1] one can show that $(t_*(\tau),y_*(\tau),z_*(\tau)=0,v_*(\tau))$ is the unique quadruple providing the minimum to the following functional

$$I(v)=\max_{a\in A}\int_0^1 v(\tau)F(a,\tau,x(\tau),u(\tau))d\tau+\int_0^1 z^2(\tau)d\tau \tag{10}$$

subject to

$$\dot{y}(\tau)=v(\tau)\max_{q\in Q}f(q,t(\tau),y(\tau),w(\tau))\ ,\ \ y(0)=x_0\ , \tag{11}$$

$$\dot{t}(\tau)=v(\tau),\ \ t(0)=t_0 \tag{12}$$

$$\dot{z}(\tau)=|v(\tau)-v_*(\tau)|,\ \ z(0)=0 \tag{13}$$

$$v(\tau)\geq 0,\ \ \tau\in[0,1] \tag{14}$$

Let a sequence of vector-functions $f^n(t,x,u)$ converge uniformly (jointly on variables) to the function $\max_{q\in Q} f(q,t,x,u)$ for $n\to\infty$ and let the following inequalities are satisfied

$$\min_{q\in R(t,x,u)} f_x(q,t,x,u)-\frac{1}{n}\le f_x{}^n(t,x,u)\le \max_{q\in R(t,x,u)} f_x(q,t,x,u)+\frac{1}{n} \qquad (15)$$

$$\min_{q\in R(t,x,u)} f_t(q,t,x,u)-\frac{1}{n}\le f_t{}^n(t,x,u)\le \max_{q\in R(t,x,u)} f_t(q,t,x,u)+\frac{1}{n} \qquad (16)$$

The problem of minimization of functional

$$I(v)=\max_{a\in A}\int_0^1 v(\tau)F(a,\tau,x(\tau),u(\tau))d\tau+\int_0^1 z^2(\tau)d\tau$$

subject to condiditions (12)-(14) and

$$\dot y(\tau)=v(\tau)\, f^n(t(\tau),y(\tau),w_*(\tau)) \quad , \quad y(0)=x_0 \qquad (17)$$

has a solution $(t_n(\tau),y_n(\tau),z_n(\tau),v_n(\tau))$.

Denote

$$p_n(\tau)=\max_{a\in A}\int_0^\tau v_n(s)F(a,s,y_n(s),u(s))ds+\int_0^\tau z_n{}^2(s)ds$$

Cleary

$$p_n(1)=\max_{a\in A}\int_0^1 v_n(\tau)F(a,t_n(\tau),y_n(\tau),w_*(\tau))d\tau+$$
$$+\int_0^1 z_n{}^2(\tau)d\tau\le\max_{a\in A}\int_0^1 v_*(\tau)F(a,t_*(\tau),y_*(\tau),w_*(\tau))d\tau$$

By Arzela criterion there exist uniformly converging sequences

$$p_k(\tau)\to p(\tau), y_k(\tau)\to y(\tau), z_k(\tau)\to z(\tau) \text{ for } k\to\infty$$

By virtue of convexity of systems

$$\dot p=v\max_{a\in A}F(a,t,y,w_*(\tau))+z^2,$$
$$\dot y=v\max_{q\in Q}f(q,t,y,w(\tau)),\quad v\ge 0$$
$$\dot t(\tau)=v(\tau),$$

$$\dot{z}(\tau)=|v(\tau)-v_*(\tau)|,$$
$$v(\tau)\geq 0$$

functions $p(\tau), y(\tau), z(\tau)$, satisfy it for some control $v(\tau)$. $(Sec[7])$. Quadruple $(t(\tau), p(\tau), y(\tau), z(\tau))$ satisfies conditions (9)-(13) woreover

$$I(v)=p(1)\leq I(v_*)$$

It follows from the uniqueness of optimal quadruple $(t_*(\tau), y_*(\tau), z_*(\tau)=0, v_*(\tau))$, that

$$y_n(\tau)\to y_*(\tau), z_n(\tau)\to z_*(\tau)=0$$

for $n\to\infty$ uniformly on τ. The last relation implies $v_n(\tau)\to v_*(\tau)$ for almost all τ.

It follows from $v_n(\tau)\to v_*(\tau)$ that $t_n(\tau)\to t_*(\tau)$ for almost all τ.

