A Certain Subclass of P-valently Analytic Functions of Bazilevič Type

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Abstract. Using extended Ruscheweyh derivatives we define a new subclass of *p*-valently analytic functions which are of *Bazilevič-type*. We denote the new subclass as $M(n,p,\alpha,\beta)$. We find some sufficient conditions and angular properties for functions belonging to the subclass $M(n,p,\alpha,\beta)$.

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1. INTRODUCTION

Let S denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. We denote by $S(p)$ the subclass of S consisting of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in N) \tag{1.2}$$

which are *p*-valently analytic in *U*. The function $f \in S$ is said to be *Bazilevič- type* [1] if it satisfies

$$\mathrm{Re}\left\{ f'(z)\left(\frac{f(z)}{z}\right)^{\alpha+i\gamma-1}\left(\frac{g(z)}{z}\right)^{-\alpha}\right\} > 0$$

where $z \in U$, $\alpha > 0$ and γ are real numbers, $g(z)$ is a starlike function with $\mathrm{Re}\left\{ zg'(z)/g(z) \right\} > 0$.

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The case where $\gamma = 0$ was also widely studied. Thomas [7] defined the class $B(\alpha)$ where $f \in B(\alpha)$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)^{1-\alpha}\,g(z)^{\alpha}}\right\} > 0$$

with $z \in U$ and $\alpha > 0$.

This subclass was later extended by Eenigenburg and Silvia [2] to a subclass of S which consists of functions f satisfying the following condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)^{1-\alpha}\,g(z)^{\alpha}}\right\} > \beta$$

where $z \in U$, $\alpha > 0$ and $0 \le \beta < 1$.

In this paper we introduce a new subclass $M(n, p, \alpha, \beta)$ of S which resembles the above mentioned subclasses. A function $f \in S(p)$ is said to be in the subclass $M(n, p, \alpha, \beta)$ if it satisfies

$$\operatorname{Re}\left(\frac{pD^{n+p} f(z)}{D^{n+p-1} f(z)}\left(\frac{D^{n+p-1} f(z)}{z^{p}}\right)^{\alpha}\right) > \beta \tag{1.3}$$

where $z \in U$, $\alpha > 0$ and $0 \le \beta < p$. $D^{n+p} f(z)$ and $D^{n+p-1} f(z)$ are extensions of the familiar operator $D^{n} f(z)$ of Ruscheweyh derivatives [5] , $n \in N_0 = N \cup \{0\}$. These operators were considered by Sekine, Owa and Obradovic [6] where

$$D^{n+p} f(z) = z^{p} + \sum_{k=1}^{\infty} C_{p,k}(n) a_{p+k} z^{p+k}$$

with $$C_{p,k}(n) = \frac{(n+p+k)\ldots(1+k)}{(n+p)!}$$

and

$$D^{n+p-1} f(z) = z^{p} + \sum_{k=1}^{\infty} C_{p-1,k}(n) a_{p+k} z^{p+k}$$

with $\quad C_{p-1,k}(n) = \dfrac{(n+p-1+k)...(1+k)}{(n+p-1)!}$.

Notice that $M(0,1,1,\beta) = C(\beta)$ is a class of close-to-convex functions of order β where a function $f \in S$ is said to be in the class $C(\beta)$ if it satisfies

$$\mathrm{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \beta \ .$$

Also, notice that the condition (1.3) implies that

$$\left|\frac{pD^{n+p}f(z)}{D^{n+p-1}f(z)}\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha} - p\right| < p-\beta \, . \tag{1.4}$$

The objective of this paper is to find sufficient conditions and angular properties for functions belonging to the subclass $M(n,p,\alpha,\beta)$.

In order to derive our main results, we have to recall the following lemmas.

Lemma 1.1 [3] **:** *Let $w(z)$ be analytic in U and such that $w(0)=0$. Then if $|w(z)|$ attains its maximum value on circle $|z| = r < 1$ at a point $z_0 \in U$ we have $z_0 w'(z) = kw(z_0)$ where $k \geq 1$ is a real number.*

