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A STOCHASTIC EPIDEMIC MODEL

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Abstract A stochastic epidemic model of SIS type has been investigated. For this, the corresponding Fokker-Planck equation has been solved for stationary or equilibrium transition probabilities. The spatial patterns of spread for the infectives for different transition stationary state probabilities have been obtained[1].

1. Introduction

The aim of the present paper is to introduce a stochastic epidemic model of SIS type. Anderson and May 1979, [1] Liu et al, 1987, [2], Hethcote and Van den Driessche 1991; [3] Mena-Lorca and Hethcote 1992, [4] and others have considered the deterministic epidemic models of SIS types. Here, a stochastic generalisation of such a model has been proposed. In the following section, the Fokker-Planck equation for this model has been derived and solved. In the subsequent section, the random spread of the infectives has been incorporated in this stochastic model

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and corresponding Fokker-Planck equation has been solved for the equilibrium transition probabilities. The spatial pattern of the spread for infectives has been obtained for a few cases of these equilibrium transition probabilities.

2. SIS Stochastic Epidemic Model

In a SIS epidemic model with a fixed population size N is divided into disjoint classes of susceptibles S(t) and infectives I(t) which depend on time t such $I(0) = I_0 > 0$ and $S(t) + I(t) = N$. The rate of increase of infectives due to the expose of susceptible to the infection agents or infectives is given by $(\beta + \beta')\ S(t)$ where β and β' are deterministic contact rate and its stochastic fluctuation respectively. It is assumed here that infection does not give rise to immunity and the rate at which infectives recover and become susceptible again is given by $(\gamma + \gamma')\ I(t)$ where γ and γ' are deterministic recovery rate and its stochastic fluctuation respectively.

Also, it is implicity assumed that births and deaths occur within the fixed population size N at equal rates and that all the new born individuals are susceptible. The rate of decrease of infective is given by $(\delta + \delta')\ I(t)$ where δ and δ' are the specific birth and death rate and its stochastic fluctuation respectively. The fluctuations β', γ', δ'

are supposed to be Wiener processes with specified properties as will be given below.

The stochastic differential equation for I(t) is, thus a generalisation of the deterministic model and is given by

$$I'(t) = (\beta + \beta')\ I(t)\ S(t) - (\gamma + \gamma')\ I(t) - (\delta + \delta')\ I(t)$$

$$= (\beta + \beta')\ I(N - 1) - (\gamma + \gamma')I - (\delta + \delta')I$$

$$= f(I) + \eta(t)\ g(I) + \eta'(t)\ g'(I) \qquad \text{(I)}$$

where

$$f(I) = \bar{\beta}I\ (N-I) - (\bar{\gamma} + \bar{\delta})I$$

$$g(I) = -I^2$$

$$g'(I) = -I$$

where $\eta(t) = \beta'$ and $\eta'(t) = \beta'N - \gamma' - \delta'$ are the Wiener processes satisfying the statistical property with

$$<\eta(t)>0 \text{ and } <\eta(t_1)\ \eta(t_2)> = 2\delta\ (t_1 - t_2)$$

$$<\eta'(t)>0 \text{ and } <\eta'(t_1)\ \eta'(t_2)> = 2\delta\ (t_1 - t_2).$$

The Fokker-Planck equation corresponding to the stochastic differential equation (I) is

$$\frac{\partial p}{\partial t} = -p\ \frac{\partial f}{\partial I} - f(I)\ \frac{\partial p}{\partial I} + \frac{\partial^2}{\partial I^2}\ \{(g^2 + g'^2)p\}$$

This partial differential equation can be solved for the equilibrium transition probabilities for large I and it is found to be

$$P(I,\ t/I_o,t_o) = \frac{K}{I^4}\, e^{\frac{2\bar{\beta}I - \bar{\beta}N + \bar{\gamma} + \bar{\delta}}{4I^2}}\ a_o \left[1 - \frac{C}{2I} + \frac{C^2}{12I^2} - \frac{D}{6I^2}\right.$$

$$+ \frac{CD}{18I^3} - \frac{B}{12I^3} - \frac{C^3}{144I^3} - \frac{C^2D}{48I^4} + \frac{7BC}{240I^4} + \frac{C^4}{2880I^4} + \frac{C^2}{120I^4} - \frac{A}{20I^4}$$