Lemma. For a control $v_n(\tau), \tau\in[0,1]$ be optimal in problem (10),(12)-(14),(17) it is necessary that the following condition be satisfied

$$\min_{a\in A(y_n(1))}$$
$$[(v-v_n(\tau))\Psi_n'(\tau,a)f^n(t_n(\tau),y_n(\tau),w_*(\tau))-F(a,t_n(\tau),y_n(\tau),w_*(\tau)+s_n(\tau,a))]$$
$$+$$
$$+\Psi_n^z(\tau)[|v-v_n(\tau)|-|v_n(\tau)-v_*(\tau)|]\leq 0 \qquad (18)$$

for almost all $\tau\in[0,1]$ and for all $v\geq 0$. There $\{\Psi_n(\tau,a), a\in A(y_n(1))\}$, $\{s_n(\tau,a), a\in A(y_n(1))\}$ is a solution to the system

$$\psi_n(\tau,a)=\int_\tau^1 v_n(s)[\psi_n'(s,a)f_y^n(t_n(s),y_n(s),w_*(s))-F(a,t_n(s),y_n(s),w_*(s))]ds, \qquad (19)$$

$$s_n(\tau,a)=\int_\tau^1 v_n(s)[\psi_n'(s,a)f_t^n(t_n(s),y_n(s),w_*(s))-F_t(a,t_n(s),y_n(s),w_*(s))]ds, \qquad (20)$$

$$\psi_n^z(s)=2\int_\tau^1 z_n(s)ds, \qquad (21)$$

$$A(y_n(1)) = (a \in A : \max_{\bar{a} \in A} \int_0^1 v_n(\tau) F(\bar{a}, t_n(\tau), y_n(\tau), w_*(\tau)) d\tau = \int_0^1 v_*(\tau) F(a, t_*(\tau), y_*(\tau), w_*(\tau)) d\tau)$$

Prof: Let $v_n(\tau), \tau \in [0,1]$ be an optimal control in problem (10),(12)-(14),(17), and let $\bar{v}_n(\tau)$ be an admizible control defined as

$$\bar{v}_n(\tau) = \begin{cases} v(\tau), \tau \in [\theta, \theta + \varepsilon], \\ v_n(\tau), \tau \in [0,1]/[\theta, \theta + \varepsilon], \end{cases} \qquad (22)$$

Where $v(\tau) \geq 0, \theta \in [0,1]$ is an arbitrary regular point of control $v_n(\tau)$, and $\varepsilon > 0$ is a sufficiently small positive number such that $\theta + \varepsilon < 1$

Denote $y_n(\tau), z_n(\tau)$ and $\bar{y}_n(\tau), \bar{z}_n(\tau)$ solutions of system (17),(13) corresponding to controls $v_n(\tau)$ and $\bar{v}_n(\tau)$, respectively.

Using the well-known scheme (see e.q.[8]) one can easly show that

$$\|\bar{y}_n(\tau) - y_n(\tau)\| = \|\Delta y_n(\tau)\| \leq k\varepsilon \quad (k = const > 0) \qquad (23)$$

for all $\tau \in [0,1]$.

Turn to calculation of the increment of quality criterion

Clearly

$$\Delta I(v_n) = \max_{b \in B} \Phi(\bar{y}_n(1), b) - \max_{b \in B} \Phi(y_n(1), b) + \int_0^1 [\bar{z}_n^2(\tau) - z_n^2(\tau)] d\tau \geq 0 \quad (24)$$

If the following expansion has a place

$$\Delta I(v_n) = I(\bar{v}_n) - I(v_n) = \varepsilon \delta I(v_n) + 0(\varepsilon)$$

then call $\delta I(v_n)$ the first variatinal of function $I(v_n)$.

Applying the modified method of increments developed in [3] for the control problems with nonsmoth quality criterion the first variation of functions $I(v_n)$ can be found as

$$\delta I(v_n) = \min_{b \in B(y_n(1))}$$
$$[(v - v_n(\tau))\Psi_n'(\tau, b) f''(t_n(\tau), y_n(\tau) y_n(\omega_1(\tau)), w_*(\tau)) + s_n(\tau, b)] +$$
$$+ \Psi_n^z(\tau)[|v - v_n(\tau)| - |v_n(\tau) - v_*(\tau)|]$$

Bu virtue of inequality (24) the assertion of Lemma follows.
Since $z_n(\tau) \to 0$ for $n \to \infty$, it follows from (21) that $\Psi_n^z(\tau) \to 0$ for $n \to \infty$ uniformly on τ. Since the sequence of matrix-functions

$$\{ f_y''(t_n(\tau), y_n(\tau), w_*(\tau)) \}, \quad \{ f_t''(t_n(\tau), y_n(\tau), w_*(\tau)) \},$$

is bounded we can choose subsequences weakly converging to some measurable functions $A(\tau)$, and $h(\tau)$, respectively .It follows then from conditions (15) and (16) that

$$\min_{q \in R(t_*(\tau), y_*(\tau), w_*(\tau))} f_y(q, t_*(\tau), y_*(\tau), w_*(\tau)) \leq A(\tau) \leq$$
$$\max_{q \in R(t_*(\tau), y_*(\tau), w_*(\tau))} f_y(q, t_*(\tau), y_*(\tau), w_*(\tau))$$

$$\min_{q \in R(t_*(\tau), y_*(\tau), \theta_*(\tau), w_*(\tau))} f_t(q, t_*(\tau), y_*(\tau), w_*(\tau)) \leq h(\tau) \leq$$
$$\max_{q \in R(t_*(\tau), y_*(\tau), \theta_*(\tau), w_*(\tau))} f_t(q, t_*(\tau), y_*(\tau), w_*(\tau))$$

We can choose a subsequence from sequence $\{\Psi_n(\tau, a), a \in A(y_n(1))\}$ which uniformly on τ converges to some function $\Psi(\tau, a)$ for each $a \in A(y_n(1))$.

