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ARITHMETICAL PROPERTIES OF THE VALUES OF SOME POWER SERIES WITH ALGEBRAIC COEFFICIENTS TAKEN FOR U_m-NUMBERS ARGUMENTS. [1]

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Abstract : In this paper it is proved that the values of some gap series for U_m-numbers arguments are either a U-number of degree $\leq m$ or an element of a certain algebraic number field. In this work the method which is used by Oryan for Liouville numbers in [9] and [10] is extended to the U_m-numbers. This extended method is used first for the gap series with rational coefficients and then for the gap series with algebraic coefficients. Further by using the similar methods for the p-adic gap series the similar results are obtained. The obtained results in the work contains the theorems in [9], [10] as special cases.

INTRODUCTION

Mahler [5] divided in 1932 the complex numbers into four classes A, S, T, U as follows.

Let $P(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial with integer coefficients. The number $H(P) = \max\{|a_n|, \ldots, |a_0|\}$ is called the height of $P(x)$. Let ξ be a complex number and

$$\omega_n(H, \xi) = \min\{|P(\xi)| \ : \text{ degree of } P \leq n,\ H(P) \leq H,\ P(\xi) \neq 0\},$$

where n and H are natural numbers. Let

$$\omega_n(\xi) = \limsup_{H\to\infty} \frac{-\log \omega_n(H,\xi)}{\log H},$$

and

$$\omega(\xi) = \limsup_{n\to\infty} \frac{\omega_n(\xi)}{n}.$$

The inequalities $0 \leq \omega_n(\xi) \leq \infty$ and $0 \leq \omega(\xi) \leq \infty$ hold. From $\omega_{n+1}(H,\xi) \leq \omega_n(H,\xi)$ we get $\omega_{n+1}(\xi) \geq \omega_n(\xi)$. So $\omega(\xi)$ is either a non-zero finite number or positive infinity.

[1] This paper is an English translation of the substance of a doctoral dissertation accepted by the Institute of Science of the University of Istanbul in October 1996. I am grateful to Prof. Dr. Mehmet H. ORYAN for his valuable help and encouragement in all stages of this work.

If for an index $\omega_n(\xi) = +\infty$, then $\mu(\xi)$ is defined as the smallest of them; otherwise $\mu(\xi) = +\infty$. So μ is uniquely determined and both of $\mu(\xi)$ and $\omega(\xi)$ cannot be finite. Therefore there are the following four possibilities for ξ. ξ is called

$$
\begin{array}{ll}
A-\text{number if} & \omega(\xi)=0\ ,\ \mu(\xi)=\infty, \\
S-\text{number if} & 0<\omega(\xi)<\infty\ ,\ \mu(\xi)=\infty, \\
T-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)=\infty, \\
U-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)<\infty.
\end{array}
$$

The class A is composed of all algebraic numbers. The transcendental numbers are divided into the classes S, T, U. ξ is called a U-number of degree m $(1 \leq m)$ if $\mu(\xi) = m$. U_m denotes the set of U-numbers of degree m. The elements of the subclass U_1 are called Liouville numbers.

Koksma [3] set up in 1939 another classification of complex numbers. He divided them into four classes A^*, S^*, T^*, U^*. Let ξ be a complex number and

$$\omega_n^*(H,\xi) = \min\{|\xi-\alpha| \ :\ \text{degree of } \alpha \leq n\, ,\, H(\alpha) \leq H\, ,\, \alpha \neq \xi\}\ ,$$

where α is an algebraic number. Let

$$\omega_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(H\omega_n^*(H,\xi))}{\log H}\ ,$$

and

$$\omega^*(\xi) = \limsup_{n\to\infty} \frac{\omega_n^*(\xi)}{n}\ .$$

We have $0 \leq \omega_n^*(\xi) \leq \infty$ and $0 \leq \omega^*(\xi) \leq \infty$. If for an index $\omega_n^*(\xi) = +\infty$, then $\mu^*(\xi)$ is defined as the smallest of them; otherwise $\mu^*(\xi) = +\infty$. So μ^* is uniquely determined and both of $\mu^*(\xi)$ and $\omega^*(\xi)$ cannot be finite. There are the following four possibilities for ξ. ξ is called

$$
\begin{array}{ll}
A^*-\text{number if} & \omega^*(\xi)=0\ ,\ \mu^*(\xi)=\infty, \\
S^*-\text{number if} & 0<\omega^*(\xi)<\infty\ ,\ \mu^*(\xi)=\infty, \\
T^*-\text{number if} & \omega^*(\xi)=\infty\ ,\ \mu^*(\xi)=\infty, \\
U^*-\text{number if} & \omega^*(\xi)=\infty\ ,\ \mu^*(\xi)<\infty.
\end{array}
$$

ξ is called a U^*-number of degree m $(1 \leq m)$ if $\mu^*(\xi) = m$. The set of U^*-numbers of degree m is denoted by U_m^*.

Wirsing [12] proved that both classifications are equivalent, i.e. A-, S-, T-, U-numbers are as same as A^*-, S^*-, T^*-, U^*-numbers. Moerover every U-number of degree m is also a U^*-number of degree m and conversely.

LeVeque [4] proved that the subclass U_m is not empty. Oryan [8] proved that a class of power series with algebraic coefficients take values in the subclass U_m for algebraic arguments under certain conditions. Zeren [13] obtained the similar results for the some gap series. Oryan [10] also proved that the values of some power series for the arguments from the set of Liouville numbers are U-numbers of degree $\leq m$.

Let p be a fixed prime number and $|\ldots|_p$ denotes the p-adic valuation of the set of rational numbers $\mathbb{Q}$. Furthermore let $\mathbb{Q}_p$ denotes the all p-adic numbers over $\mathbb{Q}$.

Mahler [6] had a classification of p-adic numbers in 1934 as follows. Let

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

be a polynomial with integer coefficients. The number

$$H(P) = \max\{|a_n|, \ldots, |a_0|\}$$

is called the height of P. Let ξ be a p-adic number and

$$\omega_n(H,\xi) = \min\{|P(\xi)|_p \ : \text{ degree of } P \leq n,\ H(P) \leq H,\ P(\xi) \neq 0\}$$

where n and H are natural numbers. Let

$$\omega_n(\xi) = \limsup_{H\to\infty} \frac{-\log \omega_n(H,\xi)}{\log H},$$

and

$$\omega(\xi) = \limsup_{n\to\infty} \frac{\omega_n(\xi)}{n}.$$

It is clear that $0 \leq \omega_n(\xi) \leq +\infty$ and $0 \leq \omega(\xi) \leq +\infty$ for $n \geq 1$. If for an index $\omega_n(\xi) = +\infty$, then $\mu(\xi)$ is defined as the smallest of them; otherwise $\mu(\xi) = +\infty$. So $\mu(\xi)$ is uniquely determined and both of $\omega(\xi)$ and $\mu(\xi)$ cannot be finite. Therefore there are the following four possibilities for p-adic ξ number. The p-adic number ξ is called

$$\begin{array}{ll} A-\text{number if} & \omega(\xi)=0\ ,\ \mu(\xi)=\infty, \\ S-\text{number if} & 0<\omega(\xi)<\infty\ ,\ \mu(\xi)=\infty, \\ T-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)=\infty, \\ U-\text{number if} & \omega(\xi)=\infty\ ,\ \mu(\xi)<\infty. \end{array}$$

ξ is called a U-number of degree m $(1 \leq m)$ if $\mu(\xi) = m$. U_m denotes the set of U-numbers of degree m. The elements of the subclass U_1 are called Liouville numbers.

The classification of complex numbers which is given by Koksma [3] can be carried over $\mathbb{Q}_p$.

Let ξ be a p-adic number and

$$\omega_n^*(H,\xi) = \min\{|\xi - \alpha|_p \ : \ \text{degree of } \alpha \le n \, , \ H(\alpha) \le H \, , \ \alpha \ne \xi\} \ ,$$

where n and H are natural numbers. Let

$$\omega_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(H\omega_n^*(H,\xi))}{\log H} \ ,$$

and

$$\omega^*(\xi) = \limsup_{n\to\infty} \frac{\omega_n^*(\xi)}{n} \ .$$

The inequalities $0 \le \omega_n^*(\xi) \le \infty$ and $0 \le \omega^*(\xi) \le \infty$ hold. If for an index $\omega_n^*(\xi) = +\infty$, then $\mu^*(\xi)$ is defined as the smallest of them; otherwise $\mu^*(\xi) = +\infty$. So $\mu^*(\xi)$ is uniquely determined and both of $\mu^*(\xi)$ and $\omega^*(\xi)$ cannot be finite. There are the following four possibilities for ξ. The p-adic number ξ is called

$$\begin{array}{ll} A^* - \text{number if} & \omega^*(\xi) = 0 \ , \ \mu^*(\xi) = \infty, \\ S^* - \text{number if} & 0 < \omega^*(\xi) < \infty \ , \ \mu^*(\xi) = \infty, \\ T^* - \text{number if} & \omega^*(\xi) = \infty \ , \ \mu^*(\xi) = \infty, \\ U^* - \text{number if} & \omega^*(\xi) = \infty \ , \ \mu^*(\xi) < \infty. \end{array}$$

ξ is called a U^*-number of degree m $(1 \le m)$ if $\mu^*(\xi) = m$. The set of p-adic U^*-numbers of degree m is denoted by U_m^*.

Both classifications are equivalent, i.e. A-, S-, T-, U-numbers are as same as A^*-, S^*-, T^*-, U^*-numbers. Moreover every U-number of degree m is also a U^*-number of degree m and conversely. Oryan [8] proved that a class of power series with algebraic coefficients takes values in the class p-adic U_m for p-adic algebraic arguments. Zeren [13] obtained the similar results for the some gap series. Furthermore Oryan [9] proved that the values of some power series for the arguments from the set of p-adic Liouville numbers are p-adic U-numbers of degree $\le m$.

