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CONVERSE THEOREM FOR MODIFIED BASKAKOV TYPE OPERATORS

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ABSTRACT. In this paper, we prove a converse theorem for Baskakov-Beta operators, using the device of Peetre's K-functional.

1.INTRODUCTION

Durrmeyer [5] introduced the integral modification of Bernstein polynomials and several researcher worked on the Durrmeyer type operators (see e.g. [1], [4], [6], [7], [9] and [10] etc.).Recently one of the authors [8] gave a new modification of Baskakov operators by taking the weight function of Beta operators to approximate Lebesgue integrable functions on $[0,\infty)$ as

(1.1) $$M_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)f(t)dt,\quad x\in[0,\infty)$$

Where

$$p_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}\quad\text{and}\quad b_{n,k}(t)=t^k[B(k+1,n)(1+t)^{n+k+1}]^{-1},$$

B(k+1,n) being the Beta function given by k!(n-!)!/(k+n)!.Direct results for the operators $M_n(f,x)$ defined by (1.1) give better estimate than the earlier modification of Baskakov operators studied in [2] , [11] and [12].

In the present paper, we study the converse behaviour of these operators.

By $C_\beta[0,\infty)$ we denote the class of continuous functions on $[0,\infty)$ satisfying $|f(t)|\le Kt^\beta, K>0$ with the norm

$$\|f\|_\beta=\sup_{0<t<\infty}|f(t)|t^{-\beta}.$$

We may rewrite operators (1.1)

$$M_n(f,x)=\int_0^{\infty}W(n,x,t)f(t)dt$$

where

$$w(n,x,t)=\sum_{k=0}^{\infty}p_{n,k}(x)b_{n,k}(t).$$

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Let C_0 denote the set of continuous functions on $(0,\infty)$ having a compact support and C_0^k the subset of C_0 of k times continuously differentiable functions, with $[a',b']\subset(a,b)$ and $[a,b]\subset(0,\infty)$. Let $G=\{g\in C^2{}_0, Supp\, g\subset[a',b']\}$ we define the Peetre's K-function by

$$K(\xi,f)=\inf\{\|f-g\|+\xi\|g''\|: g\in G\},\ \ where\, 0<\xi\le 1.$$

2.AUXILIARY RESULTS

In this section , we first prove some preliminary results .

LEMMA 2.1. For $m\in N\cup\{0\}$, we have

$$\mu_{n,m}(x)=\sum_{k=0}^{\infty}p_{n,k}(x)(\frac{k}{n}-x)^m$$

then

$$\mu_{n,1}(x)=0,\ \ \mu_{n,2}(x)=\frac{x(1+x)}{n}$$

and there holds the recurrence relation

$$n\mu_{n,m+1}(x)=x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]$$

Consequently for all $x\in[0,\infty)$

$\mu_{n,m}(x)=O(n^{-[(m+1)/2]})$, *where* $[\alpha]$ denote the integral part of α.

LEMMA2.2 [8]. If , we define

$$V_{n,m}(x)=\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)(t-x)^m\,dt$$

then

$$V_{n,0}(x)=1,\ \ V_{n,1}(x)=\frac{1+x}{n-1}$$

and there holds the recurrence relation

$$(n-m-1)V_{n,m+1}(x)=x(1+x)V'_{n,m}(x)+[(m+1)(1+2x)-x]V_{n,m}(x)+2mx(1+x)V_{n,m-1}(x),\ n>m+1.$$

Consequent for all $x\in[0,\infty)$

$$V_{n,m}(x)=O(n^{-[(m+1)/2]}).$$

LEMMA 2.3. Let

$$\phi_{n,m}(x)=\int_0^{\infty}W(n,x,t)t^m\,dt$$

then $\phi_{n,m}(x)$ is a polynomials in x of degree m and a rational function in n. Moreover for each $x\in[0,\infty)$, $\phi_{n,m}(x)=O(1)$.

