Inverse System in the Category of Šostak Fuzzy Topological Spaces

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Abstract: In this study, we first define the concept of inverse limit in category of $\overset{\vee}{S}FTS$, the category of Šostak fuzzy topological spaces and gp-maps between them. After giving the fundamental definitions we investigate series of their properties.

Keywords: Inverse system, Šostak fuzzy topological space, Mapping of inverse system

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1. INTRODUCTION

Since Chang [2] introduced fuzzy theory into topology, many authors investigated various aspects of fuzzy topology. Höhle [8] was one of the first authors who had created the notion of a topology being viewed as an $L-$subset of a powerset (in his case, 2^X). Later Kubiak [9] and Šostak [15] independently extended Höhle's notion to $L-$subsets of L^X.

In [8] , it is shown that $L-FTOP$ is a topological category over SET for each L. In [3], Chattopadhyay et al gave a definition of fuzzy topology by introducing a concept of gradation of openness of fuzzy subsets. They constructed connections between $r-$level Chang fuzzy topology and the new fuzzy topological space. In [12], Mondal et al defined the category of intuitionistic fuzzy topological spaces (briefly $IFTS$) and also established connections between a descending family of inclusive bitopologies of fuzzy subsets on X and intuitionistic fuzzy topological spaces. By using the connection, they defined product operation in the category of intuitionistic fuzzy topological spaces. Since firstly this definition was given as independently by Kubiak [8] and Šostak [15], these spaces are called as Šostak fuzzy topological spaces.

In [6], Fang gave internal characterizations of L-fuzzy topological sum spaces and examined some additional properties of L-fuzzy topological spaces. Yue [21] introduced base (subbase) of gradation of openness and by using these definitions, he presented product and quotient spaces. In [14], Shi introduced a new definition of fuzzy

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compactness in L- topological spaces when L is a complete De Morgan algebra (in case of Chang). He proved that the intersection of a fuzzy compact L-set with a closed L-set is fuzzy compact and the continuous image of a fuzzy compact L-set is fuzzy compact. In [10,11], Inverse (direct) limits are described as inverse (direct) systems in the category of fuzzy topological spaces and series of their properties are investigated. Furthermore, the mappings between two arbitrary inverse systems are defined and some of their properties are discussed for the case of category of fuzzy topological spaces. The purpose of this paper is to construct inverse system in Šostak fuzzy topological spaces. Firstly, we prove that the inverse limit is compact under some conditions. Some definitions of compactness are given in [1,12,13]. In this study, we give a new definition of compactness by using the idea in Shi [14].

2. PRELIMINARIES

Let X be a non-empty set and I be the closed unit interval $[0,1]$, $I_0=(0,1]$ and $I_1=[0,1)$. Let I^X denote a collection of all fuzzy sets in X . By $\underline{0}$ and $\underline{1}$ we denote characteristic functions $\chi_\varnothing$ and χ_X, respectively. All other notations are standart notations of fuzzy set theory. We use all notations from [12].

Definition 2.1. (Mondal, Samanta [12]) Let X be a non-empty set. An IGO of fuzzy subsets of X is an ordered pair (τ,τ^*) of functions from I^X to I such that

$(IGO1)$ $\tau(\lambda)+\tau^*(\lambda)\le 1, \forall\lambda\in I^X$,

$(IGO2)$ $\tau(\underline{0})=\tau(\underline{1})=1, \tau^*(\underline{0})=\tau^*(\underline{1})=0,$

$(IGO3)$ $\tau(\lambda_1\cap\lambda_2)\ge\tau(\lambda_1)\wedge\tau(\lambda_2)$ and $\tau^*(\lambda_1\cap\lambda_2)\le\tau^*(\lambda_1)\vee\tau^*(\lambda_2), \lambda_i\in I^X, i=1,2,$

$(IGO4)$ $\tau\left(\bigcup_{i\in\Delta}\lambda_i\right)\ge\bigwedge_{i\in\Delta}\tau(\lambda_i)$ and $\tau^*\left(\bigcup_{i\in\Delta}\lambda_i\right)\le\bigvee_{i\in\Delta}\tau^*(\lambda_i), \lambda_i\in I^X, i\in\Delta.$

The triplet (X,τ,τ^*) is called an $IFTS$. τ and τ^* may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Definition 2.2. (Mondal, Samanta [12]) Let (X,τ,τ^*) and (Y,σ,σ^*) be two $IFTSs$ and $f:X\to Y$ be a mapping. Then f is called a gp-map if for each $\mu\in I^Y$,

$$\sigma(\mu)\le\tau\left(f^{-1}(\mu)\right) \text{ and } \sigma^*(\mu)\ge\tau^*\left(f^{-1}(\mu)\right).$$

Theorem 2.3. (Mondal, Samanta [12]) Let $\left\{\left(T_r, T_r^*\right); r \in I_0\right\}$ be a descending family of inclusive bitopologies of fuzzy subsets on X. Define $\tau, \tau^* : I^X \to I$ by

$$\tau(\lambda) = \vee\{r : \lambda \in T_r\}$$

and

$$\tau^*(\lambda) = \wedge\left\{1 - r : \lambda \in T_r^*\right\}.$$

Then,

a) $\left(\tau, \tau^*\right)$ is an IGO on X,

b) $\tau_r = T_r$ iff $\bigcap\limits_{s<r} T_s = T_r \quad, \forall r \in I_0$,

c) $\tau_r^* = T_r^*$ iff $\bigcap\limits_{s<r} T_s^* = T_r^* \quad, \forall r \in I_0$.

Definition 2.4. (Shi [16]) Let (X, T) be an $L-$space. $G \in L^X$ is called fuzzy compact if for every family $U \subseteq \mathrm{T}$, it follows that

$$\bigwedge_{x \in X}\left(G'(x) \vee \bigvee_{A \in U} A(x)\right) \leq \bigvee_{V \in 2^{(U)}} \bigwedge_{x \in X}\left(G'(x) \vee \bigvee_{A \in V} A(x)\right).$$

Let (X, T) be an $L-$space (in case of Chang).

Definition 2.5. (Shi [14]) Let (X, T) be an $L-$space, $a \in L \backslash \{1\}$ and $G \in L^X$. A subfamily U in L^X is said to be

(1) an a- shading of G if for any $x \in X$, it follows $G'(x) \vee \bigvee_{A \in U} A(x) \not\leq a$.

(2) a strong a- shading of G if $\bigwedge_{x \in X}\left(G'(x) \vee \bigvee_{A \in U} A(x)\right) \not\leq a$.

It is obvious that a strong a- shading of G is an a- shading of G.

Theorem 2.6. (Shi [14]) Let (X, T) be an $L-$space and $G \in L^X$. Then the following conditions are equivalent to each other:

(1) G is fuzzy compact.

(2) For any $a \in L \backslash \{1\}$, each open strong a- shading U of G has a finite subfamily V which is a strong a- shading of G.

(3) For any $a \in L \backslash \{0\}$, each closed strong a- remote family of G has a finite subfamily F which is a strong a- remote family of G.

Theorem 2.7. (Shi [14]) If G is fuzzy compact and H is closed, then $G \wedge H$ is fuzzy compact.

3. INVERSE LIMITS IN $\overset{\vee}{S}FTS$

We called the triplet $\left(X, \tau, \tau^*\right)$, where τ and τ^* are grad functions on I^X, as Šostak fuzzy topological space. Šostak fuzzy topological spaces and gp-maps are consisted of a category. We denote this category as $\overset{\vee}{S}FTS$. In this section we give some of the necessary definitions and operations in $\overset{\vee}{S}FTSs$ that we will be using in the sequel. Later we investigate series of their properties.

