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REDUCTION OF THE GENERALIZED POPOV'S DIFFERENTIAL EQUATION TO BIRKHOFF MATRIX FORM

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Dedicated to Academician Blagoj S. Popov on his 75th anniversary

Abstract

In this paper the reduction of the generalized Popov's linear differential equation with one n-tuple regular singularity to the Birkhoff canonical matrix form is given.

1 Introduction

In [1, p. 32] B. S. Popov determined the necessary and sufficient condition for reducibility of the differential equation

$$x^2y'' + (\alpha x + \beta)xy' + (Ax^2 + Bx + C)y = 0$$

to a system of differential equations.

In the present paper we will show that the differential equation of B. S. Popov in a generalized form of n-th order can be reduced to a Birkhoff canonical matrix form. using the previous algebraic method given in [2].

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2 Preliminaries

Consider the generalized Popov's differential equation

$$x^n y^{(n)} = \sum_{i=1}^{n}\Big(\sum_{j=0}^{i} a_{ij} x^j\Big) x^{n-i} y^{(n-i)} \tag{1}$$

For the equation (1) the characteristic constants ρ_i $(1 \le i \le n)$ of the regular singularity at the coordinate origin $x = 0$ are given as roots of the equation

$$I(\rho) \doteq [\rho]_n - \sum_{i=1}^{n} a_{i0}[\rho]_{n-i} = 0,$$

where

$$[\rho]_k = \rho(\rho - 1)\cdots(\rho - k + 1), \quad [\rho]_0 = 1,$$

such that $\rho_i \neq \rho_j \pmod 1$ $(i \neq j;\ 1 \le i, j \le n)$. The last equation is derived when we look for a solution of (1) behaving as x^ρ near $x = 0$. The constants ρ_i $(1 \le i \le n)$ enter the formulation of our main result. For the sake of completeness we note that the principal characteristic constants λ_i $(1 \le i \le n)$ of the irregular singularity of rank 1 at infinity $x = \infty$ are given as roots of the equation

$$J(\lambda) \doteq \lambda^n - \sum_{i=1}^{n} a_{ii}\lambda^{n-i} = 0.$$

It is derived when looking for a solution of (1) behaving as $e^{\lambda x}$ times an arbitrary power of x at infinity.

3 Main result

Now we will prove the following result.

Theorem 3.1 *The equation (1) is reduced to the Birkhoff matrix form*

$$xI\frac{dU}{dx} = Q(x)U \equiv (Q_0 + Q_1 x)U. \tag{2}$$

where

$$(3)\quad Q(x)=\begin{bmatrix} q_{11}(x) & x & 0 & \cdots & 0 & 0 \\ q_{21}(x) & q_{22}(x) & x & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ q_{n-1,1}(x) & q_{n-1,2}(x) & q_{n-1,3}(x) & \cdots & q_{n-1,n-1}(x) & x \\ q_{n1}(x) & q_{n2}(x) & q_{n3}(x) & \cdots & q_{n,n-1}(x) & q_{nn}(x) \end{bmatrix}$$

and

$$(4)\quad q_{ij}(x) = q_{ij}^0 = \text{const}\ (j \le i \ne n),\quad q_{nj}(x) = q_{nj}^0 + a_{n+1-j,n+1-j}x\ (j \le n)$$

(in particular, $q_{ii}^0 = \rho_i - i + 1,\ 1 \le i \le n$*) by a linear transformation with polynomials in* x^{-1} *as its coefficients.*

Proof. Let us denote

$$(5)\qquad y_i(x) = y^{(i)}.$$

After a differentiation this yields

$$(6)\qquad y_i' = y_{i+1}\quad (0 \le i \le n-1).$$

If we multiply the equation (1) by x^{-n}, then it takes on the form

$$(7)\qquad y_n = \sum_{i=1}^{n} F_i(x) y_{n-i},$$

where

$$(8)\qquad F_i(x) = \sum_{j=0}^{i} a_{ij} x^{j-i}\quad (1 \le i \le n).$$

Let us denote $\mathcal{D} \equiv \frac{d}{dx}$. Now according to (6) and (7) we obtain

$$\mathcal{D}Y \equiv \mathcal{D}\begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ F_n(x) & F_{n-1}(x) & F_{n-2}(x) & \cdots & F_2(x) & F_1(x) \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \end{bmatrix}$$

$$\equiv L(x)Y.$$

i.e.