LEMMAS

Lemma 1. Let $\alpha_1, \ldots, \alpha_k$ $(k \ge 1)$ be algebraic numbers which belong to an algebraic number field K of degree g, η be an algebraic number and $F(y, x_1, \ldots, x_k)$ be a polynomial with integral coefficients so that its degree is at least one in y. Next assume that $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$. Then the degree of $\eta \le dg$ and

$$h(\eta) \le 3^{2dg+(\ell_1+\ldots+\ell_k)g} H^g h(\alpha_1)^{\ell_1 g} \ldots h(\alpha_k)^{\ell_k g} \ ,$$

where $h(\eta)$ is the height of η, $h(\alpha_i)$ $(i = 1, 2, \dots, k)$ is the height of α_i $(i = 1, 2, \dots, k)$, H is the maximum of the absolute values of coefficients of F, ℓ_i $(i = 1, 2, \dots, k)$ is the degree of F in x_i $(i = 1, 2, \dots, k)$ and d is the degree of F in y. (O. Ş. İÇEN [2], p.25)

Lemma 2. Let α be an algebraic number of height h, then

$$|\alpha| \leq h + 1$$

(Schneider, Th. [11], p.5, Hilfssatz 1)

Lemma 3. Let $\alpha_1, \dots, \alpha_k$ $(k \geq 1)$ be p-adic algebraic numbers in p-adic number field $\mathbb{Q}_p$ of degree g, η be a p-adic algebraic number and $F(y, x_1, \dots, x_k)$ be a polynomial with integral coefficients so that its degree is at least one in y. Next assume that $F(\eta, \alpha_1, \dots, \alpha_k) = 0$. Then the degree of $\eta \leq dg$ and

$$h(\eta) \leq 3^{2dg + (\ell_1 + \dots + \ell_k)g} H^g h(\alpha_1)^{\ell_1 g} \dots h(\alpha_k)^{\ell_k g} \quad ,$$

where $h(\eta)$ is the height of η, $h(\alpha_i)$ $(i = 1, \dots, k)$ is the height of α_i $(i = 1, \dots, k)$, H is the maximum of the absolute values of coefficients of F, ℓ_i $(i = 1, \dots, k)$ is the degree of F in x_i $(i = 1, \dots, k)$ and d is the degree of F in y. (Orhan Ş. İÇEN [2], p.25)

Lemma 4. Let $P(x)$ be a polynomial with integral coefficients, $\alpha \in \mathbb{Q}_p$ and $P(\alpha) = 0$. Then

$$|\alpha|_p \geq H(P)^{-1} \quad ,$$

where $H(P)$ is the height of $P(x)$. (J.F. Morrison [7], p.337)

Theorem (Baker). Let ξ be a real or complex number, $\chi > 2$ and $\alpha_1, \alpha_2, \dots$ be a sequence of distinct numbers in an algebraic number field K with field heights $H_K(\alpha_1), H_K(\alpha_2), \dots$ such that for each i

$$|\xi - \alpha_i| < (H_K(\alpha_i))^{-\chi} \qquad (i)$$

and

$$\limsup_{i \to \infty} \frac{\log H_K(\alpha_{i+1})}{\log H_K(\alpha_i)} < +\infty \quad . \qquad (ii)$$

Then ξ is either an S-number or a T-number. (Baker, A. [1], p.98, Theorem 1)

THEOREMS

Theorem 1 . *Let*

$$f(x) \;=\; \sum_{n=0}^{\infty} c_{k_n} x^{k_n} \quad (k_n \in \mathbb{Z}^+ \; (n = 0, 1, 2, \dots); \;\; k_0 < k_1 < k_2 < \dots) \tag{1.1}$$

be a series with non-zero rational coefficients $c_{k_n} = b_{k_n}/a_{k_n}$ $(a_{k_n}, b_{k_n}$ *integers;* $b_{k_n} \neq 0,$ $a_{k_n} > 0$ *and* $a_{k_n} > 1$ *for* $n \geq N_0)$ *satisfying the following conditions*

$$\lim_{n\to\infty} \frac{\log a_{k_{n+1}}}{\log a_{k_n}} = +\infty\,, \tag{1.2}$$

$$\limsup_{n\to\infty} \frac{\log |b_{k_n}|}{\log a_{k_n}} < 1 \tag{1.3}$$

and

$$\lim_{n\to\infty} \frac{\log a_{k_n}}{k_n} = +\infty. \tag{1.4}$$

Furthermore let ξ *be a* U_m*-number for which the following two properties hold.*

1°) ξ *has an approximation with algebraic numbers* α_n *of degree* m *of an algebraic number field* K *so that the following holds for sufficiently large* n

$$|\xi - \alpha_n| < \frac{1}{H(\alpha_n)^{n\omega(n)}} \qquad (\lim_{n\to\infty} \omega(n) = +\infty), \tag{1.5}$$

where $[K : \mathbb{Q}\,] = m$.

2°) *There exist two real numbers* δ_1 *and* δ_2 *with* $1 < \delta_1 \leq \delta_2$ *and*

$$a_{k_n}^{\delta_1} \leq H(\alpha_{k_n})^{k_n} \leq a_{k_n}^{\delta_2} \tag{1.6}$$

for sufficiently large n.

Then $f(x)$ *converges for every complex number* x *and* $f(\xi)$ *is either a* U*-number of degree* $\leq m$ *or an algebraic number of* K.

Proof . 1) Since the sequence $\{a_{k_n}\}$ which satisfies the conditions above is strictly increasing for sufficiently large n, we have $\lim_{n\to\infty} a_{k_n} = +\infty$. Because from (1.2) we get

$$\log a_{k_{n+1}} > 2 \log a_{k_n} > \log a_{k_n}$$

for $n \geq N_1 \geq N_0$. Hence $a_{k_{n+1}} > a_{k_n}$, that is, the sequence $\{a_{k_n}\}$ is strictly increasing. Moreover,

$$\log a_{k_n} > \log a_{k_{N_1}} 2^{n-N_1}$$

for $n \geq N_1$. It holds $\lim_{n\to\infty} \log a_{k_n} = +\infty$, since $\lim_{n\to\infty} 2^n = +\infty$. Hence we get $\lim_{n\to\infty} a_{k_n} = +\infty$.

Let

$$\theta := \limsup_{n\to\infty} \frac{\log |b_{k_n}|}{\log a_{k_n}} .$$

From (1.3) and from $\theta < \frac{1+\theta}{2} < 1$, there exists a number $N_2 \in \mathbb{N}$ such that

$$\frac{\log |b_{k_n}|}{\log a_{k_n}} < \frac{1+\theta}{2}$$

holds for $n \geq N_2 \geq N_1$. Therefore we deduce

$$|b_{k_n}| < a_{k_n}^{\frac{1+\theta}{2}} . \tag{1.7}$$

Let x be a complex number. We can show by using the Ratio Test that $f(x)$ converges. Say

$$f(x) = \sum_{n=0}^{\infty} c_{k_n} x^{k_n} = \sum_{n=0}^{\infty} u_n$$

then from (1.2), (1.4) and (1.7) we have

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{b_{k_{n+1}}}{a_{k_{n+1}}} x^{k_{n+1}}}{\frac{b_{k_n}}{a_{k_n}} x^{k_n}}\right| \leq \frac{1}{a_{k_{n+1}}^{\varepsilon}}$$

for a suitable $\varepsilon > 0$. Therefore

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 0 < 1 .$$

Now we prove an inequality which we will use later. Let $A_{k_n} := [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$ and η be a constant such that $0 < \eta < 1 - (1/\delta_1)$. We have the inequality

$$A_{k_n} < K_0\, a_{k_n}^{\frac{1}{1-\eta}} \tag{1.8}$$

for $n \geq N_3 \geq N_2$ where $K_0 > 1$ is a suitable constant. Because from (1.2) we have

$$\frac{\log a_{k_{n+1}}}{\log a_{k_n}} > \frac{1}{\eta}$$

for $n \geq N_3 \geq N_2$ and so

$$a_{k_n} \quad < \quad a_{k_{n+1}}^{\eta}. \tag{1.9}$$

Let $K_0 := a_{k_0}a_{k_1}\dots a_{k_{N_3-1}}$. From (1.9) it follows that

$$\begin{aligned}
a_{k_{N_3}} &< a_{k_{N_3+1}}^{\eta} < a_{k_n}^{\eta^{n-N_3}} \\
a_{k_{N_3+1}} &< a_{k_n}^{\eta^{n-N_3-1}} \\
&\vdots \\
a_{k_{n-1}} &< a_{k_n}^{\eta}
\end{aligned}$$

for $n \geq N_3$. So we have

$$\begin{aligned}
A_{k_n} &\leq a_{k_0}a_{k_1}\dots a_{k_{N_3-1}}a_{k_{N_3}}\dots a_{k_n} \\
&\leq K_0\; a_{k_n}^{\eta^{n-N_3}+\eta^{n-N_3-1}+\dots+\eta+1} \\
&< K_0\; a_{k_n}^{\eta^n+\dots+\eta+1} \\
&< K_0\; a_{k_n}^{1/(1-\eta)}
\end{aligned}$$

which is the inequality (1.8).

2) We consider the polynomials

$$f_n(x) = \sum_{\nu=0}^{n} c_{k_\nu} x^{k_\nu} \qquad (n = 1, 2, 3, \dots).$$

Since

$$f_n(\alpha_{k_n}) = \sum_{\nu=0}^{n} c_{k_\nu}\alpha_{k_n}^{k_\nu} = c_{k_0}\alpha_{k_n}^{k_0} + c_{k_1}\alpha_{k_n}^{k_1} + \dots + c_{k_n}\alpha_{k_n}^{k_n} \in K \;\;,$$

we have $(f_n(\alpha_{k_n}))^{\circ} \leq m$. Now we can determine an upper bound for the height of $f_n(\alpha_{k_n})$. For this, we consider the polynomial

$$F(y,x) = A_{k_n}y - \sum_{\nu=0}^{n} A_{k_n}c_{k_\nu}x^{k_\nu}\;.$$

Since $F(y,x)$ is the polynomial with integral coefficients and

$$\begin{aligned}
F(f_n(\alpha_{k_n}),\alpha_{k_n}) &= A_{k_n}f_n(\alpha_{k_n}) - \sum_{\nu=0}^{n} A_{k_n}c_{k_\nu}\alpha_{k_n}^{k_\nu} \\
&= A_{k_n}f_n(\alpha_{k_n}) - A_{k_n}\sum_{\nu=0}^{n} c_{k_\nu}\alpha_{k_n}^{k_\nu} = 0
\end{aligned}$$

applying Lemma 1 we have

$$\begin{aligned} H(f_n(\alpha_{k_n})) &\leq 3^{2.1.m+k_n.m} H(F)^m H(\alpha_{k_n})^{k_n.m} \\ &\leq 3^{3k_n m}(A_{k_n} B_{k_n})^m H(\alpha_{k_n})^{k_n.m} \end{aligned}$$

where $B_{k_n} := \max_{\nu=0}^{n}\{|b_{k_\nu}|\}$. From (1.6) we get

$$H(f_n(\alpha_{k_n})) \leq 3^{3k_n m}(A_{k_n} B_{k_n})^m a_{k_n}^{\delta_2 m} .$$

Moreover we can write

$$H(f_n(\alpha_{k_n})) \leq c^{k_n m}(A_{k_n} B_{k_n})^m a_{k_n}^{\delta_2 m}$$

where $c = 3^3 > 1$ is a constant. Since the sequence $\{a_{k_n}\}$ is monotonically increasing and $\lim_{n\to\infty} a_{k_n} = +\infty$, it follows from (1.7)

$$B_{k_n} \leq a_{k_n}^{\frac{1+\theta}{2}} \tag{1.10}$$

for $n \geq N_4 \geq N_3$. From here using (1.8) we get

$$\begin{aligned} H(f_n(\alpha_{k_n})) &\leq c^{k_n m} K_0^m a_{k_n}^{\frac{m}{1-\eta}} a_{k_n}^{\frac{1+\theta}{2}m} a_{k_n}^{\delta_2 m} \\ &\leq c^{k_n m} K_0^{k_n m} a_{k_n}^{\left(\frac{1}{1-\eta}+\frac{1+\theta}{2}+\delta_2\right)m} \\ &= (c')^{k_n m} a_{k_n}^{m\gamma} \end{aligned}$$

for $n \geq N_4$ where $c' = cK_0 > 1$ and $\gamma = \frac{1}{1-\eta} + \frac{1+\theta}{2} + \delta_2$. From (1.4) we have

$$(c')^{k_n m} = e^{k_n m \log c'} \leq e^{m \log a_{k_n}} = a_{k_n}^m$$

for $n \geq N_5 \geq N_4$. Thus it holds for $n \geq N_5$

$$H(f_n(\alpha_{k_n})) \leq a_{k_n}^{m\gamma'} \tag{1.11}$$

where $\gamma' = 1 + \gamma$.

