Istanbul Üniv. Fen Fak.
Mat. Fiz. Astro. Derg. **1**(2004-2005),31-42

RANDOM P-NORMED SPACES AND APPLICATIONS TO RANDOM FUNCTIONS

Ioan Golet

Abstract. In this paper the concept of random p-normed space is introduced. This structure is a slight enlargement of the concept of the random normed space. Some topological properties of random p-normed spaces are analyzed. Examples of random p-normed spaces together with their connections with deterministic p-normed spaces are also given. Approximation theorems for functions with values in a random p-normed spaces are proved and applications to the study of function with a random parameter are stated. [1]

1 Introduction

In [9] K. Menger proposed the probabilistic concept of distance by replacing the number $d(p,q)$, the distance between two points p,q by a distribution function $F_{p,q}$. This idea led to a large development of probabilistic analysis. Applications to systems having hysteresis, mixture processes, the measuring error, the fiabihty theory were given [3], [11]. In [12] A. N. Šerstnev used K. Menger's idea for sets endowed with an algebraic structure of linear space. So, he initiated the study of probabilistic normed spaces.

Let $\mathbb{R}$ denotes the the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $I = [0,1]$ the closed unit interval. A mapping $F : \mathbb{R} \to I$ is called a distribution function if it is non decreasing, left continuous with $\inf F = 0$ and $\sup F = 1$.

D_+ denotes the set of all distribution functions for that $F(0) = 0$. Let

[1]AMS Mathematics Subject Classification 1991. 54E70, 41A50.
Key words and phrases: probabilistic metric, random p-norm, random function.

F, G be in D_+ , then we write $F \leq G$ if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. If $a \in \mathbb{R}_+$ then H_a will be the element of D_+ for which $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$. It is obvious that $H_0 \geq F$, for all $F \in D_+$. The set D_+ will be endowed with the natural topology defined by the modified Lévy metric d_L [11].

A p-normed space is a pair $(L, ||\cdot||)$ ([7], [2]), where L is a linear space, $0 < p \leqslant 1$ and $||\cdot||$ is a real valued mapping defined on L such that the following conditions are satisfied :

(1) $||x|| \geqslant 0$ for all $x \in L$.
(2) $||x|| = 0$ if and only if $x = \theta$.
(3) $||\alpha \cdot x|| = |\alpha|^p ||x||$, whenever $x \in L$ and $\alpha \in \mathbb{R}$,
(4) $||x + y|| \leq ||x|| + ||y||$, for all $x, y \in L$.

A t-norm T is a two place function $T : I \times I \to I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$. A triangle function τ is a binary operation on D_+ which is commutative, associative, non decreasing in each place and for which H_0 is the identity, that is, $\tau(F, H_{0)}) = F$ for every $F \in D_+$. T-norms and triangle functions have had a important place in writing appropriate probabilistic triangle inequalities. The terminology and notations are standard as in [3],[11].

2 Random p-normed spaces

Definition 1. Let L be a linear space, τ a triangle function , $0 < p \leqslant 1$, and let $\mathcal{F}$ be a mapping from L into D_+. If the following conditions are satisfied :

(5) $F_x = H_0$ implies $x = \theta$,
(6) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|^p})$, for every $t > 0, \alpha \neq 0$ and $x, y \in L$,
(7) $F_{x+y} \geq \tau(F_x, F_y)$, whenever $x, y, z \in L$.

then $\mathcal{F}$ is called a probabilistic p-norm on L and ($L, \mathcal{F}, \tau$) is called a probabilistic p-normed space. If (5)-(6) are satisfied and the probabilistic triangle inequality (7) is formulated under a t-norm T :
(8) $F_{x+y,z}(t_1+t_2) \geq T(F_{xz}(t_1), F_{yz}(t_2))$, for all $x, y, z, \in L, t_1, t_2 \in \mathbb{R}_+$, then $(L, \mathcal{F}, T)$ is called a random p-normed space.

Remark 1. It is easy to check that every p-normed space $(L, ||\cdot||)$ [7], [2] can be, in a natural way, made a random p-normed space by setting $F_x(t) = H_0(t - ||x||)$, for every $x \in L$, $t \in \mathbb{R}_+$ and $T = Min$.