Lemma 1.2 [4] **:** *Let $q(z)$ be analytic in U, with $q(0)=1$ and $q(z) \neq 0$ for all $z \in U$. If there exists a point $z_0 \in U$ such that $|\arg(q(z)| < \frac{\pi}{2}\delta$ for $|z| < |z_0|$ and $|\arg(q(z_0)| = \frac{\pi}{2}\delta$ for $\delta > 0$, then we have $\frac{z_0 q'(z_0)}{q(z_0)} = i\kappa\delta$, where $\kappa \geq \frac{1}{2}\left(L + \frac{1}{L}\right) \geq 1$, when $\arg(q(z_0)) = \frac{\pi}{2}\delta$ and $\kappa \leq -\frac{1}{2}\left(L+\frac{1}{L}\right) \leq -1$, when $\arg(q(z_0)) = -\frac{\pi}{2}\delta$, $q(z_0)^{1/\delta} = \pm Li, \qquad (L>0)$.*

The following identity will also be used :

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z) \, . \tag{1.5}$$

2. SUFFICIENT CONDITION FOR CLOSE-TO-CONVEXITY

Making use of Lemma 1.1, we first prove

Theorem 2.1 : *If* $f \in S(p)$ *satisfies*

$$\left| \frac{z(D^{n+p} f(z))'}{D^{n+p} f(z)} - (1-\alpha)\frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} \right| < \frac{p-\beta}{2p-\beta} \tag{2.1}$$

for $0 \le \beta < p$ *and* $\alpha > 0$ *then* $f \in M(n, p, \alpha, \beta)$.

Proof : Define the function $w(z)$ as

$$\frac{pD^{n+p} f(z)}{D^{n+p-1} f(z)}\left(\frac{D^{n+p-1} f(z)}{z^p}\right)^{\alpha} = p + (p-\beta)w(z) \tag{2.2}$$

Differentiating (2.2) logarithmically we obtain that

$$\frac{(D^{n+p} f(z))'}{D^{n+p} f(z)} - \frac{(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} + \alpha\frac{(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} - \frac{\alpha p}{z} = \frac{(p-\beta)w'(z)}{p+(p-\beta)w(z)}$$

which gives

$$\frac{z(D^{n+p} f(z))'}{D^{n+p} f(z)} - (1-\alpha)\frac{z(D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} - \alpha p = \frac{(p-\beta)zw'(z)}{p+(p-\beta)w(z)}.$$

Suppose there exists $z_o \in U$ such that

$$\max_{z<z_0} |w(z)| = |w(z_0)| = 1.$$

Then from Lemma 1.1 we have $z_0 w'(z_0) = kw(z_0)$. Therefore letting $w(z_0) = e^{i\theta}$, with $k \ge 1$ we obtain

$$\left| \frac{z_0(D^{n+p} f(z_0))'}{D^{n+p} f(z_0)} - (1-\alpha)\frac{z_0(D^{n+p-1} f(z_0))'}{D^{n+p-1} f(z_0)} \right| = \left| \frac{(p-\beta)z_0 w'(z_0)}{p+(p-\beta)w(z_0)} + \alpha p \right|$$

$$= \left| \frac{(p-\beta)kw(z_0)}{p+(p-\beta)w(z_0)} + \alpha p \right|$$

$$\geq \frac{p-\beta}{2p-\beta}$$

which contradicts our assumption (2.1). Therefore we have $|w(z)| < 1$ in U. Finally, we have

$$\left| \frac{pD^{n+p} f(z)}{D^{n+p-1} f(z)} \left(\frac{D^{n+p-1} f(z)}{z^p} \right)^{\alpha} - p \right| = |(p-\beta)w(z)| < p-\beta$$

that is , $f \in M(n, p, \alpha, \beta)$.

Recall that a function $f \in S$ is in the class C of close-to-convex functions if $\operatorname{Re}(f'(z)) > 0$.

Letting $n = 0,\ p = 1,\ \alpha = 1$ and $\beta = 0$, from (1.3) and (2.1) we obtain

Corollary 2.2 : *If* $f \in S$ *satisfies* $\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{1}{2}$ *then* $f \in C$.

3. ANGULAR PROPERTIES

Theorem 3.1: *If* $f \in S(p)$ *satisfies the condition that*

$$-\frac{\pi}{2}\delta - \tan^{-1} \frac{\delta}{\alpha(n+p)} < \arg\left\{ \frac{pD^{n+p} f(z)}{D^{n+p-1} f(z)} \left(\frac{D^{n+p-1} f(z)}{z^p} \right)^{\alpha} - \beta \right\} < \frac{\pi}{2}\delta + \tan^{-1} \frac{\delta}{\alpha(n+p)}$$