PROOF. For m=0,1

$$\phi_{n,0}(x) = \int_0^{\infty} W(n,x,t)dt = 1.$$

and

$$\phi_{n,1}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)\, dt = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t-x)dt + x$$

Now , by using Lemma 2.2 , we have

$$\phi_{n,1}(x) = \frac{1+x}{n-1} + x = \frac{1+nx}{n-1}.$$

Next, we have

$$\phi_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) t^m dt$$

using $x(1+x)p'_{n,m}(x) = (k-nx)p_{n,k}(x),\ and\ t(1+t)b'_{n,k}(t) = [k-(n+1)t]b_{n,k}(t)$

$$x(1+x)\phi'_{n,m}(x) = \sum_{k=0}^{\infty} (k-nx)p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)t^m dt$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} [\{k-(n+1)t\} + (n+1)t - nx] b_{n,k}(t)t^m dt$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} t(1+t)b'_{n,k}(t)t^m dt + (n+1)\phi_{n,m+1}(x) - nx\phi_{n,m}(x)$$

$i.e.\ (n-m-1)\phi_{n,m+1}(x) = x(1+x)\phi'_{n,m}(x) + (nx+m+1)\phi_{n,m}(x), n > m+2.$

From the above recurrence relation, we can prove the result easily.

COROLLARY 2.4. Let β and δ be two positive numbers. Then for any m>0, there exists a constant K_m such that

$$\left\| \int_{|t-x|\geq\delta} W(n,x,t)t^{\beta}dt \right\|_{C[a,b]} \leq K_m n^{-m}.$$

PROOF. We have , by using Lemma 2.2

$$\int_{|t-x|\geq\delta} W(n,x,t)t^{\beta}dt \leq \int_{|t-x|\geq\delta} W(n,x,t)\frac{(t-x)^{2m}}{\delta^{2m}}t^{\beta}dt$$

$$\leq \frac{1}{\delta^{2m}}\left(\int_{|t-x|\geq\delta} W(n,x,t)(t-x)^{4m}dt\right)^{1/2}\left(\int_{|t-x|\geq\delta} W(n,x,t)t^{2\beta}dt\right)^{1/2}$$

$$\leq \frac{1}{\delta^{2m}}\left(\int_0^{\infty} W(n,x,t)(t-x)^{4m}dt\right)^{1/2}\left(\int_{|t-x|\geq\delta} W(n,x,t)t^{2\beta}dt\right)^{1/2}$$

$$= \frac{K_1}{\delta^{2m}}n^{-m}\left(\int_{|t-x|\geq\delta} W(n,x,t)t^{2\beta}dt\right)^{1/2}$$

and hence the corollary, since 9in view of Lemma 2.3)

$$\int_{|t-x|\geq\delta} W(n,x,t)t^{2\beta}dt = \int_{t\leq x-\delta} W(n,x,t)t^{2\beta}dt + \int_{t\geq x+\delta} W(n,x,t)t^{2\beta}dt$$

$$\leq \int_0^\infty W(n,x,t)(x-\delta)^{2\beta}dt + \int_{t\geq x+\delta} W(n,x,t)t^{2\beta}dt$$

$$\leq (x-\delta)^{2\beta} + \int_0^\infty W(n,x,t)\frac{t^m}{(x+\delta)^{m-2\beta}}dt, \quad m>2\beta$$

$$= (x-\delta)^{2\beta} + \frac{\phi_{n,m}(x)}{(x+\delta)^{m-2\beta}}$$

$$\leq K_2 \qquad \textit{for all } x\in[a,b]$$

3.MAIN RESULTS

In this section, we shall prove the converse theorem:

THEOREM 3.1. Let $0<a_1<a_2<b_2<b_1<\infty, 0<\alpha<2$ and suppose $f\in C_\beta[0,\infty)$. Then $(i)\Rightarrow(ii)$.

(i) $\|M_n(f,\cdot)-f(\cdot)\|_{C[a_1,b_1]} = O(n^{-\alpha/2})$;

(ii) $f\in Lip^*(\alpha, C[a_2,b_2])$,

where $Lip^*(\alpha,[a,b])$ denotes the Zygmund class of functions for which $\omega_2(f,h,a,b)\leq Mh^\alpha$.