Proposition 1. If T is a left continuous t-norm and τ_T is the triangle function defined by $\tau_T(F,G)(t) = \sup_{t_1+t_2<t} T(F(t_1), G(t_2)), \quad t > 0$, then $(L, \mathcal{F}, \tau_T)$ is a probabilistic p-normed space iff $(L, \mathcal{F}, T)$ is a random p-normed space.
Proposition 2. If $(L, \mathcal{F}, T)$ is random p-normed space then the random p-norm $\mathcal{F}$ has the following property :
(9) $\quad F_\theta(t) = H_0(t)$, for all $t > 0$.
Proof. Indeed, $F_\theta(t) = F_{\alpha\theta}(t) = F_\theta(\frac{t}{|\alpha|^p})$, for all $\alpha \in \mathbb{R} - \{0\}$. Then

$$F_\theta(t) = \lim_{\alpha \to 0} F_\theta(\frac{t}{|\alpha|^p}) = F_\theta(\infty) = H_0(t).$$

If we define $\mathcal{F}^m(x,y) = F_{x-y}$ then a $(L, \mathcal{F}^m, \tau)$ becomes a probabilistic metric space under the same triangle function τ. In what sequel we will consider random p-normed spaces under continuous t-norms $T \geqslant T_m$ $(T_m(a,b) = \text{Max}\,\{a+b-1, 0\})$. This condition insures the existence of a linear topology on L.
Example. We will construct a particular class of random p-normed spaces which have a large statistical disposal. Let $(L, ||\cdot||)$ be a p-normed space and $G = \{G_i\}_{i\in I}$ be an at most countable family of distribution function $(G_i \in D_+)$ and let $\Lambda = \{\lambda_i\}_{i\in I}$ an at most family of positive real numbers such that $\sum_{i\in I} \lambda_i = 1$. For every $x \in L$ we define the mapping $\mathcal{F} : L \to D_+$ by :

$$F_x(t) = \sum_{i\in I} \lambda_i G_i(\frac{t}{||x||}),$$

for all $t \in \mathbb{R}_+$. We also make the convention $F_x(\frac{t}{0}) = F(\infty) = 1$, for $t > 0$ and $F_x(\frac{0}{0}) = 0$.
The triple $(L, \mathcal{F}, T)$ becomes a random p-normed space. It is called simple generated random p-normed space by the families G and Λ. Starting from a p-normed space, for particular cases of families G and Λ different random p-normed spaces can be obtained. So, every process of measurement of vectors can be statistical interpreted by using appropriate statistical distribution functions as family G with coefficients $\{\lambda_i\}_{i\in I}$.
Theorem 1. Let $(L, \mathcal{F}, T)$ be a random p-normed space under a continuous t-norm T such that $T \geq T_m$, then :
(a) The family of subsets of L:
$\mathcal{U} = \{U(\varepsilon, \lambda)\} : \varepsilon > 0, \lambda \in (0,1)\}, \quad U(\varepsilon,\lambda) = \{(x,y) \in L \times L : F_{x-y}(\varepsilon) > 1 - \lambda\}$

is a complete system for a uniformity on L induced by the probabilistic metric $\mathcal{F}^m$.
(b) $\mathcal{V} = \{V(\varepsilon,\lambda)\} : \varepsilon > 0, \lambda \in (0,1)\}$, $\quad V(\varepsilon,\lambda) = \{x \in L : F_x(\varepsilon) > 1-\lambda\}$ is a complete system of neighbourhoods of zero for a linear topology on L generated by the p-norm $\mathcal{F}$.

Proof. We will prove only the point (b), that of (a) is similar to (b).

Let $V(\varepsilon_k,\lambda_k), k = 1,2$ be in $\mathcal{V}$. We consider $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$, $\lambda = \min\{\lambda_1,\lambda_2\}$, then $V(\varepsilon,\lambda) \subset V(\varepsilon_1,\lambda_1) \cap V(\varepsilon_2,\lambda_2)$.

Let $\alpha \in \mathbb{R}$ such that $0 \leq |\alpha| \leq 1$ and $x \in \alpha V(\varepsilon,\lambda)$, then $x = \alpha y$, where $y \in V(\varepsilon,\lambda)$ and we have

$$F_x(\varepsilon) = F_{\alpha y}(\varepsilon) = F_y(\frac{\varepsilon}{|\alpha|^p}) \geq F_y(\varepsilon) > 1-\lambda.$$

This shows us that $x \in V(\varepsilon,\lambda)$, hence $\alpha V(\varepsilon,\lambda) \subset V(\varepsilon,\lambda)$.