Definition 3.1. Let $\left(X, \tau, \tau^*\right)$ be $\overset{\vee}{S}FTS$.

(1) $\mathrm{B}, \mathrm{B}^* : X \to I$ are called a base of τ and τ^* if B and B^* satisfy the following conditions:

$$\forall A \in X, \tau(A) = \bigvee_{\underset{\lambda \in \wedge}{\vee} B_\lambda = A} \bigwedge_{\lambda \in \wedge} \mathrm{B}(B_\lambda),\ \tau^*(A) = \bigwedge_{\underset{\lambda \in \wedge}{\vee} B_\lambda = A} \bigvee_{\lambda \in \wedge} \mathrm{B}^*(B_\lambda),$$

where the expression $\bigvee_{\underset{\lambda \in \wedge}{\vee} B_\lambda = A} \bigwedge_{\lambda \in \wedge} \mathrm{B}(B_\lambda)$ and $\bigwedge_{\underset{\lambda \in \wedge}{\vee} B_\lambda = A} \bigvee_{\lambda \in \wedge} \mathrm{B}^*(B_\lambda)$ will be denoted by $\mathrm{B}^{(\text{Ш})}(A)$ and $\mathrm{B}^{*(\text{Ш})}(A)$, respectively.

(2) $\phi, \phi^* : X \to I$ are called a subbase of τ and τ^* if $\phi^{(\Pi)}, \phi^{*(\Pi)} : X \to I$ are a base for τ and τ^*, where

$$\phi^{(\Pi)}(A) = \bigvee_{(\Pi)_{\lambda \in J} B_\lambda = A} \bigwedge_{\lambda \in J} \phi(B_\lambda),\ \phi^{*(\Pi)}(A) = \bigwedge_{(\Pi)_{\lambda \in J} B_\lambda = A} \bigvee_{\lambda \in J} \phi^*(B_\lambda)$$

for all $A \in X$ with $(\prod)$ standing for "finite intersection".

Definition 3.2. Let $\left\{\left(X_t, \tau_t, \tau_t^*\right)\right\}_{t \in T}$ be a family of $\overset{\vee}{S}FTSs$ and $P_t : \prod_{t \in T} X_t \to X_t$ be the projection map for each $t \in T$. Then the grad functions on $\prod_{t \in T} X_t$ whose subbases are defined by

$$\forall A \in \prod_{t \in T} X_t,\ \phi(A) = \bigvee_{t \in T} \bigvee_{P_t^{-1}(B) = A} \tau_t(B),\ \phi^*(A) = \bigwedge_{t \in T} \bigwedge_{P_t^{-1}(B) = A} \tau_t^*(B)$$

are called the product of $\left\{\tau_t, \tau_t^*\right\}_{t\in T}$, denoted by $\prod_{t\in T}\left(\tau_t, \tau_t^*\right)$. $\left(\prod_{t\in T} X_t, \prod_{t\in T}\left(\tau_t, \tau_t^*\right)\right)$ is called the product space of $\left\{\left(X_t, \tau_t, \tau_t^*\right)\right\}_{t\in T}$ (briefly $\prod_{t\in T}\left(X_t, \tau_t, \tau_t^*\right)$).

Let $\prod_{t\in T}\tau_t = \tau, \prod_{t\in T}\tau_t^* = \tau^*$. Now we show that $P_t : \prod_{t\in T} X_t \to X_t$ is a gp-map for all $t \in T$. Since

$$\tau\left(P_t^{-1}(B)\right) = \bigvee_{t\in T} \bigvee_{P_t^{-1}(B)=P_t^{-1}(B)} \tau_t(B) \geq \tau_t(B)$$

and

$$\tau^*\left(P_t^{-1}(B)\right) = \bigwedge_{t\in T} \bigwedge_{P_t^{-1}(B)=P_t^{-1}(B)} \tau_t^*(B) \leq \tau_t^*(B),$$

$P_t : \prod_{t\in T}\left(X_t, \tau_t, \tau_t^*\right) \to \left(X_t, \tau_t, \tau_t^*\right)$ is a gp-map, for all $t \in T$.

Lemma 3.3. If $\left\{f_t : \left(X_t, \tau_t, \tau_t^*\right) \to \left(Y_t, \sigma_t, \sigma_t^*\right)\right\}_{t\in T}$ is a family of gp-maps, then

$$f = \prod f_t : \prod_{t\in T}\left(X_t, \tau_t, \tau_t^*\right) \to \prod_{t\in T}\left(Y_t, \sigma_t, \sigma_t^*\right),\ t \in T$$

is also gp-map.

Proof. We prove lemma for subbase. For $\forall A \in \prod_{t\in T} Y_t$,

$$\begin{aligned}
\phi\left(f^{-1}(A)\right) &= \bigvee_{t\in T} \bigvee_{P_t^{-1}(B)=f^{-1}(A)} \tau_t(B) = \bigvee_{t\in T} \bigvee_{P_t^{-1}(B)=\prod f_t^{-1}(A|Y_t)} \tau_t(B) \\
&= \bigvee_{t\in T} \bigvee_{B=f_t^{-1}(A|Y_t)} \tau_t(B) = \bigvee_{t\in T} \bigvee_{f_t^{-1}(A|Y_t)} \tau_t\left(f_t^{-1}(A|Y_t)\right) \\
&\geq \bigvee_{t\in T} \bigvee_{f_t^{-1}(A|Y_t)} \sigma_t(A|Y_t) \\
&= \bigvee_{t\in T} \bigvee_{P_t^{-1}(A|Y_t)=A} \sigma_t(A|Y_t) = \phi'(A)
\end{aligned}$$

Similarly, $\phi^*\left(f^{-1}(A)\right) \leq \phi^{*'}(A)$ is obtained.

Let $\check{S}FTS$ be the category of Šostak fuzzy topological spaces and J be direct poset (consider as a category).

Definition 3.4. Any functor $D: J^{op} \to \overset{\vee}{S}FTS$ is called an inverse system in $\overset{\vee}{S}FTS$, the limit of D is called an inverse limit of D.

Theorem 3.5. Every inverse system in the category of $\overset{\vee}{S}FTS$ has a unique limit.
Proof. Let

$$\underline{X} = \left(\left\{ \left(X_i, \tau_i, \tau_i^* \right) \right\}_{i \in J}, \left\{ p_i^{i'} : \left(X_{i'}, \tau_{i'}, \tau_{i'}^* \right) \to \left(X_i, \tau_i, \tau_i^* \right) \right\}_{i \prec i'} \right) \quad (1)$$

be arbitrary inverse system in Šostak fuzzy topological spaces. For each $r \in I_0$,

$$\underline{X}^{(r)} = \left(\left\{ \left(X_i, \tau_i^r, \tau_i^{*r} \right) \right\}_{i \in J}, \left\{ p_i^{i'} : \left(X_{i'}, \tau_{i'}^r, \tau_{i'}^{*r} \right) \to \left(X_i, \tau_i^r, \tau_i^{*r} \right) \right\}_{i \prec i'} \right) \quad (2)$$

is inverse system of fuzzy bitopological spaces [10, Definition 1.6, Theorem 2.3], [12, Theorem 2.13, Remark 2.14]. Thus we obtain two inverse systems of fuzzy topological spaces as follows:

$$\left(\left\{ \left(X_i, \tau_i^r \right) \right\}_{i \in J}, \left\{ p_i^{i'} : \left(X_{i'}, \tau_{i'}^r \right) \to \left(X_i, \tau_i^r \right) \right\}_{i \prec i'} \right) \quad (3)$$

$$\left(\left\{ \left(X_i, \tau_i^{*r} \right) \right\}_{i \in J}, \left\{ p_i^{i'} : \left(X_{i'}, \tau_{i'}^{*r} \right) \to \left(X_i, \tau_i^{*r} \right) \right\}_{i \prec i'} \right) \quad (4)$$