There are two major steps to prove the above theorem.

(I) We first reduce the above problem to following lemma as a special case when f has a compact support inside some interior interval [a`, b']of (a_1,b_1).

LEMMA 2.3. Let $0<a<a'<a''<b''<b'<b<\infty$. *If* $f\in C_0$ with supp $f\in[a'',b'']$ and $\|M_n(f,\cdot)-f(\cdot)\|_{C[a,b]} = O(n^{-\alpha/2})$, then

(3.1) $\qquad K(\xi,f)\leq K_0(n^{-\alpha/2}+n\xi\ K(n^{-1},f)$.

Consequently $K(\xi,f)\leq K_1\xi^{\alpha/2}$ for some constant K_1

PROOF. Since supp $f\subset[a'',b'']$, there exists $h\in G$ such that for i=0 and 2

$$\|h^{(i)}(\cdot)-M_n^{(i)}(f,\cdot)\|_{C[a,b]}\leq K_2 n^{-1}.$$

Therefore

$$K(\xi,f)\leq 3K_2n^{-1}+\|f(\cdot)-M_n(f,\cdot)\|_{C[a,b]}+\xi\|M_n''(f,\cdot)\|_{C[a,b]}.$$

Hence it is sufficient to show that there exists a constant M_3 such that for each g∈G

(3.2) $\qquad \|M_n''(f,\cdot)\|_{C[a',b']}\leq K_3n\{\|f-g\|_{C[a',b']}+n^{-1}\|g''\|_{C[a',b']}\}$.

In fact

(3.3) $\qquad \|M_n''(f,\cdot)\|_{C[a',b']}\leq\|M_n''(f-g,\cdot)\|_{C[a',b']}+\|M_n''(g,\cdot)\|_{C[a',b']}$.

Consequently, differentiation ofithe kernel W(n, x, t) gives

$$\frac{\partial^2}{\partial x^2}W(n,x,t)=\frac{1}{[x(1+x)]^2}\sum_{k=0}^{\infty}[(k-nx)^2-k-2kx+nx^2]p_{n,k}(x)b_{n,k}(t).$$

Now , using Lemma 2.1, we have

$$\int_0^{\infty}\left|\frac{\partial^2}{\partial x^2}W(n,x,t)\right|dt \le \sum_{k=0}^{\infty}\frac{|(k-nx)^2-k-2kx+nx^2|}{[x(1+x)]^2}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)dt$$

$$\le \frac{n^2}{[x(1+x)]^2}[\mu_{n,2}(x)+(1+2x)\mu_{n,1}(x)+\frac{x(1+x)}{n}]$$

$$=\frac{2n}{x(1+x)},\ using\ Lemma\ 2.1.$$

Therefore, we have

(3.4) $\qquad \|M''_n(f-g,\cdot)\|_{C[a,b]} \le \frac{2n}{a(1+a)}\|f-g\| = K_4 n\|f-g\|.$

On the other hand by Lemma 2.2 , we have

(3.5) $\qquad \int_0^{\infty}[\frac{\partial^k}{\partial x^k}W(n,x,t)](t-x)^i dt = 0\ for\ k>i$

Also by Taylor's expansion, we have

(3.6) $g(t)=g(x)+g'(x)(t-x)+g''(\xi)(t-x)^2$,

ξ lies between t and x.