$(z \in U, 0 \leq \beta < p, 0 < \delta \leq 1, \alpha > 0)$ (3.1)

then $$-\frac{\pi}{2}\delta < \arg\left\{ p\left(\frac{D^{n+p-1} f(z)}{z^p} \right)^{\alpha} - \beta \right\} < \frac{\pi}{2}\delta. \qquad (3.2)$$

Proof : Define $q(z)$ by

$$q(z)=\frac{1}{p-\beta}\left(p\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}-\beta\right) \tag{3.3}$$

Differentiating (3.3) we obtain

$$\frac{q'(z)}{q(z)}=\frac{\alpha p}{q(z)(p-\beta)}\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}\left(\frac{(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}-\frac{p}{z}\right) \tag{3.4}$$

which gives

$$\frac{zq'(z)}{\alpha}=\frac{p}{p-\beta}\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}\left(\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}-p\right)$$

Then by applying the identity (1.5) to (3.4) we obtain

$$\frac{zq'(z)}{\alpha}=\frac{p}{p-\beta}\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}\left(\frac{(n+p)D^{n+p}f(z)-nD^{n+p-1}f(z)}{D^{n+p-1}f(z)}-p\right)$$

which gives

$$\frac{zq'(z)}{\alpha}+(n+p)q(z)=\frac{p(n+p)}{p-\beta}\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right)\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}-\frac{\beta}{p-\beta}(n+p)$$

Therefore

$$\frac{zq'(z)}{\alpha(n+p)}+q(z)=\frac{1}{p-\beta}\left(p\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right)\left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{\alpha}-\beta\right)$$

Suppose there exist a point $z_0\in U$ such that $|\arg(q(z)|<\frac{\pi}{2}\delta$ for $|z|<|z_0|$ and $|\arg(q(z_0)|=\frac{\pi}{2}\delta$ where $\delta>0$. Then from Lemma 1.2, $\frac{z_0q'(z_0)}{q(z_0)}=i\kappa\delta,$ where $\kappa\geq\frac{1}{2}\left(L+\frac{1}{L}\right)\geq 1,$ *when* $\arg(q(z_0))=\frac{\pi}{2}\delta$ *and*

$$\kappa \le -\frac{1}{2}\left(L+\frac{1}{L}\right) \le -1, when \ \arg(q(z_0)) = -\frac{\pi}{2}\delta, \qquad q(z_0)^{1/\delta} = \pm Li, \qquad (L>0).$$

Suppose $\arg(q(z_0)) = \frac{\pi}{2}\delta$, then, $\kappa \ge \frac{1}{2}\left(L+\frac{1}{L}\right) \ge 1$. Therefore

$$\arg\left(p\left(\frac{D^{n+p}f(z_0)}{D^{n+p-1}f(z_0)}\right)\left(\frac{D^{n+p-1}f(z_0)}{z_0{}^p}\right)^{\alpha} - \beta\right)$$

$$= \arg\left(q(z_0)\left(1+\frac{z_0 q'(z_0)}{\alpha(n+p)q(z_0)}\right)\right)$$

$$= \arg q(z_0) + \arg\left(1+\frac{zq'(z_0)}{\alpha(n+p)q(z_0)}\right)$$

$$= \frac{\pi}{2}\delta + \arg\left(1+\frac{i\kappa\delta}{\alpha(n+p)}\right)$$

$$= \frac{\pi}{2}\delta + \tan^{-1}\frac{k\delta}{\alpha(n+p)}$$

$$\ge \frac{\pi}{2}\delta + \tan^{-1}\frac{\delta}{\alpha(n+p)}.$$

which contradicts the assumptions of the theorem.

Now suppose that $\arg(q(z_0)) = -\frac{\pi}{2}\delta$ then $\kappa \le -\frac{1}{2}\left(L+\frac{1}{L}\right) \le -1$. Therefore

$$\arg\left(p\left(\frac{D^{n+p}f(z_0)}{D^{n+p-1}f(z_0)}\right)\left(\frac{D^{n+p-1}f(z_0)}{z_0{}^p}\right)^{\alpha} - \beta\right)$$

$$= \arg q(z_0) + \arg\left(1+\frac{z_0 q'(z_0)}{\alpha(n+p)q(z_0)}\right)$$

$$= -\frac{\pi}{2}\delta + \arg\left(1+\frac{i\kappa\delta}{\alpha(n+p)}\right)$$

$$\leq -\frac{\pi}{2}\delta - \tan^{-1}\frac{\delta}{\alpha(n+p)}$$

which also contradicts the assumptions of the theorem. This completes the proof.

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