There exist limits of inverse system (3) and (4) in the category of fuzzy topological spaces [10, Theorem 2.3]. These limits are denoted by $\left(\underleftarrow{\lim} X_i, \tau^r \right), \left(\underleftarrow{\lim} X_i, \tau^{*r} \right)$, respectively. Here the fuzzy topology τ^r $\left(\tau^{*r} \right)$ is a restriction to a subspace $Y = \underleftarrow{\lim} X_i \subset \prod_{i \in J} X_i$ of product topology $\prod_{i \in J} \tau_i^r$ $\left(\prod_{i \in J} \tau_i^{*r} \right)$. Since $\tau_i^r \subset \tau_i^{*r}$, $\prod_{i \in J} \tau_i^r \subset \prod_{i \in J} \tau_i^{*r}$ is obtained. It follows that $\tau^r \subset \tau^{*r}$. Hence fuzzy bitopological space $\left(\underleftarrow{\lim} X_i, \tau^r, \tau^{*r} \right)$ is inverse limit of inverse system (2). If $r \succ r' \in I_0$, then for each $i \in J$, $\tau_i^r \subset \tau_i^{r'}$ and $\tau_i^{*r} \subset \tau_i^{*r'}$. Then $\prod_{i \in J} \tau_i^r \subset \prod_{i \in J} \tau_i^{r'}$ and $\prod_{i \in J} \tau_i^{*r} \subset \prod_{i \in J} \tau_i^{*r'}$. Thus

$$\tau^r = \prod_{i \in J} \tau_i^r \big|_Y \subset \prod_{i \in J} \tau_i^{r'} \big|_Y = \tau^{r'} \text{ and } \tau^{*r} = \prod_{i \in J} \tau_i^{*r} \big|_Y \subset \prod_{i \in J} \tau_i^{*r'} \big|_Y = \tau^{*r'}.$$

Hence $\left\{ \left(\tau^r, \tau^{*r} \right) \right\}_{r \in I_0}$ is a descending family of fuzzy bitopological spaces on Y. Then by using the family $\left\{ \left(\tau^r, \tau^{*r} \right) \right\}_{r \in I_0}$, gradation of openness $\tau : Y \to I$, $\tau^* : Y \to I$ are defined by

$$\tau(\mu) = \vee \left\{ r \in I_0 : \mu \in \tau^r \right\}, \quad \tau^*(\mu) = \wedge \left\{ 1 - r : \mu \in \tau^{*r} \right\}.$$

It follows that $\tau \subset \tau^*$.

Let
$$\left\{P_i : \left(Y, \tau, \tau^*\right) \to \left(X_i, \tau_i, \tau_i^*\right)\right\}_{i \in J} \qquad (5)$$
be a family of projection maps. Since
$$P_i : \left(Y, \tau^r, \tau^{*r}\right) \to \left(X_i, \tau_i^r, \tau_i^{*r}\right)$$
is fuzzy continuous mapping of fuzzy bitopological spaces for each $r \in I_0$, (5) is a family of gp-maps.

Now, let us show that the family $\left\{\left(Y, \tau, \tau^*\right), P_i\right\}$ is a unique limit of the inverse system (1). It suffices to show that for every $\overset{\vee}{S}FTS$ $\left(Z, \sigma, \sigma^*\right)$ and family of gp-maps $\left\{q_i : \left(Z, \sigma, \sigma^*\right) \to \left(X_i, \tau_i, \tau_i^*\right)\right\}_{i \in J}$ which satisfies the condition $q_i = p_i^{i'} \circ q_{i'}, \forall i \prec i'$, there exists a unique gp-map
$$\psi : \left(Z, \sigma, \sigma^*\right) \to \left(Y, \tau, \tau^*\right)$$
which satisfies the condition $P_i \circ \psi = q_i$.

For each $r \in I_0$,
$$\left\{q_i : \left(Z, \sigma^r, \sigma^{*r}\right) \to \left(X_i, \tau_i^r, \tau_i^{*r}\right)\right\}_{i \in J}$$
is a family of fuzzy continuous mappings of fuzzy bitopological spaces. Since the inverse systems (3) and (4) have limits in the category of fuzzy topological spaces, there exists a unique fuzzy continuous mapping $\psi : \left(Z, \sigma^r, \sigma^{*r}\right) \to \left(Y, \tau^r, \tau^{*r}\right)$ which makes up the following commutative diagram :

$$\begin{tikzcd}
Z \arrow[r, "q_i"] \arrow[d, "\psi"'] & X_i \\
Y \arrow[ur, "P_i"'] &
\end{tikzcd}$$

The mapping $\psi : \left(Z, \sigma^r, \sigma^{*r}\right) \to \left(Y, \tau^r, \tau^{*r}\right)$ is fuzzy continuous mapping of fuzzy bitopological spaces for each $r \in I_0$. Hence

$$\psi:\left(Z,\sigma,\sigma^{*}\right)\rightarrow\left(Y,\tau,\tau^{*}\right)$$

is a gp-map [12, Theorem 4.3] and $P_i \circ \psi = q_i$ is satisfied. This completes the proof.

Now let us show that the operation of inverse limit is a functor in the category of $\overset{\vee}{S}\,FTS$. For this, we define limit of morphism of inverse systems. Let

$$\underline{f}=\left(\varphi:\overline{J}\rightarrow J,\left\{f_{\bar{i}}:X_{\varphi(\bar{i})}\rightarrow Y_{\bar{i}}\right\}_{\bar{i}\in\overline{J}}\right)$$

be a morphism from the inverse system (1) to inverse system

$$\underline{Y}=\left(\left\{\left(Y_{\bar{i}},\overline{\tau}_{\bar{i}},\overline{\tau}^{*}_{\bar{i}}\right)\right\}_{\bar{i}\in\overline{J}},\left\{q_{\bar{i}}^{\bar{i}'}:\left(Y_{\bar{i}'},\overline{\tau}_{\bar{i}'},\overline{\tau}^{*}_{\bar{i}'}\right)\rightarrow\left(Y_{\bar{i}},\overline{\tau}_{\bar{i}},\overline{\tau}^{*}_{\bar{i}}\right)\right\}_{\bar{i}\prec\bar{i}'}\right)$$

in the category of $\overset{\vee}{S}\,FTS$.