(9) $$\mathcal{D}Y = L(x)Y.$$

By the linear transformation of Turrittin [3]

(10) $$U = C(x)Y,$$

where

$$C(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_{21}(x) & 1 & 0 & \cdots & 0 & 0 \\ c_{31}(x) & c_{32}(x) & 1 & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{n1}(x) & c_{n2}(x) & c_{n3}(x) & \cdots & c_{n,n-1}(x) & 1 \end{bmatrix}.$$

we obtain the equality

(11) $$x\mathcal{D}U = x[\mathcal{D}C(x) + C(x)L(x)]C^{-1}(x)U = Q(x)U.$$

and hence it follows that

(12) $$x[\mathcal{D}C(x) + C(x)L(x)] = Q(x)C(x).$$

¿From (12) $c_{ij}(x)$ $(i > j)$ can be uniquely determined so that all entries of the matrix $Q(x)$ are polynomials of the required form (4).

If we denote

$$C(x)L(x) = [p_{ij}(x)]_{n\times n} \quad \text{and} \quad Q(x)C(x) = [l_{ij}(x)]_{n\times n},$$

then we have

(13_1) $$p_{ij}(x) = c_{i,j-1}(x) \qquad (1 \le i \le n-1).$$

$$(13_2) \qquad p_{nj}(x) = c_{n,j-1}(x) + F_{n-j+1}(x)$$

and

$$(14) \qquad t_{ij}(x) = q_{ij}(x) + \sum_{\nu=j+1}^{i} q_{i\nu}(x)c_{\nu j}(x) + xc_{i+1,j}(x),$$

where $c_{ii}(x) = 1$, $c_{ij}(x) = 0$ and $q_{ij}(x) = 0$ $(j > i)$, $c_{i0}(x) = 0$ $(1 \leq i \leq n)$, $c_{n+1,j} = 0$.

Then (12) can be written as

$$(15) \qquad x[\mathcal{D}c_{ij}(x) + p_{ij}(x)] = t_{ij}(x) \quad (j < i+2).$$

These relations are identically satisfied for the entries above the diagonal because $p_{i,i+1}(x) = 1$ and $t_{i,i+1} = x$.

For the diagonal entries (15) becomes

$$xp_{ii}(x) = t_{ii}(x),$$

which implies that

$$(16) \qquad xc_{i,i-1}(x) = q_{ii}(x) + xc_{i+1,i}(x) \quad (1 \leq i \leq n-1),$$

$$(17) \qquad x[c_{n,n-1}(x) + F_1(x)] = q_{nn}(x),$$

and for the j-th subdiagonal entries we obtain

$$(18) \qquad x[\mathcal{D}c_{i,i-j}(x) + c_{i,i-j-1}(x)]$$

$$= q_{i,i-j}(x) + \sum_{\nu=0}^{j-1} q_{i,i-\nu}(x)c_{i-\nu,i-j}(x) + xc_{i+1,i-j}(x) \quad (2 \leq i \leq n-1).$$

$$(19) \qquad x[\mathcal{D}c_{n,n-j}(x) + c_{n,n-j-1}(x) + F_{j+1}(x)]$$

$$= q_{n,n-j}(x) + \sum_{\nu=0}^{j-1} q_{n,n-\nu}(x)c_{n-\nu,n-j}(x) \quad (1 \leq j \leq n-1).$$