3) Since

$$\begin{aligned} |f(\xi) - f_n(\alpha_{k_n})| &= |f(\xi) - f_n(\xi) + f_n(\xi) - f_n(\alpha_{k_n})| \\ &\leq |f(\xi) - f_n(\xi)| + |f_n(\xi) - f_n(\alpha_{k_n})| \end{aligned}$$

we can determine an upper bound for $|f(\xi)-f_n(\xi)|$ and $|f_n(\xi)-f_n(\alpha_{k_n})|$. The following equality holds.

$$f_n(\xi) - f_n(\alpha_{k_n}) = \sum_{\nu=0}^{n} c_{k_\nu}\xi^{k_\nu} - \sum_{\nu=0}^{n} c_{k_\nu}\alpha_{k_n}^{k_\nu} \tag{1.12}$$
$$= \sum_{\nu=0}^{n} c_{k_\nu}(\xi^{k_\nu} - \alpha_{k_n}^{k_\nu})$$
$$= \sum_{\nu=0}^{n} c_{k_\nu}(\xi - \alpha_{k_n})(\xi^{k_\nu-1} + \xi^{k_\nu-2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu-1}) .$$

Moreover from (1.5) we have

$$|\alpha_{k_n}| \le |\xi| + 1$$

for $n \ge N_6 \ge N_5$. Thus using (1.5) and (1.12) we get

$$|f_n(\xi) - f_n(\alpha_{k_n})| \le |\xi - \alpha_{k_n}| \sum_{\nu=0}^{n} |c_{k_\nu}||\xi^{k_\nu-1} + \xi^{k_\nu-2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu-1}| \tag{1.13}$$
$$\le H(\alpha_{k_n})^{-k_n\omega(k_n)} \sum_{\nu=0}^{n} |c_{k_\nu}|k_\nu(|\xi|+1)^{k_\nu-1}$$

for $n \ge N_6$. Since

$$\sum_{\nu=0}^{n} |c_{k_\nu}|k_\nu(|\xi|+1)^{k_\nu-1} \le k_n^2 B_{k_n}(|\xi|+1)^{k_n-1}$$

using $\lim_{n\to\infty} \omega(k_n) = +\infty$, (1.4) and (1.10) we have

$$k_n^2 B_{k_n}(|\xi|+1)^{k_n-1} \le \frac{1}{2} a_{k_n}^{\delta_1 \frac{\omega(k_n)}{2}}$$

for $n \ge N_7 \ge N_6$. From this inequality, (1.6) and (1.13) it follows that

$$|f_n(\xi) - f_n(\alpha_{k_n})| \le \frac{1}{2} H(\alpha_{k_n})^{-k_n\omega(k_n)} a_{k_n}^{\delta_1\omega(k_n)/2}$$
$$\le \frac{1}{2} H(\alpha_{k_n})^{-k_n\omega(k_n)} H(\alpha_{k_n})^{k_n\omega(k_n)/2}$$
$$= \frac{1}{2} H(\alpha_{k_n})^{-k_n\omega(k_n)/2}$$

for $n \ge N_7$. Thus using (1.6) and (1.11) we deduce that there exists a suitable sequence $\{\omega_n^*\}$ with $\lim_{n\to+\infty} \omega_n^* = +\infty$ and

$$|f_n(\xi) - f_n(\alpha_{k_n})| \le \frac{1}{2} H(f_n(\alpha_{k_n}))^{-\omega_n^*} \tag{1.14}$$

for $n \geq N_8 \geq N_7$.

4) Now we can determine an upper bound for $|f(\xi) - f_n(\xi)|$. We have

$$|f(\xi) - f_n(\xi)| = \left|\sum_{\nu=1}^{\infty} c_{k_{n+\nu}} \xi^{k_{n+\nu}}\right| \leq \sum_{\nu=1}^{\infty} \frac{|b_{k_{n+\nu}}|}{a_{k_{n+\nu}}} |\xi|^{k_{n+\nu}} .$$

From (1.7) we get

$$\frac{|b_{k_n}|}{a_{k_n}} < \frac{1}{a_{k_n}^{(1-\theta)/2}}$$

for $n \geq N_8$. Thus it follows

$$\begin{aligned} |f(\xi) - f_n(\xi)| &\leq \frac{|b_{k_{n+1}}|}{a_{k_{n+1}}} |\xi|^{k_{n+1}} + \frac{|b_{k_{n+2}}|}{a_{k_{n+2}}} |\xi|^{k_{n+2}} + \ldots \\ &< \frac{|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}} \left[1 + \left(\frac{a_{k_{n+1}}}{a_{k_{n+2}}}\right)^{(1-\theta)/2} |\xi|^{k_{n+2}-k_{n+1}} + \ldots\right] \end{aligned}$$

for $n \geq N_8$. Hence from $(1-\theta)/2 > 0$, $\lim_{n\to\infty} \log a_{k_n} = +\infty$, (1.2) and (1.4) we have

$$\left(\frac{a_{k_{n+1}}}{a_{k_{n+2}}}\right)^{(1-\theta)/2} |\xi|^{k_{n+2}-k_{n+1}} < \frac{1}{2}$$

and

$$\left(\frac{a_{k_{n+1}}}{a_{k_{n+1+\nu}}}\right)^{(1-\theta)/2} |\xi|^{k_{n+1+\nu}-k_{n+1}} < \frac{1}{2^\nu} \quad (\nu = 1, 2, 3, \ldots)$$

for $n \geq N_9 \geq N_8$. So we get

$$\begin{aligned} |f(\xi) - f_n(\xi)| &\leq \frac{|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}} \left[1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^\nu} + \ldots\right] \\ &\leq \frac{2|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}} \end{aligned}$$

for $n \geq N_9$. From (1.4) we have

$$4|\xi|^{k_{n+1}} \leq a_{k_{n+1}}^{(1-\theta)/4}$$

and

$$|f(\xi) - f_n(\xi)| \leq \frac{1}{2} a_{k_{n+1}}^{-(1-\theta)/4} \tag{1.15}$$

for $n \geq N_{10} \geq N_9$. We define now $s'(n) := (\log a_{k_{n+1}} / \log a_{k_n})$. From (1.2) $\lim_{n\to\infty} s'(n) = +\infty$. Using (1.15) we have

$$|f(\xi) - f_n(\xi)| \leq \frac{1}{2} a_{k_n}^{-s'(n)(1-\theta)/4}$$

for $n \geq N_{10}$. Since $\lim_{n\to\infty} s'(n) = +\infty$, from (1.11) we deduce that there exists a suitable sequence $\{s(n)\}$ with $\lim_{n\to\infty} s(n) = +\infty$ and

$$\frac{1}{2} a_{k_n}^{-s'(n)(1-\theta)/4} \leq \frac{1}{2} H(f_n(\alpha_{k_n}))^{-s(n)} \tag{1.16}$$

for $n \geq N_{11} \geq N_{10}$. Let now $\omega_n^{**} := \min\{s(n), \omega_n^*\}$ for $n \geq N_{11}$. So from (1.14) and (1.16) it follows that

$$|f(\xi) - f_n(\alpha_{k_n})| \leq H(f_n(\alpha_{k_n}))^{-\omega_n^{**}} \tag{1.17}$$

for $n \geq N_{11}$ where $\lim_{n\to\infty} \omega_n^{**} = +\infty$. If the sequence $\{f_n(\alpha_{k_n})\}$ is constant then $f(\xi)$ is an algebraic number of K. Otherwise $f(\xi)$ is a U-number of degree $\leq m$.

Corollary . For $k_n = n$ and $m = 1$ from Theorem 1 we obtain Theorem 1 in [10] as a special case.

Example . Let α be a constant algebraic number of degree m and c be an integer with $c > 1$. We consider the number

$$\xi = \sum_{n=0}^{\infty} \frac{1}{c^{(n!)^2}} \alpha^n .$$

Because of Theorem 1 in [8] we know that ξ is a U_m-number. We consider now the algebraic numbers

$$\alpha_n = \sum_{\nu=0}^{n} \frac{1}{c^{(\nu!)^2}} \alpha^\nu \qquad (n = 1, 2, 3, \ldots) .$$

From Lemma 1 we obtain

$$H(\alpha_n) \leq c^{k(n!)^2} ,$$

where $k > 0$ is a constant. Furthermore we get

$$\begin{aligned} |\xi - \alpha_n| &\leq c^{-((n+1)!)^2 \varepsilon} \quad (\varepsilon > 0) \\ &\leq c^{-(n!)^2 (n+1)^2 \varepsilon} \\ &\leq (H(\alpha_n))^{-\frac{(n+1)^2 \varepsilon}{k}} \\ &\leq (H(\alpha_n))^{-n \frac{(n+1)^2 \varepsilon}{kn}} \end{aligned}$$

as we have done before. If $\omega_n = \frac{(n+1)^2\varepsilon}{kn}$ then $\omega_n \to \infty$ as $n \to \infty$. From here we have

$$|\xi - \alpha_n| \leq H(\alpha_n)^{-n\omega_n} \qquad (\lim_{n\to\infty} \omega_n = +\infty) \ . \tag{1.18}$$

This is the condition (1.5). Let now choose the sequences $\{a_{n_k}\}$ and $\{b_{n_k}\}$ so that the conditions (1.2), (1.3), (1.4) and (1.6) are satisfied. We define now $f(x)$ suitably. The degrees of the terms of the sequence $\{\alpha_n\}$ are bounded. Therefore we can construct a subsequence $\{\alpha_{n_k}\}$ of this sequence so that the terms of this subsequence are different from each other and the sequence $\{H(\alpha_{n_k})\}$ is strictly increasing. For this subsequence it holds

$$\limsup_{k\to\infty} \frac{\log H(\alpha_{n_{k+1}})}{\log H(\alpha_{n_k})} = +\infty \ . \tag{1.19}$$