Let's show that, for every $V \subset \mathcal{V}$ and $x \in L$ there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$ such that $\alpha x \in V$. If $V \in \mathcal{V}$ then there exists $\varepsilon > 0$, $\lambda \in (0,1)$ such that $V = V(\varepsilon,\lambda)$. Let x be arbitrarily fixed in L and $\alpha \in \mathbb{R}$, $\alpha \neq 0$, then $F_{\alpha x,a}(\varepsilon) = F_x(\frac{\varepsilon}{|\alpha|^p})$. Since $\lim\limits_{|\alpha|^p \to 0} F_x(\frac{\varepsilon}{|\alpha|^p}) = 1$ it follows that, there exists $\alpha \in \mathbb{R}$ such that $F_x(\frac{\varepsilon}{|\alpha)|^p}) > 1-\lambda$, hence $\alpha x \in V$.

Let us prove that, for any $V \in \mathcal{V}$ there exists $V_0 \in \mathcal{V}$ such that $V_0 + V_0 \subset V$. If $V = V(\varepsilon,\lambda)$ and $x \in V(\varepsilon,\lambda)$, then there exists $\eta > 0$ such that $F_x(\varepsilon) > 1-\eta > 1-\lambda$. If $V_0 = V(\frac{\varepsilon}{2},\frac{\eta}{2})$ and $x,y \in V_0$, by triangle inequality we have

$$F_{x+y}(\varepsilon) \geq T(F_x(\frac{\varepsilon}{2}), F_y(\frac{\varepsilon}{2})) \geq T_m(1-\frac{\eta}{2}, 1-\frac{\eta}{2}) > 1-\eta > 1-\lambda.$$

The above inequalities show us that $V_0 + V_0 \subset V$.

Now, we show that $V \in \mathcal{V}$ and $\alpha \in \mathbb{R}, \alpha \neq 0$ implies $\alpha V \in \mathcal{V}$. Let us remark that $\alpha V = \alpha V(\varepsilon,\lambda) = \{\alpha x : F_{x,a}(\varepsilon) > 1-\lambda,\, a \in A\}$ and $F_x(\varepsilon) > 1-\lambda$ $\Leftrightarrow F_x(\frac{|\alpha|^p\varepsilon}{|\alpha|^p}) = F_{\alpha x,a}(|\alpha|^p\varepsilon) > 1-\lambda$. This shows that $\alpha V = V(|\alpha|^p\varepsilon,\lambda,A)$, hence $\alpha V \in \mathcal{V}$.

The above statements show us that $\mathcal{V}$ is a base for a system neighborhoods of the origin in the linear space L. It is easy to see that the uniformity generated by $\mathcal{U}$ and the topology generated by $\mathcal{V}$ are compatible.

Proposition 3. Let $\{x_n\}_{n\in\mathbb{L}}$ be a sequence in L and let $(L,\mathcal{F},T)$ be a random p-normed space under a continuous t-norm T, then the following statements are equivalent :

(c_1) $\{x_n\}$ converges to x in the topology generated by the random p-norm $\mathcal{F}$ on L.

(c_2) $F_{x_n-x}(t)$, converges to $H_0(t)$, for every $t > 0$.

(c_3) $d_L(F_{x_n}, H_0) \to 0$, $(n \to \infty)$.

Proposition 4. In the conditions of Proposition 3 the following statements are equivalent:

(f_1) $\{x_n\}$ is a Cauchy sequence in the uniformity generated by the probabilistic metric $\mathcal{F}^m$ on L.

(f_2) $F_{x_n-x_m}(t)$ converges to $H_0(t)$ for all $t > 0$.

(f_3) $d_L(F_{x_n-x_m}, H_0) \to 0$, $(n, m \to \infty)$.