¿From the equalities (16) – (19) the subdiagonal entries of $C(x)$ can be determined as polynomials of the same degree with respect to x^{-1}. Now we will substitute

$$c_{i,i-j}(x) = \sum_{m=1}^{j} c^{m}_{i,i-j} x^{-m} \quad (2 \le i \le n; \quad 1 \le j \le i-1), \tag{20}$$

where $c^{m}_{i,i-j}$ $(1 \le m \le j)$ are constants (we assume $c^{m}_{i,i-j} = 0$ for $m > j$). We recall the representation

$$F_i(x) = \sum_{m=0}^{i} a_{i,i-m} x^{-m} \qquad (1 \le i \le n), \tag{21}$$

and we write

$$q_{ij}(x) = q^{0}_{ij} + q^{1}_{ij} x. \tag{22}$$

Substituting (20), (21) and (22) into (18) and (19), we obtain

$$-jc^{j}_{i,i-j} + c^{j+1}_{i,i-j-1} = q^{0}_{ii} c^{j}_{i,i-j} + c^{j+1}_{i+1,i-j}. \tag{23_j}$$

$$-mc^{m}_{i,i-j} + c^{m+1}_{i,i-j-1} \tag{23_m}$$

$$= \sum_{\nu=0}^{j-m} q^{0}_{i,i-\nu} c^{m}_{i-\nu,i-j} + \sum_{\nu=0}^{j-m-1} q^{1}_{i,i-\nu} c^{m+1}_{i-\nu,i-j} + c^{m+1}_{i+1,i-j} \quad (1 \le m \le j-1).$$

$$c^{1}_{i,i-j-1} = q_{i,i-j}(x) + \sum_{\nu=0}^{j-1} q^{1}_{i,i-\nu} c^{1}_{i-\nu,i-j} + c^{1}_{i+1,i-j}. \tag{23_0}$$

$$-jc^{j}_{n,n-j} + c^{j+1}_{n,n-j-1} + a_{j+1,0} = q^{0}_{nn} c^{j}_{n,n-j}. \tag{24_j}$$

$$-mc^{m}_{n,n-j} + c^{m+1}_{n,n-j-1} + a_{j+1,j-m} \tag{24_m}$$

$$= \sum_{\nu=0}^{j-m} q^{0}_{n,n-\nu} c^{m}_{n-\nu,n-j} + \sum_{\nu=0}^{j-m-1} q^{1}_{n,n-\nu} c^{m+1}_{n-\nu,n-j} \quad (1 \le m \le j-1).$$

$$(24_0) \qquad c^1_{n,n-j-1} + a_{j+1,j} + a_{j+1,j+1}x = q_{n,n-j}(x) + \sum_{\nu=0}^{j-1} q^1_{n,n-\nu} c^1_{n-\nu,n-j}.$$

¿From (24_0) it just follows that

$$(25) \qquad q_{n,n-j}(x) = a_{j+1,j+1}x + q^0_{n,n-j} \quad (0 \le j \le n-1).$$

Now we are able to determine all coefficients c^m_{ij} and the polynomials $q_{ij}(x)$. We shall need the following

Lemma 3.2 *Let $\xi_0^1, \xi_0^2, \cdots, \xi_0^n$ be constants satisfying the equality*

$$(26) \qquad [\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu).$$

Denote

$$f(\rho) = \begin{vmatrix} \mu_0^1 + \rho - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho \end{vmatrix}.$$

where μ_0^j $(1 \le j \le n-1)$ are some constants. Let the unknown variable ζ satisfy the following equalities

$$\xi_0^1 + [\zeta - (n-1)] = \mu_0^1,$$

$$(27) \qquad \xi_0^j + [\zeta - (n-j)]\mu_0^{j-1} = \mu_0^j \quad (2 \le j \le n-1),$$

$$\zeta\mu_0^{n-1} + \xi_0^n = 0.$$

If ζ takes one of the values ρ_ν, say $\zeta = \rho_n$, then

$$(28) \qquad f(\rho) = [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho - \rho_\nu).$$