For each $r\in I_0$,

$$\underline{f}^{(r)}=\left(\varphi:\overline{J}\rightarrow J,\left\{f_{\bar{i}}:\left(X_{\varphi(\bar{i})},\tau^{r}_{\varphi(\bar{i})},\tau^{*r}_{\varphi(\bar{i})}\right)\rightarrow\left(Y_{\bar{i}},\overline{\tau}^{r}_{\bar{i}},\overline{\tau}^{*r}_{\bar{i}}\right)\right\}_{\bar{i}\in\overline{J}}\right)$$

is a morphism of inverse system of fuzzy bitopological spaces generated by the morphism $\underline{f}$. This morphism induces fuzzy continuous mappings of inverse limit spaces as

$$\underleftarrow{\lim}\,\underline{f}^{(r)}:\underleftarrow{\lim}\left(X_i,\tau_i^{r}\right)\rightarrow\underleftarrow{\lim}\left(Y_{\bar{i}},\overline{\tau}^{r}_{\bar{i}}\right)$$

$$\underleftarrow{\lim}\,\underline{f}^{(r)}:\underleftarrow{\lim}\left(X_i,\tau_i^{*r}\right)\rightarrow\underleftarrow{\lim}\left(Y_{\bar{i}},\overline{\tau}^{*r}_{\bar{i}}\right)$$

in the category of fuzzy topological spaces. Hence for each $r\in I_0$, $\underleftarrow{\lim}\,\underline{f}^{(r)}$ is fuzzy continuous mapping of fuzzy bitopological spaces. Thus

$$\underleftarrow{\lim}\,\underline{f}:\left(\underleftarrow{\lim}X_i,\tau,\tau^{*}\right)\rightarrow\left(\underleftarrow{\lim}Y_{\bar{i}},\overline{\tau},\overline{\tau}^{*}\right)$$

is a gp-map of $\overset{\vee}{S}\,FTSs$ [12, Theorem 4.3].

Theorem 3.6. Let $Inv(\overset{\vee}{S}\,FTS)$ be a category of all inverse systems in $\overset{\vee}{S}\,FTS$ and all mappings between them. Then $\underleftarrow{\lim}$ operation is a functor from the category of $Inv(\overset{\vee}{S}\,FTS)$ to the category of $\overset{\vee}{S}\,FTS$.

Theorem 3.7. The limit of product of inverse systems is equal to product of limits of these inverse systems in the category of $\overset{\vee}{S}\,FTS$.

Proof. Proof is done similar to [10, Theorem 3.4].

Lemma 3.8. Let $f:\left(X,\tau,\tau^*\right)\to\left(Y,\sigma,\sigma^*\right)$ be a mapping of $\overset{\vee}{S}\,FTSs$.

a) f is a Šostak fuzzy open gp-map if and only if $f:\left(X,\tau^r,\tau^{*r}\right)\to\left(Y,\sigma^r,\sigma^{*r}\right)$ is a fuzzy open mapping of fuzzy bitopological spaces for each $r\in I_0$.

b) f is a Šostak fuzzy closed gp-map if and only if $f:\left(X,\tau^r,\tau^{*r}\right)\to\left(Y,\sigma^r,\sigma^{*r}\right)$ is a fuzzy closed mapping of fuzzy bitopological spaces for each $r\in I_0$.

Proof. a) Suppose $f:\left(X,\tau,\tau^*\right)\to\left(Y,\sigma,\sigma^*\right)$ is an Šostak fuzzy open gp-map. Then for each $\lambda\in X$, $\tau(\lambda)\le\sigma\left(f(\lambda)\right)$ and $\tau^*(\lambda)\ge\sigma^*\left(f(\lambda)\right)$. Let us show that $f:\left(X,\tau^r\right)\to\left(Y,\sigma^r\right)$ and $f:\left(X,\tau^{*r}\right)\to\left(Y,\sigma^{*r}\right)$ are fuzzy open for each $r\in I_0$. For each $G\in\tau^r,\tau(G)\ge r$. We have

$$\sigma\left(f(G)\right)\ge\tau(G)\ge r,$$

i.e., $f(G)\in\sigma^r$.

For each $G\in\tau^{*r},\tau^*(G)\le 1-r$. Then

$$\sigma^*\left(f(G)\right)\le\tau^*(G)\le 1-r,$$

i.e., $f(G)\in\sigma^{*r}$.

Conversely, assume the condition holds. Let us take arbitrary fuzzy set $\lambda\in X$.

If $\tau(\lambda)=0$, then $\sigma\left(f(\lambda)\right)\ge 0$, i.e., $\sigma\left(f(\lambda)\right)\ge\tau(\lambda)$. Similarly, If $\tau^*(\lambda)=1$, then $\sigma^*\left(f(\lambda)\right)\le 1$, i.e., $\sigma^*\left(f(\lambda)\right)\le\tau^*(\lambda)$. This implies that f is a Šostak fuzzy open mapping.

Let $\tau(\lambda)=r_0>0$ and $\tau^*(\lambda)=r_1<1$, then $\lambda\in\tau^{r_0}$ and $\lambda\in\tau^{*(1-r_1)}$. Since $f:\left(X,\tau^{r_0}\right)\to\left(Y,\sigma^{r_0}\right)$ and $f:\left(X,\tau^{*(1-r_1)}\right)\to\left(Y,\sigma^{*(1-r_1)}\right)$ are fuzzy open mappings, $f(\lambda)\in\sigma^{r_0}$ and $f(\lambda)\in\sigma^{*(1-r_1)}$, hence $\sigma\left(f(\lambda)\right)\ge r_0=\tau(\lambda)$, $\sigma^*\left(f(\lambda)\right)\le 1-(1-r_1)=r_1=\tau^*(\lambda)$ i.e., f is a Šostak fuzzy open gp-map.

b) Let $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ be a Šostak fuzzy closed gp-map. Then for each $\lambda\in X$, $\tau\left(\underline{1}-\lambda\right)\le\sigma\left(\underline{1}-f\left(\lambda\right)\right)$ and $\tau^{*}\left(\underline{1}-\lambda\right)\ge\sigma^{*}\left(\underline{1}-f\left(\lambda\right)\right)$ are satisfied. Now let us show that $f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ is fuzzy closed in the fuzzy bitopological spaces for each $r\in I_0$. For arbitrary two fuzzy closed sets $F\in\left(\tau^{r}\right)'$ and $F^{*}\in\left(\tau^{*r}\right)'$, $\tau\left(\underline{1}-F\right)\ge r$ and $\tau^{*}\left(\underline{1}-F^{*}\right)\le 1-r$. To complete the proof, we only need to check if $\underline{1}-f\left(F\right)\in\sigma^{r}$ and $\underline{1}-f\left(F^{*}\right)\in\sigma^{*r}$. Since

$$\sigma\left(\underline{1}-f\left(F\right)\right)\ge\tau\left(\underline{1}-F\right)\ge r,\ \sigma^{*}\left(\underline{1}-f\left(F^{*}\right)\right)\le\tau^{*}\left(\underline{1}-F^{*}\right)\le 1-r$$

$f\left(F\right)\in\left(\sigma^{r}\right)'$ and $f\left(F^{*}\right)\in\left(\sigma^{*r}\right)'$,i.e., f is fuzzy closed mapping of fuzzy bitopological spaces.