$$\left. + \ldots\ldots\ldots\ldots \right]$$

Where

$$A = -\frac{C^2}{2}$$

$$B = \frac{3C\beta}{4}$$

$$C = -\bar{\beta}$$

$$D = \frac{7C}{2} - \frac{\beta^2}{4} \quad \text{etc.}$$

Thus we see that equilibrium transition probability for large number of infectives I is vanishingly small. For very small number of infectives we can find the solution for the equilibrium transition probability which is given by

$$P(I,\ t/I_o,\ t_o) = A\ I^{\frac{m}{\bar{\beta}N - \bar{\gamma} - \bar{\delta}}}\ e^{\frac{2\bar{\beta}I}{\bar{\beta}N - \bar{\gamma} - \bar{\delta}}}$$

For, $N > \dfrac{\gamma + \delta}{\bar{\beta}}$, $p \to 0$ as $I \to 0$

For, $N< \dfrac{\gamma + \delta}{\bar{\beta}}$, $p \to \infty$ as $I \to 0$

Therefore, for the equilibrium transition probability of complete wiping out of the epidemic

We must have , $N< \dfrac{\gamma + \delta}{\bar{\beta}}$

Also if $N = \dfrac{\gamma + \delta}{\bar{\beta}}$, then for very small number of infectives we can find the solution for the equilibrium transition probability which is given by

$$p(1,\ t/I_o,\ t_o) = \frac{C}{I^2} e^{-\frac{12I}{\beta}}$$

Here also this transition probability becomes large for small number of infectives.

Thus, we conclude that we can have an equilibrium transition probabilities for complete wiping out of epidemic for the case when

$$N \leq \frac{\bar{\gamma} + \bar{\delta}}{\bar{\beta}}$$

3. Stochastic generalisation with random dispersal of infectives

We may consider the stochastic SIS model with random dispersal of infectives.

For random dispersal of infectives the stochastic differential equation is

$$\frac{\partial I}{\partial t} = f(I) + \eta(X,t) + \alpha\Delta^2 I \qquad (2)$$

De (1987, 1991, 1995), [6, 7, 8] has introduced the stochastic partial differential equation of the following form (also other forms) from the consideration of the random nature of the contributory functions of the ecological niches or factors that directly or indirectly influence the growth and evolution of biological system.

$$\frac{\partial X_i}{\partial t}(X,t) = -\frac{\delta S}{\delta X_i} + \eta_i(X, t)$$

where $X_i(X, t)$ is the bio-densities of the ith species $(i = 1,2)$ at $X \in R^n$ $(n=1,2)$ at time t and $S \equiv S(\xi_i)$ and $\frac{\delta}{\delta X_i}$ stands for functional derivative w.r.t.X_i (ξ_i are the random variables).

For different choices of the functional S gives the different governing stochastic differential equations.

Here, $\frac{\partial I}{\partial t} = -\frac{\partial S}{\partial I} + \eta(x, t)$

where,

$$S = \int s\,dx$$

$$= \frac{1}{2}\int[\alpha(\nabla I)^2 - \beta N I^2 + \frac{2}{3}\beta I^3 + \delta I^2 + \gamma I^2]\,dx$$

The Fokker-Planck equation corresponding to the stochastic differential equation (Langevin equation) (2) can be written as

$$\frac{\partial p}{\partial t} = \int dx \; \{ \frac{\delta^2}{\delta I^2(x)} + \frac{\delta}{\delta I(x)} \; \frac{\delta}{\delta I(x)} \} P$$

$$= \int dx \; \frac{\delta}{\delta I} \; (\frac{\delta}{\delta I} + \frac{\delta S}{\delta I}) P$$

for the transition probability which can be written as a path integral as

$$P(I', t' / I'', t'') = C \int \exp \{- \frac{1}{2} \int_{t'}^{t''} dt \wedge (I, \dot{I}, t)\} DI(t)$$

With boundary conditions

$$I(x, t') = I'(x)$$

$$I(x, t'') = I''(x)$$

Where $\wedge(I, \dot{I}, t) = \frac{1}{2} \int dx \quad I(x, t) + \frac{\delta S}{\delta I(x, t)} \quad ^2$

and $\dot{I} = \frac{\partial I}{\partial t}$

For the stationary state, $\alpha \; (\nabla I)^2 - \beta N I^2 + \frac{2}{3} \; \beta I^3 + \delta I^2 + \gamma I^2$ is independent of time in the region V of R^2

$$\alpha(\nabla I)^2 - \beta N I^2 + \frac{2}{3} \beta I^3 + \gamma I^2 + \delta I^2 = f(x), \text{ say}$$

We first consider $f(x) = 0$.