Proof. Since

$$\begin{vmatrix} \mu_0^1 + y_1 & -1 & 0 & \cdots & 0 \\ \mu_0^2 & y_2 & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & y_{n-1} \end{vmatrix}$$

$$= y_1 y_2 \cdots y_{n-1} + \mu_0^1 y_2 y_3 \cdots y_{n-1} + \mu_0^2 y_3 y_4 \cdots y_{n-1} + \cdots + \mu_0^{n-1},$$

then

$$f(\rho) = [\rho]_{n-1} + \mu_0^1 [\rho]_{n-2} + \cdots + \mu_0^{n-1}$$

and

$$(\rho - \rho_n) f(\rho) =$$

$$[\rho]_n - (\rho_n - n + 1)[\rho]_{n-1} + \mu_0^1 [\rho]_{n-1} - (\rho_n - n + 2)\mu_0^1 [\rho]_{n-2} + \cdots + \mu_0^{n-1}[\rho]_1 - \rho_n \mu_0^{n-1}$$

$$= [\rho]_n + \xi_0^1 [\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n} (\rho - \rho_\nu) = (\rho - \rho_n) \prod_{\nu=1}^{n-1} (\rho - \rho_\nu). \qquad \square$$

Now we will start by the determination of $c_{n,n-j}^j$ $(1 \le j \le n-1)$ and $q_{nn}(x)$. From (17) and (25) we obtain

$$c_{n,n-1}^1 + a_{10} = q_{nn}^0. \tag{29}$$

If we substitute

$$c_{n,n-j}^j = \mu_0^j \qquad (1 \le j \le n-1),$$

then from (29) and (24_j) it follows that

$$\mu_0^1 = q_{nn}^0 - a_{10}. \tag{30}$$

$$\mu_0^{j+1} = (q_{nn}^0 + j)\mu_0^j - a_{j+1,0} \qquad (1 \le j \le n-1), \tag{31}$$

where $\mu_0^n = 0$. Applying Lemma 3.2 to (30) and (31), it follows that $q_{nn}^0 + n - 1$ is one of the roots ρ_i $(1 \le i \le n)$ of the equation

$$[\rho]_n - \sum_{i=1}^{n} a_{i0} [\rho]_{n-i} = 0.$$

By the substitution

$$q^0_{nn} + n - 1 = \rho_n \tag{32}$$

the coefficients μ_0^j $(1 \le j \le n-1)$ can be determined. In fact, according to the equation (28) the determinant

$$\begin{vmatrix} \mu_0^1 + \rho_n - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho_n - (n-3) & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho_n \end{vmatrix}$$

$$= [\rho_n]_{n-1} + \mu_0^1[\rho_n]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho_n - \rho_\nu)$$

is different from zero, which means that the coefficients $c^j_{n,n-j}$ and q^0_{nn} can be uniquely determined.

Further the coefficients $c^j_{i,i-j}$ $(1 \le j < i-1)$ and $q_{ii}(x)$ will be successively determined. ¿From (16) and (23_j), it follows that

$$c^1_{i,i-1} = q_{ii} + c^1_{i+1,i}, \tag{33_1}$$

$$jc^j_{i,i-j} + c^{j+1}_{i,i-j-1} = q^0_{ii}c^j_{i,i-j} + c^{j+1}_{i+1,i-j} \quad (1 \le j \le i-1). \tag{33_2}$$

Further we apply the mathematical induction. Let

$$c^{j+1}_{i+1,i-j}(x) = \xi_0^{j+1} \qquad (0 \le j < i-1),$$

and let the constants ξ_0^j $(1 \le j \le i)$ satisfy the equation

$$[\rho]_i + \xi_0^1[\rho]_{i-1} + \cdots + \xi_0^i = \prod_{\nu=1}^{i}(\rho - \rho_\nu). \tag{34}$$

This is so for $i = n-1$ by virtue of Lemma 3.2.

By the changes

$$c^j_{i,i-j} = \mu_0^j \qquad (1 \le j \le i-1)$$

we obtain

(35) $$\mu_0^1 = q_{ii}^0 + \xi_0^1.$$

(36) $$\mu_0^{j+1} = (q_{ii}^0 + j)\mu_0^j + \xi_0^{j+1} \qquad (1 \le j \le i-1),$$

where $\mu_0^i = 0$. Applying the Lemma to the equations (35) and (36), it follows that

(37) $$q_{ii}^0 + i - 1 = \rho_i$$

and

(38) $$[\rho]_{i-1} + \mu_0^1[\rho]_{i-2} + \cdots + \mu_0^{i-1} = \prod_{\nu=1}^{i-1}(\rho - \rho_\nu).$$

Again, assuming that (32) and (37) hold, $c_{i,i-j}^j$ $(1 \le j \le i-1)$ and $q_{ii}(x)$ can be determined. Thus by mathematical induction we proved that for each i $(1 < i \le n)$ $c_{i,i-j}^j$ $(1 \le j \le i-1)$ and $q_{ii}(x)$ can be determined. We also obtain that $q_{ii}(x)$ has the form

$$q_{ii}(x) = \rho_i - i + 1 + q_{ii}^1 x \qquad (1 \le i \le n),$$

where $q_{ii}^1 = 0$ $(i \ne n)$ and $q_{nn}^1 = a_{11}$.