Conversely, let $f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ be fuzzy closed mapping of fuzzy bitopological spaces for each $r\in I_0$. Let $\lambda\in X$ be arbitrary fuzzy set. If $\tau\left(\underline{1}-\lambda\right)=r_0$ and $\tau^{*}\left(\underline{1}-\lambda\right)=r_1$, then since $f:\left(X,\tau^{r_0}\right)\to\left(Y,\sigma^{r_0}\right)$ and $f:\left(X,\tau^{*(1-r_1)}\right)\to\left(Y,\sigma^{*(1-r_1)}\right)$ are fuzzy closed mappings, $\underline{1}-f\left(\lambda\right)\in\sigma^{r_0}$ and $\underline{1}-f\left(\lambda\right)\in\sigma^{*(1-r_1)}$. Then

$$\sigma\left(\underline{1}-f\left(\lambda\right)\right)\ge r_0=\tau\left(\underline{1}-\lambda\right),\ \sigma^{*}\left(\underline{1}-f\left(\lambda\right)\right)\le 1-\left(1-r_1\right)=r_1=\tau^{*}\left(\underline{1}-\lambda\right)$$

i.e., f is a Šostak fuzzy closed gp-map.

Lemma 3.9. Let $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ be a mapping of $\check{S}FTSs$. Then the following conditions are equivalent to each other:

a) The mapping f is a Šostak fuzzy homeomorphism;

b) The mapping f is a bijective Šostak fuzzy open gp-map;

c) The mapping f is a bijective Šostak fuzzy closed gp-map.

Proof. a)$\Rightarrow$b) Let $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ be a Šostak fuzzy homeomorphism. Then for each $r\in I_0$, $f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ is a fuzzy homeomorphism of bitopological spaces. Every fuzzy homeomorphism ise a fuzzy open and fuzzy continuous mapping. From Lemma 3.8, for each $r\in I_0$ since

$f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ is fuzzy open and fuzzy continuous mapping, $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ is a bijective Šostak fuzzy open gp-map.

b) $\Rightarrow$ **a)** Let $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ be a bijective Šostak fuzzy open gp-map. From Lemma 3.8, $f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ is a bijective fuzzy open and fuzzy continuous mapping (f is a fuzzy homeomorphism of fuzzy bitopological spaces). For each $r\in I_0$, since $f:\left(X,\tau^{r},\tau^{*r}\right)\to\left(Y,\sigma^{r},\sigma^{*r}\right)$ is a fuzzy homeomorphism, $f:\left(X,\tau,\tau^{*}\right)\to\left(Y,\sigma,\sigma^{*}\right)$ is a Šostak fuzzy homeomorphism.

Similarly, a)$\Rightarrow$c) and c) $\Rightarrow$a) are proved.

Theorem 3.10. Let $\underline{f}=\left(\varphi:\overline{J}\to J,\left\{f_{\bar{i}}:X_{\varphi(\bar{i})}\to Y_{\bar{i}}\right\}_{\bar{i}\in\overline{J}}\right)$ be a morphism from the inverse system $\underline{X}=\left\{X_i\right\}_{i\in J}$ to the inverse system $\underline{Y}=\left\{Y_{\bar{i}}\right\}_{\bar{i}\in\overline{J}}$ in the category of $Inv(\check{S}\,FTS)$. If $f_{\bar{i}}$ is injective (bijective) gp-map for each $\bar{i}\in\overline{J}$, then

$$\underleftarrow{\lim}\underline{f}:\underleftarrow{\lim}\underline{X}\to\underleftarrow{\lim}\underline{Y}$$

is injective (bijective) gp-map.

Proof. For each $r\in I_0$,

$$\underline{f}_r=\left(\varphi:\overline{J}\to J,\left\{f_{\bar{i},r}:\left(X_{\varphi(\bar{i})},\tau^{r}_{\varphi(\bar{i})},\tau^{*r}_{\varphi(\bar{i})}\right)\to\left(Y_{\bar{i}},\sigma^{r}_{\bar{i}},\sigma^{*r}_{\bar{i}}\right)\right\}_{\bar{i}\in\overline{J}}\right)$$

is a morphism of inverse systems of fuzzy bitopological spaces. Since $f_{\bar{i},r}$ is injective (bijective) mapping for each $\bar{i}\in\overline{J}$, $\underleftarrow{\lim}\underline{f}_r$ is injective (bijective) fuzzy continuous mapping from [11, Theorem2.1, Theorem 2.4]. Then

$$\underleftarrow{\lim}\underline{f}:\underleftarrow{\lim}\underline{X}\to\underleftarrow{\lim}\underline{Y}$$

is injective (bijective) gp-map.

Corallary 3.11. Let us take the mapping $\underline{f}$ in Theorem 3.10. If $f_{\bar{i}}$ is Šostak fuzzy homeomorphism for each $\bar{i}\in\overline{J}$, then

$$\underleftarrow{\lim}\underline{f}:\underleftarrow{\lim}\underline{X}\to\underleftarrow{\lim}\underline{Y}$$

is also Šostak fuzzy homeomorphism.

Proof. From Theorem 3.10, $\underleftarrow{\lim}\underline{f}$ is bijective gp-map. From Lemma 3.9, each $f_{\bar{i}}$ is Šostak fuzzy open gp-map and from Lemma 3.8.

$$f_{\bar{i},r}:\left(X_{\varphi(\bar{i})},\tau^{r}_{\varphi(\bar{i})},\tau^{*r}_{\varphi(\bar{i})}\right)\to\left(Y_{\bar{i}},\sigma^{r}_{\bar{i}},\sigma^{*r}_{\bar{i}}\right)$$

is fuzzy open mapping for each $r\in I_0$. Then $\underleftarrow{\lim}\underline{f}_r$ is fuzzy homeomorphism for each $r\in I_0$. Hence $\underleftarrow{\lim}\underline{f}$ is Šostak fuzzy homeomorphism.

Theorem 3.12. Let $\underline{X}$ be an inverse system in the category of $\overset{\vee}{S}FTS$.

a) If each $p_i^{i'}:\left(X_{i'},\tau_{i'},\tau^*_{i'}\right)\to\left(X_i,\tau_i,\tau^*_i\right)$ $(i\prec i')$ is injective gp-map, then each mapping

$$\pi_i:\left(\underleftarrow{\lim}\underline{X},\tau,\tau^*\right)\to\left(X_i,\tau_i,\tau^*_i\right)$$

is also an injective gp-map.

b) If each $p_i^{i'}:\left(X_{i'},\tau_{i'},\tau^*_{i'}\right)\to\left(X_i,\tau_i,\tau^*_i\right)$ $(i\prec i')$ is bijective gp-map, then each mapping

$$\pi_i:\left(\underleftarrow{\lim}\underline{X},\tau,\tau^*\right)\to\left(X_i,\tau_i,\tau^*_i\right)$$

is also a bijective gp-map.

Proof. a) For two arbitrary fuzzy points $x_\lambda=\left\{x^i_{\lambda_i}\right\}\neq\left\{y^i_{\mu_i}\right\}=y_\mu$, let

$$\pi_{i_1}(x_\lambda)=x^{i_1}_{\lambda_{i_1}}=y^{i_1}_{\mu_{i_1}}=\pi_{i_1}(y_\mu).$$

Then

$$x^{i_1}_{\lambda_{i_1}}=y^{i_1}_{\mu_{i_1}}\Leftrightarrow x^{i_1}=y^{i_1}\text{ and }\lambda_{i_1}=\mu_{i_1}.$$

Since $p_i^{i'}:\left(X_{i'},\tau_{i'},\tau^*_{i'}\right)\to\left(X_i,\tau_i,\tau^*_i\right)$ is injective and for each $i'\succ i_1$

$$p_{i_1}^{i'}\left(x^{i'}_{\lambda_{i'}}\right)=x^{i_1}_{\lambda_{i_1}}=p_{i_1}^{i'}\left(y^{i'}_{\mu_{i'}}\right)=y^{i_1}_{\mu_{i_1}},$$