Because if this lim sup was finite, from (ii) in Baker' s Theorem and from (1.18) the condition (i) would be satisfied and because of Baker' s Theorem ξ would be an S-number or a T-number. This would contradict the fact that ξ is a U_m-number. Hence (1.19) is true. On the other hand because of (1.19) there exists an index subsequence $\{n_{k_j}\}$ of the sequence $\{n_k\}$ such that

$$\lim_{j\to\infty} \frac{\log H(\alpha_{n_{k_j+1}})}{\log H(\alpha_{n_{k_j}})} = +\infty \ . \tag{1.20}$$

Since $\{H(\alpha_{n_k})\}$ is monotonically increasing, we have

$$\frac{\log H(\alpha_{n_{k_j+1}})}{\log H(\alpha_{n_{k_j}})} \leq \frac{\log H(\alpha_{n_{k_{j+1}}})}{\log H(\alpha_{n_{k_j}})} \ .$$

From here using (1.20) we get

$$\lim_{j\to\infty} \frac{\log H(\alpha_{n_{k_{j+1}}})}{\log H(\alpha_{n_{k_j}})} = +\infty \ . \tag{1.21}$$

Let

$$a_{n_{k_j}} := H(\alpha_{n_{k_j}})^{\left[\!\left[\frac{n_{k_j}}{2}\right]\!\right]} \qquad (j = 1, 2, 3, \ldots)$$

where $[\![x]\!]$ denotes the integral part of x. For the sequence $\{a_{n_{k_j}}\}$ we show that the condition (1.6) is satisfied for $\delta_1 = 2$, $\delta_2 = 3$. It is clear that

$$a_{n_{k_j}}^2 = H(\alpha_{n_{k_j}})^{\left[\!\left[\frac{n_{k_j}}{2}\right]\!\right]2} \leq H(\alpha_{n_{k_j}})^{n_{k_j}} \leq a_{n_{k_j}}^3 \ .$$

Because it holds

$$\left[\!\left[\frac{n_{k_j}}{2}\right]\!\right] 2 \leq \frac{n_{k_j}}{2} 2 = n_{k_j}$$

and on the other hand

$$\frac{n_{k_j}}{3} \le \frac{n_{k_j}}{2} - 1 < \left[\left[\frac{n_{k_j}}{2}\right]\right]$$

for $n_{k_j} \geq 6$. Thus we have

$$n_{k_j} \le 3\left[\left[\frac{n_{k_j}}{2}\right]\right] .$$

Now we show that the condition (1.2) is satisfied. From (1.21) we obtain

$$\frac{\log a_{n_{k_{j+1}}}}{\log a_{n_{k_j}}} = \frac{\left[\left[\frac{n_{k_{j+1}}}{2}\right]\right] \log H(\alpha_{n_{k_{j+1}}})}{\left[\left[\frac{n_{k_j}}{2}\right]\right] \log H(\alpha_{n_{k_j}})} \to +\infty$$

as $j \to \infty$, since

$$\left[\left[\frac{n_{k_{j+1}}}{2}\right]\right] \ge \left[\left[\frac{n_{k_j}}{2}\right]\right]$$

and $H(\alpha_{n_{k_j}})$ is monotonically increasing to infinity as $j \to \infty$. Furthermore since

$$\lim_{j\to\infty} \frac{\left[\left[\frac{n_{k_j}}{2}\right]\right]}{n_{k_j}} = \frac{1}{2}$$

we obtain

$$\lim_{j\to\infty} \frac{\log a_{n_{k_j}}}{n_{k_j}} = \lim_{j\to\infty} \frac{\left[\left[\frac{n_{k_j}}{2}\right]\right] \log H(\alpha_{n_{k_j}})}{n_{k_j}} = +\infty .$$

From here we have the condition (1.4). For $b_{n_{k_j}} = 1$ $(j = 0, 1, 2, \ldots)$ the condition (1.3) is satisfied. Thus the conditions of Theorem 1 are satisfied for ξ and

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{a_{n_{k_j}}} x^{n_{k_j}} .$$

Therefore either $\mu(f(\xi)) \le m$ or $f(\xi)$ belongs to K. Using the above ideas it is possible to construct many other ξ and $f(x)$ so that the conditions of Theorem 1 are satisfied.

Theorem 2. *Let*

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_{k_n}}{a_{k_n}} x^{k_n} \qquad (k_n \in \mathbb{Z}^+ \quad (n = 0, 1, 2, \ldots) \ ; \ k_0 < k_1 < k_2 < \ldots) \quad (2.1)$$

be a series with non-zero algebraic integer η_{k_n} $(n = 0, 1, 2, \ldots)$ of a number field K of degree q and with positive integers a_{k_n} $(a_{k_n} > 1$ for $n \ge N_0)$ satisfying the following conditions

$$\lim_{n\to\infty} \frac{\log a_{k_{n+1}}}{\log a_{k_n}} = +\infty , \qquad (2.2)$$

$$\limsup_{n\to\infty} \frac{\log H(\eta_{k_n})}{\log a_{k_n}} < 1 \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{\log a_{k_n}}{k_n} = +\infty \ , \tag{2.4}$$

where $H(\eta_{k_n})$ $(n = 0, 1, 2, \ldots)$ *is the height of* η_{k_n} $(n = 0, 1, 2, \ldots)$. *Furthermore let* ξ *be a* U_m*-number for which the following two properties hold.*

1°) ξ *has an approximation with algebraic numbers* α_n *of degree* m *of an algebraic number field* L *so that the following holds for sufficiently large* n

$$|\xi - \alpha_n| < \frac{1}{H(\alpha_n)^{n\omega(n)}} \qquad (\lim_{n\to\infty} \omega(n) = +\infty) \,, \tag{2.5}$$

where $[L : \mathbb{Q}\,] = m$.

2°) *There exist two real numbers* c_1 *and* c_2 *with* $1 < c_1 \leq c_2$ *and*

$$a_{k_n}^{c_1} \leq H(\alpha_{k_n})^{k_n} \leq a_{k_n}^{c_2} \tag{2.6}$$

for sufficiently large n. *Let* M *be a smallest number field which contains* K *and* L *with* $[M : \mathbb{Q}\,] = t$.

Then $f(x)$ *converges for every complex number* x *and* $f(\xi)$ *is either a* U*-number of degree* $\leq t$ *or an algebraic number of* M.

Proof . 1) Since the sequence $\{a_{k_n}\}$ which satisfies the conditions above is strictly increasing for sufficiently large n, we have $\lim_{n\to\infty} a_{k_n} = +\infty$. Because from (2.2) we have

$$\log a_{k_{n+1}} > 2 \log a_{k_n} > \log a_{k_n}$$

for $n \geq N_1 \geq N_0$. Hence $a_{k_{n+1}} > a_{k_n}$, that is, the sequence $\{a_{k_n}\}$ is strictly increasing. Moreover,

$$\log a_{k_n} > \log a_{k_{N_1}} 2^{n-N_1}$$

for $n \geq N_1$. It holds $\lim_{n\to\infty} \log a_{k_n} = +\infty$, since $\lim_{n\to\infty} 2^n = +\infty$. Thus we get $\lim_{n\to\infty} a_{k_n} = +\infty$.

Let

$$\theta := \limsup_{n\to\infty} \frac{\log H(\eta_{k_n})}{\log a_{k_n}} \ .$$

From (2.3) and from $\theta < \frac{1+\theta}{2} < 1$, there exists a number $N_2 \in \mathbb{N}$ such that

$$\frac{\log H(\eta_{k_n})}{\log a_{k_n}} < \frac{1+\theta}{2}$$

holds for $n \geq N_2 \geq N_1$. Thus we deduce

$$H(\eta_{k_n}) < a_{k_n}^{\frac{1+\theta}{2}} \tag{2.7}$$

for $n \geq N_2$. Applying Lemma 2 we have

$$|\eta_{k_n}| \leq H(\eta_{k_n}) + 1 \leq 2H(\eta_{k_n}) < 2a_{k_n}^{\frac{1+\theta}{2}} \ . \tag{2.8}$$

Let x be a complex number. We can show by using the Ratio Test that $f(x)$ converges. Say

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_{k_n}}{a_{k_n}} x^{k_n} = \sum_{n=0}^{\infty} u_n$$

then from (2.2), (2.4) and (2.8) we have

$$\left|\frac{u_{n+1}}{u_n}\right| \leq \frac{1}{a_{k_{n+1}}^{\varepsilon_0}}$$

for a suitable $\varepsilon_0 > 0$. Therefore

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 0 < 1 \ .$$

Now we prove an inequality which we will use later. Let $A_{k_n} := [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$ and let η be a constant such that $0 < \eta < 1 - (1/c_1)$. We have the inequality

$$A_{k_n} \leq a_{k_0} \ldots a_{k_n} \leq a_{k_n}^{\varepsilon + (\frac{1}{1-\eta})} \tag{2.9}$$

for $n \geq N_3 \geq N_2$ where $0 < \varepsilon < c_1 - 1/(1-\eta)$. From (2.2) we have

$$\frac{\log a_{k_{n+1}}}{\log a_{k_n}} > \frac{1}{\eta}$$

for $n \geq N_3$ and so

$$a_{k_n} \ < \ a_{k_{n+1}}^{\eta} \ . \tag{2.10}$$

Let $K_0 := a_{k_0} a_{k_1} \ldots a_{k_{N_3 - 1}}$. From (2.10) it follows

$$\begin{aligned} a_{k_{N_3}} &< a_{k_{N_3+1}}^{\eta} < a_{k_n}^{\eta^{n-N_3}} \\ a_{k_{N_3+1}} &< a_{k_n}^{\eta^{n-N_3-1}} \\ &\vdots \\ a_{k_{n-1}} &< a_{k_n}^{\eta} \end{aligned}$$

for $n \geq N_3$. Thus we have

$$\begin{aligned} A_{k_n} &\leq a_{k_0}a_{k_1}\dots a_{k_{N_3-1}}a_{k_{N_3}}\dots a_{k_n} \\ &\leq K_0\ a_{k_n}^{\eta^{n-N_3}+\eta^{n-N_3-1}+\dots+\eta+1} \\ &< K_0\ a_{k_n}^{\eta^n+\dots+\eta+1} \\ &< K_0\ a_{k_n}^{1/(1-\eta)} \end{aligned}$$

for $n \geq N_3$. Since $\lim_{n\to\infty} a_{k_n} = +\infty$, it follows

$$K_0 \leq a_{k_n}^{\varepsilon}$$

for sufficiently large n. Thus we have inequality (2.9).