3 Functions with values in random p-normed spaces

Proposition 5. Let f be a function and let $(f_n)_{n\in N}$ be a sequence of functions defined on a non-empty subset A of real line with values in a random p-normed spaces $(L, \mathcal{F}, T)$. Then:

(a) The function f is continuous in $t_0 \in A$ iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $\delta(\varepsilon, \lambda) > 0$ such that for all $t \in A$ with $|t - t_0| < \delta(\varepsilon, \lambda)$ we have

$$F_{f(t)-f(t_0)}(\varepsilon) > 1 - \lambda$$

(b) The function f is uniformly continuous on the set A iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $\delta(\varepsilon, \lambda) > 0$ such that, for all $t', t" \in A$ with property $|t' - t"| < \delta(\varepsilon, \lambda)$ we have

$$F_{f(t')-f(t")}(\varepsilon) > 1 - \lambda$$

(c) The sequence $\{f_n\}_{n\in N}$ converges on the set A to the function f iff for every $\varepsilon > 0, \lambda \in (0,1)$ and $t \in A$ there is an integer $N(\varepsilon, \lambda, t)$ such that, for all $n > N(\varepsilon, \lambda, t)$ we have $F_{f_n(t)-f(t)}(\varepsilon) > 1 - \lambda$.

d) The sequence $\{f_n\}_{n\in N}$ is uniformly convergent, on the set A to the function f iff the condition of the point (c) is satisfied for a $N(\varepsilon, \lambda, t)$ that doesn't depend of t.

Proof. The above statements are valid because the family $\{V_x(\varepsilon, \lambda) = \{y \in L| \quad F_{x-y}(\varepsilon) > 1 - \lambda\}, \quad \varepsilon > 0, \lambda \in (0,1)\}$ is a complete system of neighbourhoods for the point x in the topology generated by the random p-norm $\mathcal{F}$:

The above proposition show us that topological properties of functions with values in the uniform space $(L, \mathcal{U})$ associated to a random p-normed space $(L, \mathcal{F}, T)$ may be expressed by the random p-norm $\mathcal{F}$. So, we obtain a convenient manner to utilize some known results from the uniform space theory to the study of some properties of functions with values in random p-normed spaces. The above statements will also permit us to utilize some results from the study of functions with values in a random p-normed space to functions with values in a p-normed space depending of a random parameter.

Definition 2. A sequence $\{f_n\}_{n\in N}$ of functions defined on a set $A \subset \mathbb{R}$ with values in the random p-normed space $(L, \mathcal{F}, T)$ is called a Cauchy sequence if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is an integer $N(\varepsilon, \lambda) > 0$ such that $F_{f_n(t)-f_m(t)}(\varepsilon) > 1 - \lambda$ for all $t \in A$ and $n.m \geq N(\varepsilon, \lambda)$.

Theorem 2. A sequence $\{f_n\}_{n\in N}$ of functions defined on the set A with values in a complete random p-normed space $(L, \mathcal{F}, T)$ is uniformly convergent on A iff $\{f_n\}_{n\in N}$ is a Chauchy sequence on A.

Proof. Let us suppose that the sequence $\{f_n\}_{n\in N}$ is uniformly convergent on A to the function f. Let $\varepsilon > 0$ and $\lambda \in (0, l)$ be given, then there is a natural integer $N(\varepsilon, \lambda)$ and a real number $\eta > 0$ such that $F_{f_n(t)-f(t)}(\frac{\varepsilon}{2}) > 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2}$ for all $t \in A$ and $n \geq N(\varepsilon, \lambda)$.

Let us consider $n, m \geq N(\varepsilon, \lambda)$, then we have

$$F_{f_n(t)-f_m(t)}(\varepsilon) \geq T(F_{f_n(t)-f(t)}(\frac{\varepsilon}{2}), F_{f(t)-f_m(t)}(\frac{\varepsilon}{2})) \geq$$

$$T_m(1 - \frac{\eta}{2}, 1 - \frac{\eta}{2}) \geq 1 - \eta > 1 - \lambda$$

for all $t \in A$. This implies that the sequence $\{f_n\}_{n\in N}$ is Cauchy on the set A.

Conversely, if $\{f_n\}_{n\in N}$ is a Cauchy sequence on A, then for every fixed $t \in A$ the sequence $\{f_n(t)\}_{n\in N}$ is a Cauchy sequence in $(L, \mathcal{F}, T)$. Since $(L, \mathcal{F}, T)$ is complete it follows that the sequence $\{f_n(t)\}_{n\in N}$ is convergent to an element in L denoted by $f(t)$. We will prove that the sequence $\{f_n\}_{n\in N}$ is uniformly convergent on the set A to the function f .