$x^{i'}_{\lambda_{i'}}=y^{i'}_{\mu_{i'}}$ is satisfied. Since J is directed poset for each $i\in J$, there exists $i'\in J$ such that $i'\succ i$ and $i'\succ i_1$ for $i,i_1\in J$. Since $x^{i'}_{\lambda_{i'}}=y^{i'}_{\mu_{i'}}$, $x^i_{\lambda_i}=p_i^{i'}\left(x^{i'}_{\lambda_{i'}}\right)=p_i^{i'}\left(y^{i'}_{\mu_{i'}}\right)=y^i_{\mu_i}$. Hence $x_\lambda=y_\mu$ is obtained.

b) Let us show that π_{i_1} is surjective mapping. Let $x^{i_1}_{\lambda_{i_1}}\in I^{X_{i_1}}$ be an arbitrary fuzzy point. For each $i'\succ i_1$, there exists a unique fuzzy point $x^{i'}_{\lambda_{i'}}$ such that $p_{i_1}^{i'}\left(x^{i'}_{\lambda_{i'}}\right)=x^{i_1}_{\lambda_{i_1}}$. For each $i\in J$, there exists $i'\in J$ such that $i'\succ i$ and $i'\succ i_1$. Then we get $p_i^{i'}\left(x^{i'}_{\lambda_{i'}}\right)=x^i_{\lambda_i}$. Now, we show that $x_\lambda = \left\{ x^i_{\lambda_i} \right\}$ belongs to $\varprojlim \underline{X}$. Let $i' \succ i, i' \succ i_1$ and $\tilde{i}' \succ \tilde{i}, \tilde{i}' \succ i_1$ be for each $\tilde{i} \succ i'$. Then

$$x^i_{\lambda_i} = p^{i'}_i \left(x^{i'}_{\lambda_{i'}} \right),\ x^{\tilde{i}}_{\lambda_{\tilde{i}}} = p^{\tilde{i}'}_{\tilde{i}} \left(x^{\tilde{i}'}_{\lambda_{\tilde{i}'}} \right).$$

We can choose $i'' \in J$ such that $i'' \succ i', i'' \succ \tilde{i}'$ for the elements $i', \tilde{i}' \in J$. Then

$$x^{i_1}_{\lambda_{i_1}} = p^{i''}_{i_1} \left(x^{i''}_{\lambda_{i''}} \right) = \left(p^{i'}_{i_1} \circ p^{i''}_{i'} \left(x^{i''}_{\lambda_{i''}} \right) \right) = \left(p^{\tilde{i}'}_{i_1} \circ p^{i''}_{\tilde{i}'} \left(x^{i''}_{\lambda_{i''}} \right) \right)$$

and

$$x^{i_1}_{\lambda_{i_1}} = p^{i'}_{i_1} \left(x^{i'}_{\lambda_{i'}} \right) = p^{\tilde{i}'}_{i_1} \left(x^{\tilde{i}'}_{\lambda_{\tilde{i}'}} \right).$$

Since the mappings $p^{i'}_{i_1}, p^{\tilde{i}'}_{i_1}$ are bijective,

$$p^{i''}_{i'} \left(x^{i''}_{\lambda_{i''}} \right) = x^{i'}_{\lambda_{i'}} \quad \text{and} \quad p^{i''}_{\tilde{i}'} \left(x^{i''}_{\lambda_{i''}} \right) = x^{\tilde{i}'}_{\lambda_{\tilde{i}'}}.$$

Hence $p^{i''}_i \left(x^{i''}_{\lambda_{i''}} \right) = \left(p^{i'}_i \circ p^{i''}_{i'} \left(x^{i''}_{\lambda_{i''}} \right) \right) = x^i_{\lambda_i}$, $p^{i''}_{\tilde{i}} \left(x^{i''}_{\lambda_{i''}} \right) = \left(p^{\tilde{i}'}_{\tilde{i}} \circ p^{i''}_{\tilde{i}'} \left(x^{i''}_{\lambda_{i''}} \right) \right) = x^{\tilde{i}}_{\lambda_{\tilde{i}}}$ and thus

$$p^{\tilde{i}}_i \left(x^{\tilde{i}}_{\lambda_{\tilde{i}}} \right) = p^{i''}_i \left(x^{i''}_{\lambda_{i''}} \right) = x^i_{\lambda_i}$$

is obtained. It is clear that

$$\pi_{i_1} \left(x_\lambda \right) = x^{i_1}_{\lambda_{i_1}}.$$

Corallary 3.13. If $p^{i'}_i : \left(X_{i'}, \tau_{i'}, \tau^*_{i'} \right) \to \left(X_i, \tau_i, \tau^*_i \right)$ is a Šostak fuzzy homeomorphism in the inverse system $\underline{X}$ in the category of $\check{S}FTS$, then

$$\pi_i : \left(\varprojlim \underline{X}, \tau, \tau^* \right) \to \left(X_i, \tau_i, \tau^*_i \right)$$

is a Šostak fuzzy homeomorphism.

Let $\left(X, T, T^* \right)$ $\left(T \subset T^* \right)$ be bitopological space. By using the topologies T and T^*, let us convert the space X to fuzzy bitopological space. For arbitrary fuzzy set $\mu \in X$, $\sup p\, \mu = \left\{ x \in X : \mu(x) > 0 \right\}$.

It is clear that the families

$$\tilde{T} = \{\mu \in X : \operatorname{supp} \mu \in T\}, \quad \tilde{T}^* = \{\mu \in X : \operatorname{supp} \mu \in T^*\}$$

generate fuzzy topology in the space X and since $T \subset T^*$, $\tilde{T} \subset \tilde{T}^*$ is satisfied. Thus $\left(X, \tilde{T}, \tilde{T}^*\right)$ is a fuzzy bitopological space.

Lemma 3.14. If $\left(X, \tau, \tau^*\right)$ is a fuzzy compact space, then $\left(X, T, T^*\right)$ is a compact space.

Proof. If fuzzy topological space $\left(X, \tilde{T}^*\right)$ is fuzzy compact space, then $\left(X, \tilde{T}\right)$ is also fuzzy compact space. Let $\left(X, T^*\right)$ be a fuzzy compact space and $\{G_\lambda\}_{\lambda \in \wedge}$ be an arbitrary open cover of $\left(X, T^*\right)$. Then $X = \bigcup_{\lambda \in \wedge} G_\lambda$. For each $G_\lambda \in T^*$, let us consider fuzzy set $\chi_{G_\lambda} \in X$. Since $\operatorname{supp}\left(\chi_{G_\lambda}\right) = G_\lambda \in T^* \Rightarrow \chi_{G_\lambda} \in \tilde{T}^*$. Hence the family of fuzzy open sets $\left\{\chi_{G_\lambda}\right\}_{\lambda \in \wedge}$ is fuzzy open cover of $\left(X, \tilde{T}^*\right)$. Since $\left(X, \tilde{T}^*\right)$ is fuzzy compact space, $\left\{\chi_{G_\lambda}\right\}_{\lambda \in \wedge}$ has a finite subcover, say $\left\{\chi_{G_j}\right\}_{j \in J}$ where J is finite. Then

$$X = \bigvee_{j \in J} \chi_{G_j} \Rightarrow X = \bigcup_{j \in J} G_j,$$

i.e., $\left(X, T^*\right)$ is a compact space. Since $T \subset T^*$, the space $\left(X, T\right)$ is also compact, i.e., bitopological space $\left(X, T, T^*\right)$ is a compact.