Then wehove from (19) and (20)

$$\psi(\tau, a) = \int_\tau^1 v_*(s)[A'(s)\psi_n(s, a) - F_y(a, t_n(s), y_n(s), w_*(s))]ds, \qquad (25)$$

$$s(\tau, a) = \int_\tau^1 v_*(s)[h'(s)s_n(s, a) - F_t(a, t_n(s), y_n(s), w_*(s))]ds, \qquad (26)$$

Besides, passing in (18) to limit for $n \to \infty$, we obtain that the maximum principle is satisfied for $(t_*(\tau), y_*(\tau), v_*(\tau))$

$$\min_{b\in B(y_*(1))}\left[(v-v_n(\tau))\Psi'(\tau,a)\max_{q\in Q}f(q,t_*(\tau),y_*(\tau),w_*(\tau))-F(a,t_*(\tau),y_*(\tau),w_*(\tau))+s(\tau,b)\right]\le 0 \tag{27}$$

(27) implies,that

$$\min_{a\in A(y_*(1))}\left[\Psi'(\tau,a)\max_{q\in Q}f(q,t_*(\tau),y_*(\tau),w_*(\tau))-F(a,t_*(\tau),y_*(\tau)w_*(\tau))+s(\tau,a)\right]\le 0 \tag{28}$$

$$\max_{a\in A(y_*(1))}\left[\Psi'(\tau,a)\max_{q\in Q}f(q,t_*(\tau),y_*(\tau),w_*(\tau))-F(a,t_*(\tau),y_*(\tau)w_*(\tau))+s(\tau,a)\right]\ge 0$$

for almost all $\tau\in\Delta(v_*)$,

$$\min_{a\in A(y_*(1))}\left[\Psi'(\tau,a)\max_{q\in Q}f(q,t_*(\tau),y_*(\tau),w_*(\tau))-F(a,t_*(\tau),y_*(\tau)w_*(\tau))+s(\tau,a)\right]\le 0 \tag{29}$$

for almost all $\tau\in[0,1]/\Delta(v_*)$

Theorem: For the optimality of a control $u_*(t), t\in[t_0,t_{1*}]$ in problem (1)-(4) it is necessary that the following conditions be sasisfied

$$\min_{a\in A(x_*(t_{1*}))}\left[p'(t,b)\max_{q\in Q}f(q,t,x_*(t),u_*(t))-F(a,t,x_*(t),u_*(t))+r(t,a)\right]\le 0$$

$$\max_{a\in A(x_*(t_{1*}))}\left[p'(t,b)\max_{q\in Q}f(q,t,x_*(t),u_*(t))-F(a,t,x_*(t),u_*(t))+r(t,a)\right]\ge 0 \tag{32}$$

$$\min_{a\in A(x_*(t_{1*}))}\left[p'(t,b)\max_{q\in Q}f(q,t,x_*(t),u_*(t))-F(a,t,x_*(t),u_*(t))+r(t,a)\right]\le 0 \tag{33}$$

where $p(t,b)$ and $r(t,b)$ are a solution to the problem

$$\dot{p}(t,a) = -A'(t)p(t,a) - F(a,t,x_*(t),u_*(t)), \qquad p(t_{1*},a) = -\Phi_x(x_*(t_{1*}),a),$$
$$a \in A(x_*(t_{1*})) \tag{34}$$

$$\dot{r}(t,a) = -h'(t)p(t,a) - F_t(a,t,x_*(t),u_*(t)), \quad r(t_{1*},a) == \quad a \in A(x_*(t_{1*})) \tag{35}$$

$$\min_{q \in R(t,x_*(t),u_*(t))} f_x(q,t,x_*(t),u_*(t)) \le A(t) \le \max_{q \in R(t,x_*(t),u_*(t))} f_x(q,t,x_*(t),u_*(t))$$

$$\min_{q \in R(t,x_*(t),u_*(t))} f_t(q,t,x_*(t),u_*(t)) \le h(t) \le \le \max_{q \in R(t,x_*(t),u_*(t))} f_t(q,t,x_*(t),u_*(t))$$

for almost all $t \in [t_0, t_{1*}]$ and all $u \in U$.

$$A(x_*(t_{1*})) = (a \in A : \max_{\bar{a} \in A} \int_0^{t_{1*}} F(\bar{a},t,x_*(t),u_*(t))d\tau = \int_0^{t_{1*}} F(a,t,x_*(t),u_*(t))dt).$$

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