Since $\{f_n\}_{n\in N}$, is a Cauchy sequence it follows that for every $\varepsilon >$ and $\lambda \in (0,1)$ there is $N(\varepsilon, \lambda)$ such that $F_{f_n(t)-f_m(t)}(\varepsilon) > 1 - \frac{\varepsilon}{2}$ for all $n, m \geq N(\varepsilon, \lambda) \in N$ and $t \in A$. By the triangle inequality and by the fact that $T \geq T_m$ we have

$$F_{f_n(t)-f(t)}(\varepsilon) \geq T_m(F_{f_n(t)-f_m(t)}(\frac{\varepsilon}{2}), F_{f_m(t)-f(t)}(\frac{\varepsilon}{2})),$$

for all $n, m \geq N(\varepsilon, \lambda)$ and $t \in A$.

Letting $m \longrightarrow \infty$ it follows that the sequence $\{f_n\}_{n\in N}$ is uniformly convergent on the set A to the function f. This completes the proof.

In what follows we analyze the approximation of continuous functions defined on a compact interval with values in a random p-normed spaces. First, we give a general approximation theorem based on so-called Borel functions. Using this result we will extend the Weierstrass approximation theorem to such functions.

These results can be applied to approximation of random functions with values in a p-normed space.

As a consequence of these results it is obtained a result with regard to the problem of the approximation of continuous functions with values in linear metric spaces.

For each pair of integers $m, n, 0 \leq m \leq n, n \neq 0$, let $g_{m,n}$ be the Borel functions defined on the unit interval $[0,1]$ by:

$$(10) \qquad g_{m,n} = \begin{cases} 0 & if \ \ 0 \leq t \leq \frac{m-1}{n} \ \text{ or } \ t \geq \frac{m+1}{n} \\ nt - m + 1 & if \ \ \frac{m-1}{n} \leq t \leq \frac{m}{n} \\ -nt + m + 1 & if \ \ \frac{m}{n} \leq t \leq \frac{m+1}{n} \end{cases}$$

For a function f defined on $[0,1]$ with values in a random normed space $(L, \mathcal{F}, T)$ we consider the sequence of functions $\{f_n\}_{n\in N}$ defined by

$$(11) \qquad f_n(t) = \sum_{m=0}^{n} g_{mn}(t) f(\frac{m}{n})$$

Theorem 3. If the function $f : [0,1] \longrightarrow (L, \mathcal{F}, T)$ is continuous on $[0,1]$, then the sequence of functions $\{f_n\}_{n\in N}$ defined by (11) is uniformly convergent on $[0,1]$ to the function f.

Proof. Let $\varepsilon > 0$ and $\lambda \in (0,1)$. Then by continuity of the function f it follows that there exist $N(\varepsilon, \lambda) \in N$ and $\eta > 0$ such that $F_{f(t')-f(t'')}(\frac{\varepsilon}{2} \geq 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2}$ for all $t', t'' \in [0,1]$ with property that $|t' - t''| < \frac{1}{N(\varepsilon,\lambda)}$.

Let $n \geq N(\varepsilon, \lambda)$ and $s \in [0,1]$, we choose the integer k such that $0 \leq k \leq n$ and $\frac{k-1}{n} \leq s \leq \frac{k}{n^n}$. Then we have $s = \frac{k}{n} - \frac{u}{n}$, where $0 \leq u \leq 1$ and $f_n(s) = \sum\limits_{m=0}^{n} g_{m,n}(s) f(\frac{m}{n}) = uf(\frac{k-1}{n}) + (1-u) f(\frac{k}{n})$.