The converse of Lemma 3.14. is not true.

Example 3.15. Let $I = [0,1]$ be a closed unit interval and $T = T^*$ be Euclid topology. Let us consider open sets $U_n = \left[0, 1 - \frac{1}{n}\right), n = \overline{2, \infty}$ and $V = \left(\frac{1}{2}, 1\right]$ in I. Then the family $\{U_n, V\}_{n = \overline{2,\infty}}$ is open cover of I. Define fuzzy sets $f_n : I \to I$ and $g : I \to I$ as follow respectively,

$$f_n(x)=\begin{cases}1-\dfrac{1}{n}, x\in U_n\\ 0 \quad , x\notin U_n\end{cases} \text{ and } g(x)=\begin{cases}\dfrac{1}{5}, x\in V\\ 0, x\notin V\end{cases}.$$

Thus for $\forall n=\overline{2,\infty}$, $\operatorname{supp} f_n = U_n, \operatorname{supp} g = V$. It is clear that $\{f_n, g\}_{n=\overline{2,\infty}}$ is the family of fuzzy open sets in I^I. Since $(\overset{\infty}{\underset{n=2}{\vee}} f_n)\vee g = \underline{1}$, the family $\{f_n, g\}_{n=\overline{2,\infty}}$ is fuzzy open cover of I^I. But the fuzzy open cover $\{f_n, g\}_{n=\overline{2,\infty}}$ has no finite subcover. Although I is compact space, I^I is not fuzzy compact space.

Lemma 3.16. Let (X,T,T^*) be bitopological space. Then (X,T,T^*) is Hausdorff space if and only if fuzzy bitopological space (X,τ,τ^*) is fuzzy Hausdorff space.

Proof. If the space (X,T) is Hausdorff, then (X,T^*) is also Hausdorff space. Similarly, if $\left(X,\tilde{T}\right)$ is fuzzy Hausdorff space, then $\left(X,\tilde{T}^*\right)$ is also fuzzy Hausdorff space. Hence proof of the lemma is given for the spaces (X,T) and $\left(X,\tilde{T}\right)$.

Let (X,T) be Hausdorff space and $x_\lambda, y_\mu\, (x\neq y)$ be arbitrary fuzzy points in X such that $x_\lambda \neq y_\mu$. Since (X,T) is Hausdorff space and $x\neq y\in X$, there exist $G,H\in T$ such that $x\in G, y\in H$ and $G\cap H=\varnothing$. Then χ_G and χ_H are two fuzzy open sets in X. It is clear that $x_\lambda \le \chi_G$ and $y_\mu \le \chi_H$, i.e., χ_G and χ_H are fuzzy open neighborhood of fuzzy points x_λ and y_μ, respectively. Since $\chi_G \wedge \chi_H = \chi_{G\cap H} = \chi_\varnothing = \underline{0}$, $\left(X,\tilde{T}\right)$ is fuzzy Hausdorff space.

Conversely, suppose $\left(X,\tilde{T}\right)$ is fuzzy Hausdorff space and $x\neq y\in X$ are two points. For each $\lambda,\mu\in I/\{0\}$, fuzzy points x_λ and y_μ belong to X. Since $x\neq y$, $x_\lambda \neq y_\mu$. Also since $\left(X,\tilde{T}\right)$ is fuzzy Hausdorff space, there exist $A,B\in\tilde{T}$ such that $x_\lambda \le A, y_\mu \le B$ and $A\wedge B=\underline{0}$. If $x_\lambda \le A$, then $A(x)\ge\lambda$ and $x\in \operatorname{supp} A$. If $y_\mu \le B$, then $B(y) \geq \mu$ and $y \in \sup pB$. Since A, B are fuzzy open sets, $\sup pA \in T, \sup pB \in T$. Since $A \wedge B = \underline{0}$, $\sup pA \cap \sup pB = \varnothing$, i.e., (X,T) is Hausdorff space.

Lemma 3.17. $f:(X,T,T^*) \to (Y,T',T^{*'})$ is continuous if and only if $\vec{f}:(X,\tau,\tau^*) \to (Y,\tau',\tau^{*'})$ is fuzzy continuous.

Proof. Let us prove this lemma for the functions $f:(X,T) \to (Y,T')$ and $\vec{f}:\left(X,\tilde{T}\right) \to \left(Y,\tilde{T}'\right)$

Let $f:(X,T) \to (Y,T')$ be continuous and $\mu \in \tilde{T}'$. We show that $\overleftarrow{f}(\mu) = \mu \circ f \in \tilde{T}$. Here

$$\sup p \overleftarrow{f}(\mu) = \sup p(\mu \circ f) = \{x \in X : (\mu \circ f)(x) > 0\}.$$

Since $\mu \in \tilde{T}'$, $\sup p\mu \in T'$. Also since f is continuous mapping, $f^{-1}(\sup p\mu) \in T$.

$$f^{-1}(\sup p\mu) = \{x \in X : f(x) \in \sup p\mu\} = \{x \in X : \mu(f(x)) > 0\}.$$

Hence $\sup p \overleftarrow{f}(\mu) = f^{-1}(\sup p\mu) \in T$, i.e., $\vec{f}$ is fuzzy continuous.

Conversely, let us assume that $\vec{f}:\left(X,\tilde{T}\right) \to \left(Y,\tilde{T}'\right)$ be fuzzy continuous and $G \in T'$ be an arbitrary open set. Now, let us show that $f^{-1}(G)$ is open set. The fuzzy set χ_G is fuzzy open in I^Y. Since $\vec{f}$ is fuzzy continuous, $\overleftarrow{f}(\chi_G)$ is fuzzy open, where $\overleftarrow{f}(\chi_G) = \chi_G \circ f$. Then

$$(\chi_G \circ f)(x) = \chi_G(f(x)) = \begin{cases} 1, & f(x) \in G \\ 0, & f(x) \notin G \end{cases} = \begin{cases} 1, & x \in f^{-1}(G) \\ 0, & x \notin f^{-1}(G) \end{cases}.$$

Hence $f^{-1}(G) = \sup p(\chi_G \circ f)$, i.e., f is continuous.

Similarly, proof of the lemma can be done for the functions $f:(X,T^*) \to (Y,T^{*'})$ and $\vec{f}:\left(X,\tilde{T}^*\right) \to \left(Y,\tilde{T}^{*'}\right)$.

If

$$\left(\{(X_i,T_i,T_i^*)\}_{i \in J}, \{p_i^{i'}\}_{i \prec i'}\right) \quad (6)$$

is an inverse system of bitopological spaces, then

$$\left(\left\{\left(X_i,\tilde{T}_i,\tilde{T}_i^*\right)\right\}_{i\in J},\left\{\overrightarrow{p}_i^{i'}\right\}_{i\prec i'}\right) \quad (7)$$

is an inverse system of fuzzy bitopological spaces.

Theorem 3.18. If (7) is an inverse system of fuzzy compact Hausdorff spaces, then fuzzy bitopological space $\underleftarrow{\lim} I^{X_i}$ is also fuzzy compact space.