Finally, we will prove that the sets

$$\{q_{n,n-j}(x),\ c_{n-j-m}^m;\quad 1 \le m \le n-j-1\}$$

and

$$\{q_{i,i-j}(x),\ c_{i,i-j-m}^m;\quad 1 \le m \le i-j-1\} \quad (n-1 \ge i \ge 1)$$

can be determined successively for the values n and i $(1 \le i \le n-1)$. It will be proved by induction on j. Let the sets $\{q_{n,n-\nu}, c_{n-\nu-m}^m\}$ and $\{q_{i,i-\nu}, c_{i-\nu-m}^m\}$ $(1 \le i \le n-1)$ be known for $0 \le \nu \le j-1$. Then from (24_m) we obtain

(39) $$-mc_{n,n-j-m}^m + c_{n,n-j-m-1}^{m+1} + a_{j+m+1,j}$$

$$= \sum_{\nu=0}^{j} q_{n,n-\nu}^0 c_{n-\nu,n-j-m}^m + \sum_{\nu=0}^{j-1} q_{n,n-\nu}^1 c_{n-\nu,n-j-m}^{m+1} \qquad (1 \le m \le n-j-1).$$

By the change

$$c^{j}_{n-j,n-j-m} = \xi_0^m \qquad (1 \le m \le n-j-1)$$

and using (34) and (38), we obtain

$$[\rho]_{n-j-1} + \xi_0^1[\rho]_{n-j-2} + \cdots + \xi_0^{n-j-1} = \prod_{i=1}^{n-j-1} (\rho - \rho_i). \tag{40}$$

If we substitute

$$c^{m}_{n,n-j-m} = \mu_0^m \qquad (1 \le m \le n-j-1),$$

then from (24_0) there follow the equalities

$$q^1_{n,n-j} = a_{j+1,j+1}, \tag{41}$$

$$q^0_{n,n-j} = \mu_0^1 + \text{known terms}, \tag{42}$$

and from (39) we obtain

$$-m\mu_0^m + \mu_0^{m+1} = q^0_{nn}\mu_0^m + q^0_{n,n-j}\xi_0^m + \cdots \qquad (1 \le m \le n-j-1). \tag{43}$$

The equation (43) can be represented in the following form

$$\mu_0^{m+1} = [\rho_n - j - (n-j-m-1)]\mu_0^m + q^0_{n,n-j}\xi_0^m + \cdots$$

and hence we obtain the determinant

$$\begin{vmatrix} \xi_0^1 + \rho_n - j - (n-j-2) & -1 & 0 & \cdots & 0 \\ \xi_0^2 & \rho_n - j - (n-j-3) & -1 & \cdots & 0 \\ \vdots & & & & \\ \xi_0^{n-j-2} & 0 & 0 & \cdots & -1 \\ \xi_0^{n-j-1} & 0 & 0 & \cdots & \rho_n - j \end{vmatrix}$$

$$= [\rho_n - j]_{n-j-1} + \xi_0^1[\rho_n - j]_{n-j-2} + \cdots + \xi_0^{n-j-1}$$

$$= \prod_{i=1}^{n-j-1} (\rho_n - j - \rho_i) \neq 0.$$

Thus, the set $\{q_{n,n-j}, c^m_{n,n-j-m}\}$ is uniquely determined. In order to prove that the sets $\{q_{i,i-j}, c^m_{i,i-j-m}\}$ $(1 \le i \le n-1)$ can be uniquely determined, we again apply mathematical induction. The arguments are similar to those above, so we omit the details. □

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