Then we have:

$$F_{f(s)-f_n(s)}(\varepsilon) = F_{(1-u)f(s)+uf(s)-f_n(s)}(\varepsilon) \geq T(F_{(1-u)[f(s)-f(\frac{k}{n}]}(\frac{\varepsilon}{2})),$$

$$F_{u[f(s)-f(\frac{k-1}{n})]}(\frac{\varepsilon}{2}) = T(F_{f(s)-f(\frac{k}{n})}(\frac{\varepsilon}{2|1-u|^p})),$$

$$F_{f(s)-f(\frac{k-1}{n})}(\frac{\varepsilon}{2|u|^p})) \geq T(F_{f(s)-f(\frac{k}{n})}(\frac{\varepsilon}{2}), F_{f(s)-f(\frac{k-1}{n})}(\frac{\varepsilon}{2})) \geq T_m(1-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}) >$$

$$> 1-\eta > 1-\lambda.$$

By these inequalities it follows the conclusion of the theorem.

Definition 3. A polynomial with values in a random p-normed space $(L, \mathcal{F}, T)$ is a function $P : R \longrightarrow (L, \mathcal{F}, T)$ given by $P(t) = \sum_{k=0}^{n} a_k t^k$, where $a_k \in L, a_n \neq 0$ and $n \in N$.

Theorem 4. Let $(L, \mathcal{F}, T)$ be a random p-normed space. If $f : [0,1] \rightarrow (L, \mathcal{F}, T)$ is a continuous function, then there exists a sequence of polynomial with values in $(L, \mathcal{F}, T)$ uniformly convergent on $[0,1]$ to f.

Proof. Let us consider the sequence of function $\{f_n\}_{n\in N}$, $f_n(t) = \sum_{m=0}^{n} g_{m,n}(t) f(\frac{m}{n})$ where $g_{m,n}$ $0 \leq m \leq n, n \neq 0$ are Borel functions. By Theorem 3 the sequence $\{f_n\}_{n\in N}$ is uniformly convergent on $[0,1]$ to the function f, that is, for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exist an positive integer $N_1(\varepsilon, \lambda)$ and $\eta > 0$ such that

$$F_{f(t)-f_n(t)}(\frac{\varepsilon}{2}) > 1 - \frac{\eta}{2} > 1 - \frac{\lambda}{2}$$

for all $t \in [0,1]$ and $n \geqslant N_1(\varepsilon, \lambda)$.

By the classical Weierstrass approximation theorem there exists a polynomial $P_{r(k;m,n,p)}$ such that

$$|g_{m,n}(t) - P_{r(k,m,n)}(t)|^p < \frac{\varepsilon^{n+k+1}}{2n+1}$$

for all $t \in [0,1]$. Let us consider the following sequence of polynomials with values in $(L, \mathcal{F}, T)$; $\{P_{n,k}\}_{n,k\in N}$ defined by $P_{n,k}(t) = \sum_{m=0}^{n} P_{r(k,m,n)}(t) f(\frac{m}{n})$. In the sequel we shall prove that the sequence $\{P_{n,k}\}_{n,k\in N}$ contains a subsequence uniformly convergent on $[0,1]$ to f.

Let $\varepsilon > 0$ and $\lambda \in (0,1)$, we have:

$$F_{f(t)-P_{n,k}(t)}(\varepsilon) \geq T(F_{f(t)-f_n(t)}(\frac{\varepsilon}{2}), F_{f_n(t)-P_{n,k}(t)}(\frac{\varepsilon}{2})) \geq$$

$$\geq T(F_{f(t)-f_n(t)}(\frac{\varepsilon}{2}), \underbrace{T(T(\cdots T}_{n\ \ times}(F_{0,n}(\frac{\varepsilon}{2(n+1)|g_{0,n}(t)-P_{r(k,0,n)}|^p}),$$

$$F_{f(\frac{1}{n})}(\frac{\varepsilon}{2(n+1)|g_{1,n}(t)-P_{r(k,i,n)}|^p}),\cdots$$

$$\cdots, F_{f(\frac{n}{n})}(\frac{\varepsilon}{2(n+1)|g_{n,n}(t)-P_{r(k,n,n)}|^p}))).$$