Proof. It is enough to prove for inverse system of fuzzy topological spaces $\left\{\left(X_i,\tilde{T}_i^*\right)\right\}_{i\in J}$. Since the inverse system $\left\{\left(X_i,\tilde{T}_i^*\right)\right\}_{i\in J}$ is fuzzy compact Hausdorff, $\left\{\left(X_i,T_i^*\right)\right\}_{i\in J}$ is inverse system of compact Hausdorff spaces from Lemma 3.19., Lemma 3.21 . Limit of this inverse system is compact [4]. Since this limit is compact subset of Hausdorff space $\prod_{i\in J} X_i$, it is closed.

Now, let us show that $\underleftarrow{\lim} I^{X_i}$ is a fuzzy compact space. Let $\{\mu_\alpha\}_{\alpha\in\wedge}$ be arbitrary fuzzy open cover of fuzzy space $\underleftarrow{\lim} I^{X_i}$. Since $\underleftarrow{\lim} I^{X_i} = I^{\underleftarrow{\lim} X_i}$ [11], this space is fuzzy subspace of product space. Then $\mu_\alpha = \prod_{i\in J} I^{X_i} \wedge \lambda_\alpha$ for each $\alpha\in\wedge$, where λ_α is fuzzy open set in $\prod_{i\in J} I^{X_i}$. Hence $\underleftarrow{\lim} I^{X_i} \leq \vee_{\alpha\in\wedge} \lambda_\alpha$. Since $\underleftarrow{\lim} X_i$ is closed set, $U = \prod_{i\in J} X_i / \underleftarrow{\lim} X_i$ is open set. Then the family $\{\lambda_\alpha, \chi_U\}_{\alpha\in\wedge}$ is a fuzzy open cover of fuzzy space $\prod_{i\in J} I^{X_i}$. Since this space is fuzzy compact, there exists a finite subcover as $\left\{\lambda_{\alpha_i}, \chi_U\right\}_{i\in J}$ of this cover, where J is finite. Hence the family $\left\{\lambda_{\alpha_i}\right\}_{i\in J}$ is finite subcover of $\prod_{i\in J} I^{X_i}$ and so $\left\{\mu_{\alpha_i}\right\}_{i\in J}$ is a finite subcover of $\underleftarrow{\lim} I^{X_i}$.

Definition 3.19. Let (X,τ,τ^*) be a $\check{S}\,FTS$.

a) The space (X,τ,τ^*) is called $r-$ Hausdorff if and only if for each fuzzy points $x_\alpha, y_\beta \in X$, there exist fuzzy set λ,μ such that

$$x_\alpha \leq \lambda,\ y_\beta \leq \mu,\ \lambda\wedge\mu = \underline{0} \text{ and } \tau(\lambda),\tau(\mu) \geq r; \quad \tau^*(\lambda),\tau^*(\mu) \leq 1-r \ .$$

b) If $\left(X,\tau,\tau^{*}\right)$ is $r-$ Hausdorff for each $r\in I_0$, then this space is called strong Hausdorff space.

It is clear that if $\left(X,\tau,\tau^{*}\right)$ is strong Hausdorff space, then the fuzzy topological space $\left(X,\tau^{r}\right)$ is fuzzy Hausdorff space for each $r\in I_0$. Since $\tau^{r}\subset\tau^{*r}$, $\left(X,\tau^{*r}\right)$ is fuzzy Hausdorff space. Thus the fuzzy bitopological space $\left(X,\tau^{r},\tau^{*r}\right)$ is fuzzy Hausdorff space with respect to τ^{r} and τ^{*r}. Similarly if the fuzzy bitopological space $\left(X,\tau^{r},\tau^{*r}\right)$ is fuzzy Hausdorff space for each $r\in I_0$, then $\overset{\vee}{S}\,FTS$ $\left(X,\tau,\tau^{*}\right)$ generated by bitopologies $\left\{\left(\tau^{r},\tau^{*r}\right)\right\}$ is strong Hausdorff space.

Definition 3.20. a) Let $\left(X,\tau,\tau^{*}\right)$ be $\overset{\vee}{S}\,FTS$ and $G\in X$ be a fuzzy set. G is said to be smooth compact space if for each family of fuzzy sets $U=\{A:\tau(A)>0$, or $\tau^{*}(A)<1\}$ which satisfies the condition $\underset{x\in X}{\wedge}\left(G'(x)\vee\underset{A\in U}{\vee}A(x)\right)>p$ $\forall p\in I_1$, there exists a finite subfamily F of U such that

$$\underset{x\in X}{\wedge}(G'(x)\vee\underset{A\in F}{\vee}A(x))>p.$$

If $G=\underline{1}$, then the space $\left(X,\tau,\tau^{*}\right)$ is called smooth compact space.

b) If G is fuzzy compact space in $\left(X,\tau^{r},\tau^{*r}\right)$ for each $r\in I_0$, then $G\in X$ is said to be strong smooth compact space.

Lemma 3.21. If the fuzzy set G is smooth compact in $\overset{\vee}{S}\,FTS$ $\left(X,\tau,\tau^{*}\right)$, then the fuzzy set G is fuzzy compact space in the fuzzy topological spaces $\left(X,\tau^{r}\right)$ and $\left(X,\tau^{*r}\right)$ for each $r\in I_0$.

Proof. Let the fuzzy set G be smooth compact in $\left(X,\tau,\tau^{*}\right)$. For each $r\in I_0$, let us consider $U=\{A\}$ which are strong p - shading fuzzy sets of G in $\left(X,\tau^{*r}\right)$ for $\forall p\in I_1$, i.e., the following condition

$$\underset{x\in X}{\wedge}\left(G'(x)\vee\underset{A\in U}{\vee}A(x)\right)>p$$

is satisfied. Since $U=\{A\}$ is a family of fuzzy open sets, $\tau^*(A)\le 1-r$ for each $A\in U$. Since $r>0$, $\tau^*(A)<1$ or $\tau(A)>0$ are satisfied. Then by using definition of smooth compact, it can be found a finite subfamily F of U such that

$$\underset{x\in X}{\wedge}\left(G'(x)\vee \underset{A\in F}{\vee} A(x)\right)>p,$$

i.e., F is strong p- shading of fuzzy set G. Hence from [14], the fuzzy set G is fuzzy compact in the fuzzy space$\left(X,\tau^{*r}\right)$. Since $\tau^r\subset\tau^{*r}$, the fuzzy set G is fuzzy compact in the fuzzy space $\left(X,\tau^r\right)$, too.

Theorem 3.22. If the inverse system $\left(\left\{\left(X_i,\tau_i,\tau_i^*\right)\right\}_{i\in J},\left\{\overrightarrow{p_i}^{i'}\right\}_{i\prec i'}\right)$ is smooth compact, strong Hausdorff in the category of $\overset{\vee}{S}FTS$ and fuzzy topologies $\left(\tau_i^r,\tau_i^{*r}\right)$ are given by sup p , for each $i\in J, r\in I_0$, then $\underleftarrow{\lim} I^{X_i}$ is a strong smooth compact space.

Proof. From Definition 3.19, Definition 3.20 and Lemma 3.21, $\left\{\left(I^{X_i},\tau_i^r\right)\right\}$ is an inverse system of fuzzy compact Hausdorff spaces for each $r\in I_0$. From Theorem 3.18, $\underleftarrow{\lim}\left(I^{X_i},\tau_i^r\right)$ is a fuzzy compact space. Since $\underleftarrow{\lim}\left(I^{X_i},\tau_i^r\right)$ is fuzzy compact space for each $r\in I_0$, $\underleftarrow{\lim}\left(I^{X_i},\tau_i\right)$ is strong smooth compact space in the category of $\overset{\vee}{S}FTS$.

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