2) We consider the polynomials

$$f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} x^{k_\nu} \qquad (n = 1, 2, 3, \dots) \ .$$

Let

$$\gamma_n := \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \alpha_{k_n}^{k_\nu} = f_n(\alpha_{k_n}) \ .$$

Since $\gamma_n \in M \quad (n = 1, 2, 3, \dots)$, we have $(\gamma_n)^\circ \leq t \quad (n = 1, 2, 3, \dots)$. Now we can determine an upper bound for the height of γ_n. For this, we consider the polynomial

$$F(y, x_0, x_1, \dots, x_n, x_{n+1}) = A_{k_n} y - \sum_{\nu=0}^{n} \frac{A_{k_n}}{a_{k_\nu}} x_\nu\, x_{n+1}^{k_\nu} \ .$$

Since $F(y, x_0, x_1, \dots, x_n, x_{n+1})$ is the polynomial with integral coefficients and

$$\begin{aligned} F(\gamma_n, \eta_{k_0}, \eta_{k_1}, \dots, \eta_{k_n}, \alpha_{k_n}) &= A_{k_n}\gamma_n - \sum_{\nu=0}^{n} \frac{A_{k_n}}{a_{k_\nu}} \eta_{k_\nu} \alpha_{k_n}^{k_\nu} \\ &= A_{k_n}\gamma_n - A_{k_n} \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \alpha_{k_n}^{k_\nu} = 0 \ , \end{aligned}$$

applying Lemma 1 we have

$$H(\gamma_n) \ \leq \ 3^{2.t.1+[(1+1+\dots+1)+k_n]t} H^t H(\eta_{k_0})^t \dots H(\eta_{k_n})^t H(\alpha_{k_n})^{k_n.t}$$

where H is the height of the polynomial $F(y, x_0, x_1, \dots, x_n, x_{n+1})$, $g = t$, $d = 1$, $\ell_0 = 1, \dots, \ell_n = 1$, $\ell_{n+1} = k_n$. Since $H = \max_{\nu=0}^{n} \left\{ A_{k_n} , \frac{A_{k_n}}{a_{k_\nu}} \right\} = A_{k_n}$, using (2.6) we get

$$\begin{aligned} H(\gamma_n) &\leq 3^{2t+3k_n t} A_{k_n}^t H(\eta_{k_0})^t \dots H(\eta_{k_n})^t H(\alpha_{k_n})^{k_n.t} \\ &\leq 3^{5k_n t} A_{k_n}^t H(\eta_{k_0})^t \dots H(\eta_{k_n})^t a_{k_n}^{c_2 t} \end{aligned}$$

for $n \geq N_3$.

Let $K_1 := H(\eta_{k_0}) \dots H(\eta_{k_{N_3-1}})$. From (2.7) it follows that

$$\begin{aligned} H(\eta_{k_0}) \dots H(\eta_{k_n}) &\leq K_1 (a_{k_{N_3}} \dots a_{k_n})^{(1+\theta)/2} \\ &\leq K_1 (a_{k_0} a_{k_1} \dots a_{k_n})^{(1+\theta)/2} \end{aligned}$$

for $n \geq N_3$. Thus using (2.9) we have

$$\begin{aligned} H(\gamma_n) &\leq c^{k_n t} A_{k_n}^t (a_{k_0} a_{k_1} \dots a_{k_n})^{t(1+\theta)/2} a_{k_n}^{c_2 t} \\ &\leq c^{k_n t} (a_{k_0} a_{k_1} \dots a_{k_n})^{t(1+\theta)/2+t} a_{k_n}^{c_2 t} \\ &\leq c^{k_n t} a_{k_n}^{[\varepsilon + (1/(1-\eta))][t(1+\theta)/2+t]} a_{k_n}^{c_2 t} \\ &= c^{k_n t} a_{k_n}^{[\varepsilon + (1/(1-\eta))][t(1+\theta)/2+t]+c_2 t} \\ &= c^{k_n t} a_{k_n}^{\gamma t} \end{aligned}$$

where $\gamma = [\varepsilon + (1/(1-\eta))][(1+\theta)/2+1] + c_2$ and $c > 1$ is a suitable constant. On the other hand from (2.4) we obtain

$$c^{k_n t} = e^{k_n t \log c} \leq e^{t \log a_{k_n}} = a_{k_n}^t$$

for $n \geq N_4 \geq N_3$. Thus we have

$$H(\gamma_n) \leq a_{k_n}^{t\gamma'} \tag{2.11}$$

for $n \geq N_4$ where $\gamma' = 1 + \gamma$.

3) Now we can determine an upper bound for $|f(\xi) - \gamma_n|$. Since

$$\begin{aligned} |f(\xi) - \gamma_n| &= |f(\xi) - f_n(\xi) + f_n(\xi) - \gamma_n| \\ &\leq |f(\xi) - f_n(\xi)| + |f_n(\xi) - \gamma_n| \ , \end{aligned}$$

we must determine an upper bound for $|f(\xi) - f_n(\xi)|$ and $|f_n(\xi) - \gamma_n|$. We have

$$|f(\xi) - f_n(\xi)| = \left| \sum_{\nu=n+1}^{\infty} \frac{\eta_{k_\nu}}{a_{k_\nu}} \xi^{k_\nu} \right| \leq \sum_{\nu=n+1}^{\infty} \frac{|\eta_{k_\nu}|}{a_{k_\nu}} |\xi|^{k_\nu}$$

and from (2.8)

$$\frac{|\eta_{k_n}|}{a_{k_n}} \leq \frac{2a_{k_n}^{(1+\theta)/2}}{a_{k_n}} = 2a_{k_n}^{(\theta-1)/2}$$

for $n \geq N_4$. Thus it follows that

$$\begin{aligned}|f(\xi)-f_n(\xi)| &\le \sum_{\nu=n+1}^{\infty}\frac{|\eta_{k_\nu}|}{a_{k_\nu}}|\xi|^{k_\nu} \le \sum_{\nu=n+1}^{\infty} 2a_{k_\nu}^{(\theta-1)/2}|\xi|^{k_\nu}\\ &= \frac{2}{a_{k_{n+1}}^{(1-\theta)/2}}|\xi|^{k_{n+1}} + \frac{2}{a_{k_{n+2}}^{(1-\theta)/2}}|\xi|^{k_{n+2}} + \dots\\ &= \frac{2|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}}\left[1+\left(\frac{a_{k_{n+1}}}{a_{k_{n+2}}}\right)^{(1-\theta)/2}|\xi|^{k_{n+2}-k_{n+1}}+\dots\right]\end{aligned}$$

for $n \ge N_4$. Hence from $(1-\theta)/2 > 0$, $\lim_{n\to\infty}\log a_{k_n} = +\infty$, (2.2) and (2.4) we can obtain

$$\left(\frac{a_{k_{n+1}}}{a_{k_{n+1+\nu}}}\right)^{(1-\theta)/2}|\xi|^{k_{n+1+\nu}-k_{n+1}} < \frac{1}{2^\nu} \quad (\nu = 1,2,3,\dots)$$

for $n \ge N_5 \ge N_4$. From here we have

$$\begin{aligned}|f(\xi)-f_n(\xi)| &\le \frac{2|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}}\left[1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^\nu}+\dots\right]\\ &\le \frac{4|\xi|^{k_{n+1}}}{a_{k_{n+1}}^{(1-\theta)/2}}\end{aligned}$$

for $n \ge N_5$. From (2.4) it follows that

$$8|\xi|^{k_{n+1}} \le a_{k_{n+1}}^{(1-\theta)/4}$$

and here also

$$|f(\xi)-f_n(\xi)| \le \frac{1}{2}a_{k_{n+1}}^{-(1-\theta)/4} \tag{2.12}$$

for $n \ge N_6 \ge N_5$. We define now $s'(n) := (\log a_{k_{n+1}}/\log a_{k_n})$. From (2.2) $\lim_{n\to\infty} s'(n) = +\infty$. Using (2.12) we have

$$|f(\xi)-f_n(\xi)| \le \frac{1}{2}a_{k_n}^{-s'(n)(1-\theta)/4} \tag{2.13}$$

for $n \ge N_6$. Since $\lim_{n\to\infty} s'(n) = +\infty$, from (2.11) we deduce that there exists a suitable sequence $\{s(n)\}$ with $\lim_{n\to\infty} s(n) = +\infty$ and

$$\frac{1}{2}a_{k_n}^{-s'(n)(1-\theta)/4} \le \frac{1}{2}H(\gamma_n)^{-s(n)}$$

for $n \geq N_7 \geq N_6$. From here using (2.13) we have

$$|f(\xi) - f_n(\xi)| \leq \frac{1}{2} H(\gamma_n)^{-s(n)} \tag{2.14}$$

for $n \geq N_7$.

4) Now we can determine an upper bound for $|f_n(\xi) - \gamma_n)|$. The following equalities hold.

$$\begin{aligned} f_n(\xi) - \gamma_n &= \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \xi^{k_\nu} - \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \alpha_{k_n}^{k_\nu} \\ &= \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} (\xi^{k_\nu} - \alpha_{k_n}^{k_\nu}) \\ &= \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} (\xi - \alpha_{k_n})(\xi^{k_\nu - 1} + \xi^{k_\nu - 2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu - 1}) \ . \end{aligned} \tag{2.15}$$

From (2.5) we have

$$|\alpha_{k_n}| \leq |\xi| + 1$$

for $n \geq N_8 \geq N_7$. Thus using (2.5) and (2.15) we get

$$\begin{aligned} |f_n(\xi) - \gamma_n| &\leq |\xi - \alpha_{k_n}| \sum_{\nu=0}^{n} \frac{|\eta_{k_\nu}|}{a_{k_\nu}} |\xi^{k_\nu - 1} + \xi^{k_\nu - 2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu - 1}| \\ &\leq H(\alpha_{k_n})^{-k_n \omega(k_n)} \sum_{\nu=0}^{n} \frac{|\eta_{k_\nu}|}{a_{k_\nu}} k_\nu (|\xi| + 1)^{k_\nu - 1} \end{aligned} \tag{2.16}$$

for $n \geq N_8$. Moreover we can obtain that

$$\sum_{\nu=0}^{n} \frac{|\eta_{k_\nu}|}{a_{k_\nu}} k_\nu (|\xi| + 1)^{k_\nu - 1} \leq k_n^2 \beta_{k_n} (|\xi| + 1)^{k_n - 1} \tag{2.17}$$

where $\beta_{k_n} := \max_{\nu=0}^{n} |\eta_{k_\nu}|$. Since the sequence $\{a_{k_n}\}$ is monotonically increasing and $\lim_{n \to \infty} a_{k_n} = +\infty$, from (2.8) it follows that

$$\beta_{k_n} \leq 2a_{k_n}^{(1+\theta)/2}$$

for $n \geq N_9 \geq N_8$. Thus we have

$$k_n^2 \beta_{k_n} (|\xi| + 1)^{k_n - 1} \leq 2{k_n}^2 (|\xi| + 1)^{k_n - 1} a_{k_n}^{(1+\theta)/2}$$

for $n \geq N_9$. From (2.16) and (2.17) we obtain that

$$|f_n(\xi) - \gamma_n| \leq 2H(\alpha_{k_n})^{-k_n \omega(k_n)} {k_n}^2 (|\xi| + 1)^{k_n - 1} a_{k_n}^{(1+\theta)/2}$$

for $n \geq N_9$. Then using (2.6) it follows that

$$|f_n(\xi) - \gamma_n| \leq \frac{2k_n^2(|\xi|+1)^{k_n-1}}{a_{k_n}^{c_1\omega(k_n)-(1+\theta)/2}} . \tag{2.18}$$

for sufficiently large n. Using (2.4) and $\lim_{n\to\infty} \omega(k_n) = +\infty$ we deduce that there exists a suitable sequence $\{s''(n)\}$ with $\lim_{n\to\infty} s''(n) = +\infty$ and

$$\frac{2k_n^2(|\xi|+1)^{k_n-1}}{a_{k_n}^{c_1\omega(k_n)-(1+\theta)/2}} \leq \frac{1}{2}(a_{k_n}^{t\gamma'})^{-s''(n)} \tag{2.19}$$

for $n \geq N_{10} \geq N_9$. From (2.11), (2.18) and (2.19) we have

$$|f_n(\xi) - \gamma_n| \leq \frac{1}{2}H(\gamma_n)^{-s''(n)} \tag{2.20}$$

for $n \geq N_{10}$. Let now $s'''(n) := \min\{s''(n), s(n)\}$ for $n \geq N_{10}$. Thus from (2.14) and (2.20) it follows that

$$|f(\xi) - \gamma_n| \leq H(\gamma_n)^{-s'''(n)} \tag{2.21}$$

for $n \geq N_{10}$ where $\lim_{n\to\infty} s'''(n) = +\infty$.