Then

$$\alpha(\nabla I)^2 - \beta N I^2 + \frac{2}{3} \beta I^3 + \gamma I^2 + \delta I^2 = 0 \qquad (4)$$

The first order non-linear partial differential equation can be solved by Jacobi's method (Sneddon, 1957) [7]. The subsidiary equations are (for two dimensional).

$$\frac{dx_1}{2\alpha\ u_1} = \frac{dx_2}{2\alpha\ u_2} = \frac{dx_3}{2u_3\{I^2\ (-\beta N + \gamma + \delta + \frac{2}{3}\beta I^3\}} = \frac{du_1}{0} = \frac{du_2}{0}$$

$$= \frac{du_3}{-u_3^2\ (-2\beta NI + 2\beta I^2 + 2\gamma I + 2\delta I)} \qquad (5)$$

where

$$u_i = \frac{\partial u}{\partial x_i}\ (i = 1,\ 2)$$

$x = (x_1, x_2)$, $I = x_3$ and u being assumed relation between x

and x_3 that is $u(x, x_3) = 0$ or $u(x_1, x_2, x_3) = 0$

From (4)

$$\alpha\ (u_1^2 + u_2^2) + u_3^2\ (-\ \beta NI^2 + \frac{2}{3}\beta I^3 + \gamma I^2 + \delta I^2) = 0 \qquad (6)$$

Solving u_i's we get u_i = constant = a_i $(i = 1,2)$ and $u_3^2\{-I^2\beta N$

$+ (\gamma + \delta)\ I^2 + \frac{2}{3}\beta I^3\}$ = constant = a(say)

$\alpha\ (a_1^2 + a_2^2) + a = 0$ or, $a = -\alpha\ (a_1^2 + a_2^2)$

We can find the complete integral of the equation (6) and hence integrating the Pfaffian equation

$$du = \sum_{i=1}^{3} u_i dx_i = \sum_{i=1}^{2} u_1 dx_1 + u_3\, dI$$

$$= \sum_{i=1}^{2} u_i dx_i \pm \frac{\sqrt{\alpha(a_1^2 + a_2^2)}}{\sqrt{I^2\beta N - (\gamma+\delta) I^2 - \frac{2}{3}\beta I^3}} \times dI$$

The solution u = 0 is given by

$$= (a_1x_1 + a_2x_2) \pm \int \frac{\sqrt{\alpha(a_1^2 + a_2^2)}}{I^2(\beta N - \gamma - \delta) - \frac{2}{3}\beta I^3} dI = \text{constant}$$

After performing the integration we have

$$I = \frac{3}{2} (\beta N - \gamma - \delta) \sec h^2 \phi/2$$

where

$$\phi = \frac{A - (a_1x_1 + a_2x_2)}{\sqrt{\alpha(a_1^2 + a_2^2)}}$$

For this case, the spatial pattern has been depicted in Fig. 1.

For non zero constant f(x), the complete integral can be found in the similar fashion.

for, $f(x) = -\beta/3(N-\rho)^3$, where $\rho = \frac{\gamma + \delta}{\beta}$

we have

$$I = \frac{3}{2\beta} (\beta N-\gamma-\delta) \operatorname{sech}^2 \frac{\sqrt{(\beta N-\gamma-\delta)}}{2} \times \frac{A - (A_1x_1 + A_2x_2)}{\sqrt{\alpha (a_1^2 + a_2^2) + \frac{1}{3}(N-\rho)^3}} \quad (8)$$

The spatial pattern in this case has been shown in Fig. 2.

In this way we can find the equilibrium spatial distribution of the infected population.

Concluding remark :

In this consideration a stochastic epidemic model has been proposed. The Fokker-Planck equation corresponding to the stochastic partial differential equations has been found and solved for the equilibrium or stationary transition probabilities. For several values of the transitin probabilities including the maximum one the spatial patterns have been calculated and depicted in the figures 1-2. These patterns thus corresponds to have some fixed probabilities. One is the maximum probability actually fig 1. corresponds to a spatial pattern having the maximum transition probability.