using (3.5) and (3.6), we obtain

$$M_n''(g,x)=\int_0^{\infty}[\frac{\partial^2}{\partial x^2}W(n,x,t)]g(t)dt$$

$$=\int_0^{\infty}[\cdots](g(x)+g'(x)(t-x)+g''(\xi)(t-x)^2)dt$$

$$=\int_0^{\infty}[\cdots]g''(\xi)(t-x)^2 dt$$

and , using Lemma 2.1 , Lemma 2.2 and Schwarz inequality, we get

$$\|M_n''(g,\cdot)\|_{C[a,b]} \le \|g''\|\cdot\left\|\int_0^{\infty}\left|\frac{\partial^2}{\partial x^2}W(n,x,t)\right|(t-x)^2 dt\right\|_{C[a,b]}$$

$$\le \|g''\|\cdot\left\|\sum_{k=0}^{\infty}[(k-nx)^2+(1+2x)(k-nx)+nx(1+x)].p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)(t-x)^2 dt\right\|$$

$$\le K_5\|g''\|$$

Hence (3.2) follows, by combining (3.3) , (3.4) and (3.7). This completes the proof of (3.1).

LEMMA 3.3. Relation (3.1) implies

$f\in L^*_p(\alpha,C[a,b])$.

PROOF. Proceeding along the lines of the proof from [3] we have

(3.8) $\qquad K(\xi,f)\le K_6\xi^{\alpha/2}$, for some constant $K_6>0$.

Now let $0<|\delta|\le h$. Then for any g∈y, we have

$$|\Delta^2{}_\delta f(x)|\le|\Delta^2{}_\delta(f(x)-g(x)|+|\Delta^2{}_\delta g(x)|.$$
$$\le 4\|f-g\|+\delta^2\|g''\|.$$

Therefore, using (3.8) we get

$$\omega_2(f,h,a,b)\le 4K(h^2,f)\le 4K_6h^\alpha\ ,\qquad i.e.\ \ f\in Lip^*(\alpha,C[a,b])..$$

(II) In this step we show that on using Lemma 3.2 and 3.3 the required results follows.

Let us choose a`,a``,b`,b`` in such a way that a_1<a`<a``<$a_2$ and $b_2$<b``,b`,b_1.Also let $g\in C_0^\infty$ be such that supp g⊂[a``, b``] and g(x)=1 on [a_2, b_2].

First assume that $0<\alpha\le 1$. For x∈[a`, b`] we have

$$M_n(fg,x)-f(x)g(x)=g(x)[M_n(f,x)-f(x)]+\int_{a_1}^{b_1}W(n,x,t)f(t)[g(t)-g(x)]dt+o(n^{-1})$$

(3.9) $=I_1+I_2+o(n^{-1}),\quad say.$

where $o(n^{-1})$. term is uniform for x∈[a`, b`] by corollary 2.4.

By making use of the assumption $\|M_n(f,\cdot)-f(\cdot)\|_{C[a_1,b_1]}=o(n^{-\alpha/2})$ we have

(3.10) $\|I_1\|_{C[a',b']}\le\|g\|_\infty\cdot\|M_n(f,\cdot)-f(\cdot)\|_{C[a_1,b_1]}\le K_7n^{-\alpha/2}.$

Also by mean value theorem, we get

$$I_2=\int_{a_1}^{b_1}W(n,x,t)f(t)[g'(\xi)(t-x)]dt.$$

Hence , by Lemma 2.2 and Cauchy-Schwarz inequality

(3.11) $\|I_2\|_{C[a',b']}=o(n^{-1/2})\le o(n^{-\alpha/2}).$

Combining (3.9), (3.10) and (3.11) we get

$$\|M_n(fg,\cdot)-fg(\cdot)\|_{C[a',b']}=o(n^{-\alpha/2}).$$

Thus by Lemma 3.2 and Lemma 3.3 , we have $fg\in Lip^*(\alpha,[a',b'])$. Since g(x)=1 on [$a_2$,$b_2$] it follows that $f\in Lip^*(\alpha,[a_2,b_2])$ proving the implication $(i)\Rightarrow(ii)$ *when* $0<\alpha\le 1$.

Now assume that $1<\alpha<2$. We also choose two points a^*_1 and b^*_1 satisfying a_1<a^*_1<a` and b`< b^*_1<b_1. Let δ∈(0,1), we shall prove the assertion for $1<\alpha<2-\delta$. Since δ is arbitrary, we may conclude that the result holds for α<2.