If the sequence $\{\gamma_n\}$ is constant then $f(\xi)$ is an algebraic number of M. Otherwise $f(\xi)$ is a U-number of degree $\leq t$.

Corollary . For $k_n = n$ and $t = 1$ from Theorem 2 we obtain Theorem **3** in [10] as a special case.

Example . Let α be a constant algebraic number of degree m and c be an integer with $c > 1$. We consider the number

$$\xi = \sum_{n=0}^{\infty} \frac{1}{c^{(n!)^2}}\alpha^n .$$

Because of Theorem 1 in [8] ξ is a U_m-number. We consider now the algebraic numbers

$$\alpha_n = \sum_{\nu=0}^{n} \frac{1}{c^{(\nu!)^2}}\alpha^\nu \qquad (n = 1, 2, 3, \ldots) .$$

From Lemma 1 we obtain

$$H(\alpha_n) \leq c^{k(n!)^2}$$

where $k > 0$ is a constant. From the above we get

$$|\xi - \alpha_n| \leq (H(\alpha_n))^{-n\omega_n} \qquad (\omega_n = \frac{(n+1)^2\varepsilon}{kn} \to \infty) .$$

This is the condition (2.5). We can now choose the sequence $\{a_{n_k}\}$ and $\{\eta_{n_k}\}$ so that the conditions (2.2), (2.3), (2.4) and (2.6) are satisfied. As in the example of Theorem 1 we can construct a subsequence $\{\alpha_{n_{k_j}}\}$ of the sequence $\{\alpha_n\}$ so that the terms of this subsequence are different from each other and for the sequence $\{H(\alpha_{n_{k_j}})\}$ the conditions (1.19), (1.20) and (1.21) are satisfied.

Let

$$a_{n_{k_j}} := H(\alpha_{n_{k_j}})^{\left\|\left[\frac{n_{k_j}}{2}\right]\right\|} \qquad (j = 1, 2, 3, \dots)$$

and β be a constant algebraic integer of a number field K of degree q. If

$$\eta_{n_{k_j}} = \beta^{n_{k_j}} \qquad (j = 1, 2, 3, \dots)$$

the conditions (2.2), (2.3), (2.4) and (2.6) are satisfied for $c_1 = 2, \quad c_2 = 3$. So the conditions of Theorem 2 hold for ξ and

$$f(x) = \sum_{j=0}^{\infty} \frac{\beta^{n_{k_j}}}{a_{n_{k_j}}} x^{n_{k_j}} \ .$$

Therefore either $\mu(f(\xi)) \leq t$ or $f(\xi)$ belongs to a smallest number field which contains K and $\mathbb{Q}(\alpha)$.

Theorem 3 . *In the p-adic field* $\mathbb{Q}_p$*, let*

$$f(x) \;=\; \sum_{n=0}^{\infty} c_{k_n} x^{k_n} \qquad (k_n \in \mathbb{Z}^+ \ (n = 0, 1, 2, \dots);\ k_0 < k_1 < k_2 < \dots) \qquad (3.1)$$

be a series with non-zero rational coefficients $c_{k_n} = b_{k_n}/a_{k_n}$ $(a_{k_n}, \quad b_{k_n}$ *integers;* $b_{k_n} \neq 0$, $a_{k_n} > 0$, $(a_{k_n}, b_{k_n}) = 1$ *and* $a_{k_n} > 1$ *for* $n \geq N_0)$ *satisfying the following conditions*

$$\lim_{n\to\infty} \frac{u_{k_{n+1}}}{u_{k_n}} = +\infty\,, \qquad (3.2)$$

$$0 \leq \limsup_{n\to\infty} \frac{\log_p A_{k_n} B_{k_n}}{u_{k_n}} < \infty \qquad (3.3)$$

and

$$\lim_{n\to\infty} \frac{u_{k_n}}{k_n} = +\infty \qquad (3.4)$$

where $|c_{k_n}|_p = p^{-u_{k_n}}$, $A_{k_n} = [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$, $B_{k_n} = \max_{\nu=0}^{n} |b_{k_\nu}|$. *Furthermore let* ξ *be a p-adic* U_m*-number for which the following two properties hold.*

1°) ξ *has an approximation with p-adic algebraic numbers* α_n *of degree* m *of a p-adic algebraic number field* K *so that the following holds for sufficiently large* n.

$$|\xi - \alpha_n|_p \leq H(\alpha_n)^{-n\omega(n)} \qquad (\lim_{n\to\infty} \omega(n) = +\infty) \ , \tag{3.5}$$

where $[K : \mathbb{Q}] = m$.

2°) *There exist two real numbers* δ_1 *and* δ_2 *with* $1 < \delta_1 \leq \delta_2$ *and*

$$p^{u_{k_n}\delta_1} \leq H(\alpha_{k_n})^{k_n} \leq p^{u_{k_n}\delta_2} \tag{3.6}$$

for sufficiently large n *where* $H(\alpha_{k_n})$ $(n = 0, 1, 2, \ldots)$ *is the height of* α_{k_n} $(n = 0, 1, 2, \ldots)$.

Then the radius of convergence of $f(x)$ *is infinity and* $f(\xi)$ *is either a p-adic U-number of degree* $\leq m$ *or a p-adic algebraic number of* K.

Proof. 1) Since

$$r = \frac{1}{\limsup\limits_{k_n\to\infty} \sqrt[k_n]{|c_{k_n}|_p}} = \frac{1}{\limsup\limits_{k_n\to\infty} p^{-\frac{u_{k_n}}{k_n}}} = \liminf_{k_n\to\infty} p^{\frac{u_{k_n}}{k_n}} = +\infty \ ,$$

it follows that the radius of convergence of $f(x)$ is infinity. We consider the polynomials

$$f_n(x) = \sum_{\nu=0}^{n} c_{k_\nu} x^{k_\nu} \quad (n = 1, 2, \ldots) \ .$$

Since

$$f_n(\alpha_{k_n}) = \sum_{\nu=0}^{n} c_{k_\nu} \alpha_{k_n}^{k_\nu} = c_{k_0}\alpha_{k_n}^{k_0} + c_{k_1}\alpha_{k_n}^{k_1} + \ldots + c_{k_n}\alpha_{k_n}^{k_n} \in K,$$

we have $(f_n(\alpha_{k_n}))^\circ \leq m$. Now we can determine an upper bound for the height of $f_n(\alpha_{k_n})$. For this, we consider the polynomial

$$F(y, x) = A_{k_n} y - \sum_{\nu=0}^{n} A_{k_n} c_{k_\nu} x^{k_\nu} \ .$$

Since $F(y,x)$ is the polynomial with integral coefficients and

$$F(f_n(\alpha_{k_n}),\alpha_{k_n}) = A_{k_n} f_n(\alpha_{k_n}) - \sum_{\nu=0}^{n} A_{k_n} c_{k_\nu} \alpha_{k_n}^{k_\nu} = 0 \ ,$$

applying Lemma 3 we have

$$\begin{aligned} H(f_n(\alpha_{k_n})) &\le 3^{2.1.m+k_n.m} H(F)^m H(\alpha_{k_n})^{k_n.m} \\ &\le 3^{3k_n m} (A_{k_n} B_{k_n})^m H(\alpha_{k_n})^{k_n.m} \ . \end{aligned}$$

Thus using (3.6) we get

$$H(f_n(\alpha_{k_n})) \le 3^{3k_n m} (A_{k_n} B_{k_n})^m p^{u_{k_n}.m.\delta_2} \ .$$

Moreover we can write

$$H(f_n(\alpha_{k_n})) \le c_1^{k_n m} (A_{k_n} B_{k_n})^m p^{u_{k_n}.m.\delta_2} \tag{3.7}$$

where $c_1 > 1$ is a constant.

Let $\theta := \limsup_{n\to\infty} \frac{\log_p A_{k_n} B_{k_n}}{u_{k_n}}$. From (3.3) there exists a number $N_1 \in \mathbb{N}$ such that

$$\frac{\log_p A_{k_n} B_{k_n}}{u_{k_n}} < \frac{1+\theta}{2}$$

for $n \ge N_1 \ge N_0$. Thus we have

$$(A_{k_n} B_{k_n})^m < p^{c_2 u_{k_n}} \tag{3.8}$$

for $n \ge N_1$ where $c_2 = \frac{1+\theta}{2} m$. From (3.4) we obtain

$$c_1^{k_n m} = p^{k_n m \log_p c_1} \le p^{m u_{k_n}} \tag{3.9}$$

for $n \ge N_2 \ge N_1$. Combining (3.7), (3.8) and (3.9) it follows that

$$H(f_n(\alpha_{k_n})) \le p^{c_3 u_{k_n}} \tag{3.10}$$

for $n \ge N_2$ where $c_3 = c_2 + m + m\delta_2$.