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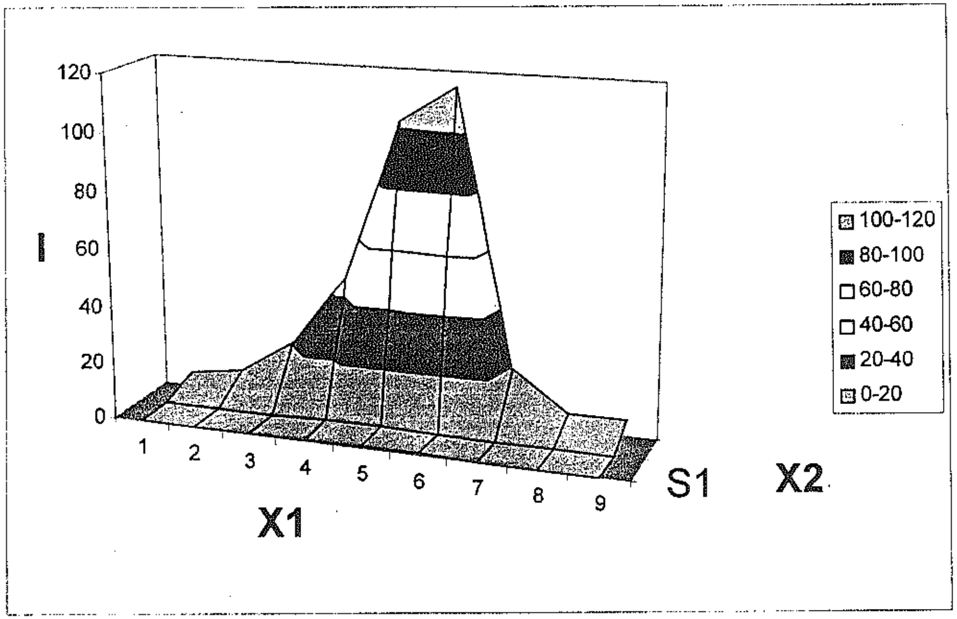
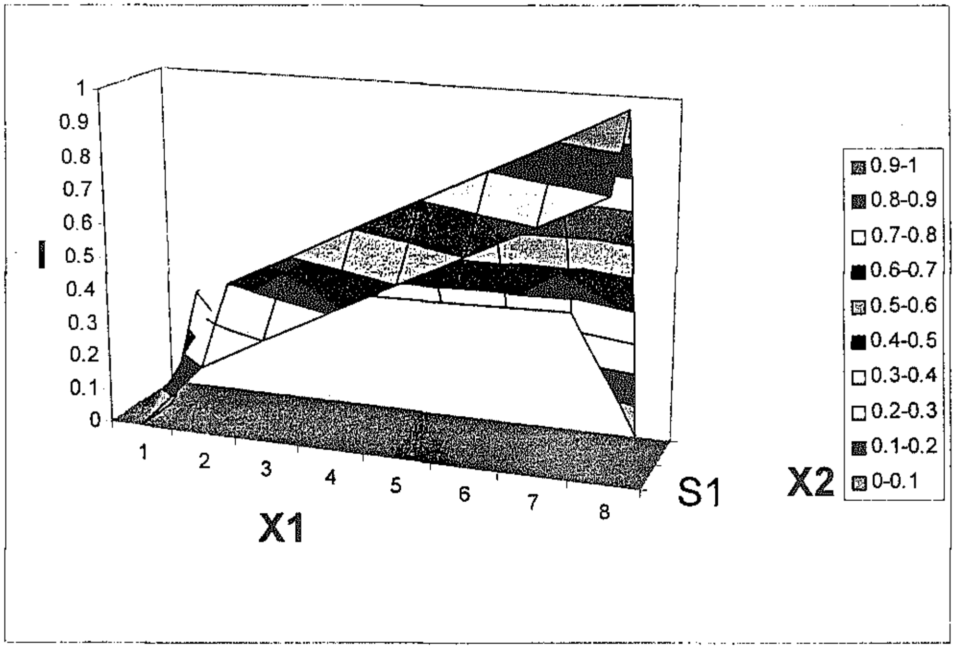


Fig. 1 : The spatial pattern of the epidemic given by the equation (7) by taking the values of the parameters. α = I, β = .8, γ = .4, δ= .2, a_1 = .3, a_2=.5, A= 5, N = 1.



<u>Fig. 2</u> : The spatial pattern of the epidemic given by the equation (8) by taking the values of the parameters. α = I, β = .8, γ = .4, δ= .2, a_1 = .3, a_2=.5, A=5, N = 1.