In view of the inequalities (3) and (4) we have

$$F_{f(t)-P_{n,k}(t)}(\varepsilon) \geq T(1-\frac{\eta}{2}, T(T(...(T(F_{f(\frac{0}{n})}(\frac{1}{\varepsilon^{n+k}}),\cdots$$

$$\cdots,)F_{f(n,n)}(\frac{1}{\varepsilon^{n+k}})))))$$

Since the function $f:[0,1]\longrightarrow(L,\mathcal{F},T)$ is continuous it follows that f is bounded. Taking into account that f is bounded and that $\frac{1}{n+k}\longrightarrow\infty$ when $k\longrightarrow\infty$ it follows that $\sup\limits_{k>0}\inf\limits_{t\in[0,1]}F_{f(t)}(\frac{1}{\varepsilon^{n+k}})=1$. This means that for every $n\geq 1$ there exists $N(n)\in N$ such that, if $k\geq N(n)$ we have $\inf\limits_{t\in[0,1]}F_{f(t)}(\frac{1}{\varepsilon^{n+k}})>1-\frac{1}{(n+1)^2}$. Hence for all $k\geq N(n)$ we have

$$F_{f(t)-P_{n,k}(t)}(\varepsilon)\geq T_m(1-\frac{\eta}{2}, T_m^n(1-\frac{1}{(n+1)^2}))\geq$$

$$T_m(1-\frac{\eta}{2}, Max\{\sum_{k=1}^{n+1}(1-\frac{1}{(n+1)^2}-n,0)\})=$$

$$=T_m(1-\frac{\eta}{2}, Max\{1-\frac{1}{(n+1)^2},0\})$$

Now, let $N(\eta)\in N$ such that $\frac{1}{N(\eta)}<\frac{\eta}{2}$. From the above it follows that for every $n\geq Max\{N(\varepsilon,\lambda),N(\eta)\}$ we have

$$F_{f(t)-P_{n,N(n)(t)}}(\varepsilon)\geq T_m(1-\frac{\eta}{2},1-\frac{\eta}{2})\geq 1-\eta>1-\lambda$$

for all $t\in[0,1]$. By the arbitrariness of $\varepsilon>0$ and $\lambda\in(0,l)$ is follows that the sequence of polynomials $\{P_{n,N(n)}\}_{n\in N}$ converges uniformly on $[0,1]$ to the function f. This completes the proof of theorem.

Remark 2. If in Theorem 4 we consider $p=1$ and the t-norm T, under that

$(L, \mathcal{F}, T)$ is a random 1-normed (normed) space, is a t-norm of Hadzić type [4],[5] it is known that the random normed space $(L, \mathcal{F}, T)$ endowed with the (ε, λ)-topology defined by the random norm $\mathcal{F}$ is a locally convex topological linear space. In these conditions the statement of Theorem 4. can be deduced by using the approximation theorem of Weierstrass for continuous functions with values in $\mathbb{R}^n$. Indeed, because f is continuous it results that $f([0,1])$ is a compact set in $(L, \mathcal{F}, T)$. In this case $(L, \mathcal{F}, T)$ endowed with the (ε, λ)-topology generated by $\mathcal{F}$ is in the same time a locally convex topological linear space. So, it results that $f([0,1])$ is homeomorphic with a compact subset of $\mathbb{R}^n$. Theorem 4. offers a generalization in the following sense. It is known [3], [4] that for a t-norm which is not of Hadzič type there exist a random normed space $(L, \mathcal{F}, T)$ which is not locally convex topological linear space in the (ε, λ)-topology generated by the random norm $\mathcal{F}$.

It is also known that any linear metric space (L, d) can be considered a random normed space $(L, \mathcal{F}, T)$, under the t-norm $T = T_m$. Using thise statements we can give the following consequence of Theorem 4.

Corollary 1 . If f is a continuous function on $[0,1]$ with values in a linear metric space (L, d) then, there exists a sequence of polynomials with values in the linear metric space (L, d) which converges uniformly on $[0,1]$ to the function f.

In what follows we shall assume that $(\Omega, \mathcal{K}, P)$ is a complete probability measure space, i.e. the set Ω is a nonempty abstract set, $\mathcal{K}$ is a σ-algebra of subsets of Ω and P is a complete probability measure on $\mathcal{K}$.

First we present some basic definitions of the probability theory in Banach spaces [1].

Let $(X, \mathcal{B})$ be a measurable space, where $(X, ||\cdot||)$ is a separable Banach space and $\mathcal{B}$ is the σ-algebra of the Borel subsets of $(X, ||\cdot||)$.

A mapping $x : \Omega \to X$ is said to be random variable (X-valued r.v.) with values in X if $x^{-1}(B)$ for all $B \in \mathcal{B}$.