2) It holds that

$$\begin{aligned} |f(\xi) - f_n(\alpha_{k_n})|_p &= |f(\xi) - f_n(\xi) + f_n(\xi) - f_n(\alpha_{k_n})|_p \\ &\le \max\{|f(\xi) - f_n(\xi)|_p, |f_n(\xi) - f_n(\alpha_{k_n})|_p\} \ . \end{aligned} \tag{3.11}$$

We can determine an upper bound for $|f(\xi) - f_n(\xi)|_p$ and $|f_n(\xi) - f_n(\alpha_{k_n})|_p$. It holds

$$\begin{aligned} |f(\xi) - f_n(\xi)|_p &= |\sum_{\nu=n+1}^{\infty} c_{k_\nu}\xi^{k_\nu}|_p \\ &\leq \max\{|c_{k_{n+1}}|_p|\xi|_p^{k_{n+1}}, |c_{k_{n+2}}|_p|\xi|_p^{k_{n+2}}, \ldots\} . \end{aligned}$$

We can find an upper bound for $|c_{k_n}\xi^{k_n}|_p$ as follows

$$|c_{k_n}\xi^{k_n}|_p = |c_{k_n}|_p|\xi|_p^{k_n} = p^{-u_{k_n}+k_n \log_p|\xi|_p} .$$

From (3.4) we have

$$u_{k_n}/2 \leq u_{k_n} - k_n \log_p|\xi|_p$$

and

$$|c_{k_n}\xi^{k_n}|_p \leq p^{-u_{k_n}/2}$$

for $n \geq N_3 \geq N_2$. According to (3.2), since the sequence $\{u_{k_n}\}$ is monotonically increasing for sufficiently large n we obtain

$$|f(\xi) - f_n(\xi)|_p \leq \max\{p^{-u_{k_{n+1}}/2}, p^{-u_{k_{n+2}}/2}, \ldots\} = p^{-u_{k_{n+1}}/2} \tag{3.12}$$

for $n \geq N_4 \geq N_3$.

3) We have

$$\begin{aligned} |f_n(\xi) - f_n(\alpha_{k_n})|_p &= \left|\sum_{\nu=0}^{n} c_{k_\nu}(\xi^{k_\nu} - \alpha_{k_n}^{k_\nu})\right|_p \leq \max_{\nu=0}^{n} |c_{k_\nu}(\xi^{k_\nu} - \alpha_{k_n}^{k_\nu})|_p \qquad (3.13) \\ &= \max_{\nu=0}^{n}\{|c_{k_\nu}|_p|\xi - \alpha_{k_n}|_p|\xi^{k_\nu-1} + \xi^{k_\nu-2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu-1}|_p\} . \end{aligned}$$

Since

$$|\alpha_{k_n}|_p = |\xi - (\xi - \alpha_{k_n})|_p \leq \max\{|\xi|_p, |\xi - \alpha_{k_n}|_p\} \leq |\xi|_p + 1$$

for sufficiently large n, it follows that

$$|\xi^{k_\nu-1} + \xi^{k_\nu-2}\alpha_{k_n} + \ldots + \alpha_{k_n}^{k_\nu-1}|_p \leq (|\xi|_p + 1)^{k_\nu-1} .$$

Hence using (3.13) we get

$$|f_n(\xi) - f_n(\alpha_{k_n})|_p \leq \max_{\nu=0}^{n}\{p^{-u_{k_\nu}}\}|\xi - \alpha_{k_n}|_p(|\xi|_p + 1)^{k_n-1} .$$

Since the sequence $\{u_{k_n}\}$is monotonically increasing for $n \geq N_4$, $\max_{\nu=0}^{n}\{p^{-u_{k_\nu}}\}$ is bounded. Thus there exists a constant $c_4 > 0$ such that

$$|f_n(\xi) - f_n(\alpha_{k_n})|_p \leq c_4|\xi - \alpha_{k_n}|_p(|\xi|_p + 1)^{k_n-1}$$

for $n \geq N_4$. From (3.5) and (3.6) we have

$$|f_n(\xi) - f_n(\alpha_{k_n})|_p \leq c_5^{k_n} H(\alpha_{k_n})^{-k_n\omega(k_n)} \leq c_5^{k_n} p^{-u_{k_n}\delta_1\omega(k_n)} \tag{3.14}$$

for $n \geq N_4$ where $c_5 > 0$ is a constant. Since $\lim_{n\to\infty} \omega(k_n) = +\infty$, from (3.2), (3.4) and (3.10) we deduce that there exist two suitable sequences $\{s'_n\}$ and $\{s''_n\}$ with $\lim_{n\to\infty} s'_n = +\infty$, $\lim_{n\to\infty} s''_n = +\infty$,

$$p^{-u_{k_{n+1}}/2} \leq H(f_n(\alpha_{k_n}))^{-s'_n} \tag{3.15}$$

and

$$c_5^{k_n} p^{-u_{k_n}\delta_1\omega(k_n)} \leq H(f_n(\alpha_{k_n}))^{-s''_n} \tag{3.16}$$

for $n \geq N_5 \geq N_4$. Therefore from (3.11), (3.12) and (3.14) we obtain

$$|f(\xi) - f_n(\alpha_{k_n})|_p \leq \max\{p^{-u_{k_{n+1}}/2}, c_5^{k_n} p^{-u_{k_n}\delta_1\omega(k_n)}\} \tag{3.17}$$

for $n \geq N_5$. Thus combining (3.15), (3.16) and (3.17) we have

$$|f(\xi) - f_n(\alpha_{k_n})|_p \leq \max\{H(f_n(\alpha_{k_n}))^{-s'_n}, H(f_n(\alpha_{k_n}))^{-s''_n}\}$$

for $n \geq N_5$. Let $s_n := \min\{s'_n, s''_n\}$. From the inequality above we get

$$|f(\xi) - f_n(\alpha_{k_n})|_p \leq H(f_n(\alpha_{k_n}))^{-s_n}$$

for $n \geq N_5$ where $\lim_{n\to\infty} s_n = +\infty$. If the sequence $\{f_n(\alpha_{k_n})\}$ is not a constant sequence then $\mu(f(\xi)) \leq m$ for $f(\xi)$, that is, $f(\xi)$ is a p-adic U-number of degree $\leq m$. Otherwise $f(\xi)$ is a p-adic algebraic number of K.

Corollary . For $k_n = n$ and $m = 1$ from Theorem 3 we obtain Theorem 1 in [9] as a special case.

Theorem 4 . *In the p-adic field* $\mathbb{Q}_p$, *let*

$$f(x) = \sum_{n=0}^{\infty} \frac{\eta_{k_n}}{a_{k_n}} x^{k_n} \qquad (k_n \in \mathbb{Z}^+ \ (n = 0, 1, 2, \dots);\ k_0 < k_1 < k_2 < \dots) \tag{4.1}$$

be a series with non-zero p-adic algebraic integers η_{k_n} $(n = 0, 1, 2, \ldots)$ *of a p-adic number field* K *of degree* q *and with positive integers* a_{k_n} $(a_{k_n} > 1$ *for* $n \geq N_0)$, $|\eta_{k_n}/a_{k_n}|_p = p^{-t_{k_n}}$ *and* $A_{k_n} = [a_{k_0}, a_{k_1}, \ldots, a_{k_n}]$ *satisfying the following conditions*

$$\lim_{n\to\infty} \frac{t_{k_{n+1}}}{t_{k_n}} = +\infty , \tag{4.2}$$

$$0 \leq \limsup_{n\to\infty} \frac{\log_p A_{k_n} H(\eta_{k_n})}{t_{k_n}} < \infty \tag{4.3}$$

and

$$\lim_{n\to\infty} \frac{t_{k_n}}{k_n} = +\infty , \tag{4.4}$$

where $H(\eta_{k_n})$ $(n = 0, 1, 2, \ldots)$ *is the height of* η_{k_n} $(n = 0, 1, 2, \ldots)$. *Furthermore* ξ *be a p-adic* U_m*-number for which the following two properties hold.*

1°) ξ *has an approximation with p-adic algebraic numbers* α_n *of degree* m *of a p-adic number field* L *so that the following holds for sufficiently large* n

$$|\xi - \alpha_n|_p \leq H(\alpha_n)^{-n\omega(n)} \qquad (\lim_{n\to\infty} \omega(n) = +\infty) , \tag{4.5}$$

where $[L : \mathbb{Q}] = m$.

2°) *There exist two real numbers* c_1 *and* c_2 *with* $1 < c_1 \leq c_2$ *and*

$$p^{t_{k_n} c_1} \leq H(\alpha_{k_n})^{k_n} \leq p^{t_{k_n} c_2} \tag{4.6}$$

for sufficiently large n *where* $H(\alpha_{k_n})$ $(n = 0, 1, 2, \ldots)$ *is the height of* α_{k_n} $(n = 0, 1, 2, \ldots)$. *Let* M *be a smallest number field which contain* K *and* L *with* $[M : \mathbb{Q}] = t$.

Then the radius of convergence of $f(x)$ *is infinity and* $f(\xi)$ *is either a p-adic U-number of degree* $\leq t$ *or a p-adic algebraic number of* M.

Proof . 1) It can be satisfied that the radius of convergence of $f(x)$ is infinity as Theorem 3. We consider the polynomials

$$f_n(x) = \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} x^{k_\nu} \qquad (n = 1, 2, \ldots) .$$

Let

$$\gamma_n := f_n(\alpha_{k_n}) = \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \alpha_{k_n}^{k_\nu} \ .$$

Since $\gamma_n \in M$, $(\gamma_n)^\circ \leq t$ $(n = 1, 2, \ldots)$. We can now determine an upper bound for the height of γ_n. For this, we consider the polynomial

$$F(y, x_0, x_1, \ldots, x_n, x_{n+1}) = A_{k_n} y - \sum_{\nu=0}^{n} \frac{A_{k_n}}{a_{k_\nu}} x_\nu x_{n+1}^{k_\nu} \ .$$

Since $F(y, x_0, x_1, \ldots, x_n, x_{n+1})$ is the polynomial with integral coefficients and

$$\begin{aligned} F(\gamma_n, \eta_{k_0}, \eta_{k_1}, \ldots, \eta_{k_n}, \alpha_{k_n}) &= A_{k_n}\gamma_n - \sum_{\nu=0}^{n} \frac{A_{k_n}}{a_{k_\nu}} \eta_{k_\nu} \alpha_{k_n}^{k_\nu} \\ &= A_{k_n}\gamma_n - A_{k_n} \sum_{\nu=0}^{n} \frac{\eta_{k_\nu}}{a_{k_\nu}} \alpha_{k_n}^{k_\nu} = 0 \ , \end{aligned}$$

applying Lemma 3 we have

$$H(\gamma_n) \leq 3^{2.t.1+[(1+1+\ldots+1)+k_n]t} H^t H(\eta_{k_0})^t \ldots H(\eta_{k_n})^t H(\alpha_{k_n})^{k_n t}$$

where H is the height of the polynomial $F(y, x_0, x_1, \ldots, x_n, x_{n+1})$, $g = t$, $d = 1$, $\ell_0 = 1, \ldots, \ell_n = 1, \ell_{n+1} = k_n$. Since

$$H = \max_{\nu=0}^{n} \{A_{k_n}, A_{k_n}/a_{k_\nu}\} = A_{k_n} \ ,$$

using (4.6) we have

$$\begin{aligned} H(\gamma_n) &\leq 3^{2t+3k_n t} A_{k_n}^t H(\eta_{k_0})^t \ldots H(\eta_{k_n})^t p^{t_{k_n} t c_2} \\ &\leq l_0^{k_n t} A_{k_n}^t H(\eta_{k_0})^t \ldots H(\eta_{k_n})^t p^{t_{k_n} t c_2} \end{aligned} \tag{4.7}$$

for sufficiently large n where $l_0 > 0$ is a suitable constant. From (4.2) and (4.3) it follows

$$\lim_{n\to\infty} t_{k_{n+1}} / \log_p(A_{k_n} H(\eta_{k_n})) = +\infty \tag{4.8}$$

for $n \geq N_1 \geq N_0$. Since $|a_{k_{n+1}}|_p \leq 1$, from Lemma 4 we obtain

$$H(\eta_{k_{n+1}})^{-1} \leq |\eta_{k_{n+1}}|_p \leq p^{-t_{k_{n+1}}} |a_{k_{n+1}}|_p \leq p^{-t_{k_{n+1}}}$$

and from here

$$t_{k_{n+1}} \leq \log_p H(\eta_{k_{n+1}}) \ .$$

Furthermore since $A_{k_n} \geq 1$, we can write

$$\frac{t_{k_{n+1}}}{\log_p(A_{k_n} H(\eta_{k_n}))} \leq \frac{\log_p H(\eta_{k_{n+1}})}{\log_p H(\eta_{k_n})} \quad .$$

Thus using (4.8) we obtain

$$\lim_{n\to\infty} \frac{\log_p H(\eta_{k_{n+1}})}{\log_p H(\eta_{k_n})} = +\infty \quad .$$

It is satisfied

$$H(\eta_{k_{n+1}})^{\nu} > H(\eta_{k_n}) \tag{4.9}$$

for $n \geq N_2 \geq N_1$ where ν is a constant with $0 < \nu < 1/2$.

Let $K_0 := H(\eta_{k_0})H(\eta_{k_1})\ldots H(\eta_{k_{N_2-1}})$. From (4.9) we have

$$\begin{aligned} H(\eta_{k_{N_2}}) &< H(\eta_{k_{N_2+1}})^{\nu} < H(\eta_{k_n})^{\nu^{n-N_2}} \\ H(\eta_{k_{N_2+1}}) &< H(\eta_{k_n})^{\nu^{n-N_2-1}} \\ &\vdots \\ H(\eta_{k_{n-1}}) &< H(\eta_{k_n})^{\nu} \end{aligned}$$

for $n \geq N_2$. We also get

$$\begin{aligned} H(\eta_{k_0})\ldots H(\eta_{k_n}) &\leq H(\eta_{k_0})\ldots H(\eta_{k_{N_2-1}})H(\eta_{k_{N_2}})\ldots H(\eta_{k_n}) \\ &\leq K_0\, H(\eta_{k_n})^{\nu^{n-N_2}+\nu^{n-N_2-1}+\ldots+\nu+1} \\ &< K_0\, H(\eta_{k_n})^{\nu^{n}+\ldots+\nu+1} \\ &< K_0\, H(\eta_{k_n})^{1/1-\nu} < K_0\, H(\eta_{k_n})^2 \end{aligned}$$

for $n \geq N_2$. Combining this inequality with (4.7) it follows that

$$\begin{aligned} H(\gamma_n) &\leq l_0^{k_n t} A_{k_n}^t K_0^t H(\eta_{k_n})^{2t} p^{t_{k_n} t c_2} \\ &\leq l_1^{k_n t} (A_{k_n} H(\eta_{k_n}))^{2t} p^{t_{k_n} t c_2} \end{aligned} \tag{4.10}$$

where l_1 is a constant with $l_1 = l_0 K_0 > 0$. From (4.4) we obtain

$$l_1^{k_n t} = p^{k_n t \log_p l_1} \leq p^{t_{k_n}} \tag{4.11}$$

for $n \geq N_3 \geq N_2$. On the other hand from (4.3) we have

$$A_{k_n} H(\eta_{k_n}) \leq p^{t_{k_n} l_2} \tag{4.12}$$

for $n \geq N_4 \geq N_3$ where $l_2 > 0$ is a suitable constant. Combining (4.10),(4.11) and (4.12) it follows that

$$H(\gamma_n) \leq p^{t_{k_n}+2tl_2t_{k_n}+t_{k_n}tc_2} = p^{t_{k_n}l_3} \tag{4.13}$$

for $n \geq N_4$ where l_3 is a constant with $l_3 = 1 + t(2l_2 + c_2)$.

2) It holds

$$\begin{aligned} |f(\xi) - \gamma_n|_p &= |f(\xi) - f_n(\xi) + f_n(\xi) - \gamma_n|_p \\ &\leq max\{|f(\xi) - f_n(\xi)|_p, |f_n(\xi) - \gamma_n|_p\} \ . \end{aligned} \tag{4.14}$$

We can determine an upper bound for $|f(\xi) - f_n(\xi)|_p$ and $|f_n(\xi) - \gamma_n|_p$.

$$\begin{aligned} |f(\xi) - f_n(\xi)|_p &= \left| \sum_{\nu=n+1}^{\infty} \frac{\eta_{k_\nu}}{a_{k_\nu}} \xi^{k_\nu} \right|_p \\ &\leq \max\left\{ \left|\frac{\eta_{k_{n+1}}}{a_{k_{n+1}}}\right|_p |\xi|_p^{k_{n+1}}, \left|\frac{\eta_{k_{n+2}}}{a_{k_{n+2}}}\right|_p |\xi|_p^{k_{n+2}}, \ldots \right\} \end{aligned}$$

and

$$\left|\frac{\eta_{k_n}}{a_{k_n}} \xi^{k_n}\right|_p = \left|\frac{\eta_{k_n}}{a_{k_n}}\right|_p |\xi|_p^{k_n} = p^{-t_{k_n}+k_n \log_p |\xi|_p}$$

are hold. From (4.4) it follows that

$$\frac{t_{k_n}}{2} \leq t_{k_n} - k_n \log_p |\xi|_p$$

for $n \geq N_5 \geq N_4$.So we have

$$\left|\frac{\eta_{k_n}}{a_{k_n}} \xi^{k_n}\right|_p \leq p^{\frac{-t_{k_n}}{2}}$$

for $n \geq N_5$. According to (4.2) since the sequence $\{t_{k_n}\}$ is monotonically increasing for sufficiently large n, we obtain

$$|f(\xi) - f_n(\xi)|_p \leq \max\{p^{-t_{k_{n+1}}/2}, p^{-t_{k_{n+2}}/2}, \ldots\} = p^{-t_{k_{n+1}}/2} \tag{4.15}$$

for $n \geq N_6 \geq N_5$.

3) Furthermore it is clear that

$$|f_n(\xi)-\gamma_n|_p = \left|\sum_{\nu=0}^{n}\frac{\eta_{k_\nu}}{a_{k_\nu}}(\xi^{k_\nu}-\alpha_{k_n}^{k_\nu})\right|_p \le \max_{\nu=0}^{n}\left|\frac{\eta_{k_\nu}}{a_{k_\nu}}(\xi^{k_\nu}-\alpha_{k_n}^{k_\nu})\right|_p \tag{4.16}$$

$$= \max_{\nu=0}^{n}\left\{\left|\frac{\eta_{k_\nu}}{a_{k_\nu}}\right|_p |\xi-\alpha_{k_n}|_p|\xi^{k_\nu-1}+\xi^{k_\nu-2}\alpha_{k_n}+\ldots+\alpha_{k_n}^{k_\nu-1}|_p\right\} .$$

Since

$$|\alpha_{k_n}|_p = |\xi-(\xi-\alpha_{k_n})|_p \le \max\{|\xi|_p, |\xi-\alpha_{k_n}|_p\} \le |\xi|_p+1$$

for sufficiently large n, we get

$$|\xi^{k_\nu-1}+\xi^{k_\nu-2}\alpha_{k_n}+\ldots+\alpha_{k_n}^{k_\nu-1}|_p \le (|\xi|_p+1)^{k_\nu-1} .$$

From here using (4.16) we obtain

$$|f_n(\xi)-\gamma_n|_p \le \max_{\nu=0}^{n}\{p^{-t_{k_\nu}}\}|\xi-\alpha_{k_n}|_p(|\xi|_p+1)^{k_n-1} .$$

Since the sequence $\{t_{k_n}\}$ is monotonically increasing for $n \ge N_6$, $\max_{\nu=0}^{n}\{p^{-t_{k_\nu}}\}$ is bounded. Therefore there exists a positive constant l_4 sucb that

$$|f_n(\xi)-\gamma_n|_p \le l_4|\xi-\alpha_{k_n}|_p(|\xi|_p+1)^{k_n-1}$$

for $n \ge N_6$ Thus from (4.5) and (4.6) we have

$$\begin{aligned} |f_n(\xi)-\gamma_n|_p &\le l_5^{k_n}H(\alpha_{k_n})^{-k_n\omega(k_n)} \\ &\le l_5^{k_n}p^{-t_{k_n}c_1\omega(k_n)} \end{aligned} \tag{4.17}$$

for $n \ge N_6$ where l_5 is a suitable constant with $l_5 > 0$. Furthermore from $\lim_{n\to\infty}\omega(k_n) = +\infty$, (4.2), (4.4) and (4.13) we deduce that there exist suitable sequences $\{s'_n\}$ and $\{s''_n\}$ such that

$$p^{-t_{k_{n+1}}/2} \le H(\gamma_n)^{-s'_n} \tag{4.18}$$

and

$$l_5^{k_n}p^{-t_{k_n}c_1\omega(k_n)} \le H(\gamma_n)^{-s''_n} \tag{4.19}$$

for $n \ge N_7 \ge N_6$ where $\lim_{n\to\infty} s'_n = +\infty$ and $\lim_{n\to\infty} s''_n = +\infty$. Combining (4.14), (4.15) and (4.17) we obtain

$$|f(\xi)-\gamma_n|_p \le \max\{p^{-t_{k_{n+1}}/2}, l_5^{k_n}p^{-t_{k_n}c_1\omega(k_n)}\} \tag{4.20}$$

for $n \geq N_7$. From here using (4.18), (4.19) and (4.20) we also have

$$|f(\xi) - \gamma_n|_p \leq \max\{H(\gamma_n)^{-s'_n}, H(\gamma_n)^{-s''_n}\}$$

for $n \geq N_7$. Let $s_n := \min\{s'_n, s''_n\}$. From the inequality above we obtain

$$|f(\xi) - \gamma_n|_p \leq H(\gamma_n)^{-s_n}$$

for sufficiently large n where $\lim_{n\to\infty} s_n = +\infty$. If the sequence $\{\gamma_n\}$ is not a constant sequence then $\mu(f(\xi)) \leq t$ for $f(\xi)$, that is, $f(\xi)$ is a p-adic U-number of degree $\leq t$ Otherwise $f(\xi)$ is a p-adic algebraic number of K.

Corollary . For $k_n = n$ ve $t = 1$ from Theorem 4 we obtain Theorem 3 in [9] as a special case.

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