Two X-valued r.v. $x(\omega)$ and $y(\omega)$ are said to be equivalent if $x(\omega)$ and $y(\omega)$ are equal with the probability one, i.e.

$$P(\{\omega \in \Omega | x(\omega) = y(\omega)\}) = 1.$$

Let us note by $\mathcal{X}$ the space of all classes of equivalent random variables with values in a separable Banach space X. It is known that $(\mathcal{X}, \mathcal{F}, T_m)$ is a complete random normed space, where the random norm $\mathcal{F}$ is defined by

$$F_x(t) = P(\{\omega \in \Omega : \quad ||x(\omega)|| < t\})$$

Furthermore,the (ε, λ)-topology on $\mathcal{X}$ induced by the random norm $\mathcal{F}$ is equivalent to the topology of the convergence in probability on $\mathcal{X}$.

Let A be a subset of the set of real numbers and let f be a mapping of $A \times X$ into X.

The mapping f is said to be a random function defined on A with values in the separable Banach space X(briefly X-valued r.f.), if for any $t \in A$ the mapping $f(t, \cdot) : \Omega \to X$ is a X-valued r.v..

For every $\omega \in \Omega$ the mapping $f(\cdot, \omega) : A \to X$ is called a realization of X-valued r.f. $f(t, \omega)$.

Two X-valued random functions f and g are said to be equivalent if $f(t, \omega)$ and $g(t, \omega)$ are equal almost surely for any $t \in A$, i.e

$$P(\{\omega \in \Omega | \ f(t, \omega) = g(t, \omega)\}) = 1$$

for any $t \in A$.

For important results related, to the probability theory in Banach spaces we refer to [1].

Now, let f be a X-valued random function defined on $A \subset \mathbb{R}$, then one can define the mapping $\widetilde{f}$ on A with values in the random normed space $(\mathcal{X}, \mathcal{F}, T_m)$ by $A \ni t \to \widetilde{f}(t)$, where $[\widetilde{f}(t)](\omega) = f(t, \omega)$. Conversely, for each function $\widetilde{f} : A \to (\mathcal{X}, \mathcal{F}, T_m)$ one can define the X-valued random function on A by $f(t, \omega) = [\widetilde{f}(t)](\omega)$ for any $t \in A$ and $\omega \in \Omega$. Furthermore the correspondence $f \to \widetilde{f}$ is one to one and onto. By this correspondence results obtained for functions with values in random normed spaces can be applied to the study of random functions with values in separable Banach spaces.

References

[1] A.T.Bharucha-Reid, *Random Integral Requations*, Academic Press, 1972.

[2] A.Bano, A.R. Khan, A. Ladif, *Coincidence points and best appromaximations in p-normed spaces*,Radovi Matematički, Vol. 12 (2003), 27-36.

[3] G. Constantin and Ioana Istrăţescu, *Elements of Probabilistic Analysis*, Kluwer Academic Publishers, 1989.

[4] O. Had žić, *Fixed point theory in topological vector spaces*, Novi Sad, 1984.

[5] O. Had žić, Endre Pap, *Fixed point theory in probabilistic metric spaces*, Kluver Academic Publishers,Dordrecht, 2001.

[6] T.L. Hicks, *Random normed linear structures*, Math. Japonica No. 3, 1996, pp. 483-486.

[7] G.Köthe, *Topological vector spaces I* ,Springer-Verlag,Berlin,1969.

[8] M.S. Matveichuk, *Random norm and characteristic probabilistics on orthoprojections associated with factors*, Probabiistic methods and Cybernetics, Kazan University No. 9, 1971, pp. 73-77.

[9] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci., USA, no.28, 1942, pp.535-537.

[10] D.Kh. Mushtari, *On the linearity of isometric mappings of random spaces*, Kazan Gos. Univ. Ucen.Zap., no.128, 1968, pp. 86-90.

[11] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North Holland, New York, Amsterdam, Oxford, 1983.

[12] A.N. Šerstnev, *Random normed spaces : Problems of completeness* Kazan Gos. Univ. Ucen. Zap., no. 122, 1962, pp. 3-20.

Assoc. prof. dr. Goleţ Ioan
"Politehnica" University of Timişoara
Dept. of Mathematics
R-1900 Timişoara
E-mail : igolet@etv.utt.ro