We notice from the previous result that the condition $\|M_n(f,\cdot)-f(\cdot)\|_{C[a_1,b_1]}=o(n^{-\alpha/2})$ implies $f'\in Lip(1-\delta,C[a^*_1,b^*_1])$.

Now for $x\in[a',b']$.

$$M_n(fg,x)-f(x)g(x)=g(x)[M_n(f,x)-f(x)]+f(x)[M_n(g,x)-g(x)]$$
$$+\int_{a_1^*}^{b_1^*}W(n,x,t)[f(t)-f(x)][g(t)-g(x)]dt+o(n^{-1})$$
$$=J_1+J_2+J_3+o(n^{-1}).$$

where $o(n^{-1})$ term holds uniformly for $x\in[a',b']$ (by corollary 2.4).

In fact $\|J_1\|_{C[a',b']} = O(n^{-\alpha/2})$ follows from the assumption,

$\|J_2\|_{C[a',b']} = O(n^{-\alpha/2}) \le O(n^{-\alpha/2})$, by Lemma 2.2.

Also since $|f(t) - f(x)| \le K|t-x|^{1+\delta}$ *and* $g(t) - g(x) = g'(\xi)(t-x)$,. using Jensen's inequality and Lemma 2.2 , we obtain

$\|J_3\|_{C[a',b']} = O(n^{-(2+\delta)/2}) \le O(n^{-\alpha/2})$.

Combining the above estimates of J_1,$J_2$and J_3 , we get

$$\|M_n(fg,\cdot) - fg(\cdot)\|_{C[a',b']} = O(n^{-\alpha/2}).$$

As in the first case using Lemma 3.2 and Lemma 3.3, the results follows.

This completes the proof of converse theorem.

REFERENCES

1. P.N.Agrawal and Vijay Gupta, Simultaneous approximation by linear combination of modified Bernstein polynomials,Bull. Soc. Math. Greece, 30 (1989),21-29.
2. P.N.Agrawal, Vijay Gupta and A.Sahai, On convergence of derivatives of linear combinations of modified Lupas operators.Publ.de L'Insl Math. 45 (59) (1989), 147-154.
3. H.Berens and G.G.Lorentz, Inverse theorem for Bernstein polynomials, Indiana Univ. Math. J. 21 (1972) 693-708.
4. Z.Ditzian and K.Ivanov, Bernsteian-type operators and their derivatives, J.Approx.Theory 56 (1989), 72-90.
5. J.L. Durrmeyer, Une Formule d'inversion de a transformee de laplace, Applications a la theory des Moments, These de 3e cycle, Faculte des sciences de 'Universite' de Paris 1967.
6. Vijay Gupta, P.N.Agrawal, A.Sahi and T.A.K.Sinha. Lp-approximation by combination of modified Szasz-Mirakyan operators, Demons. Math. XXIII, (3) (1990) 577-591.
7. Vijay Gupta and P.N. Agrawal , An estimate of the rate of convergence for modified Szasz-Mirakyan operators of function of bounded variation, Publ de L' Inst. Math. 49 (63) (1991),97-103
8. Vijay Gupta, A note on the modified Baskakov operators, Approx. Theory and its Appl. 10:3(1994), 74-78.
9. Vijay Gupta, Approximation by Szasz-Durrmeyer operators,Proc. 57th Annual Conf Indian Math. Soc., Aligarh, India (1991).
10. Vijay Gupta and P.N.Agrawal, Lp -approximation by iterative combination of Phillips operators, Publ.de L'Insl. Math. 52(66)(1992),101-109.
11. M.Heilmann and M.W.Muller, On simultaneous approximation by the method of Baskakov-Durrmeyer operators, Numer. Funct. Anal. and Optimiz. 10 (1989) 127-138.
12. R.P.Sinha, P.N.Agrawal and Vijay Gupta , On simultaneous approximation by modified Baskakov operators , Bull.Soc.Math. Belg.Ser.B 43(2) (1991)217-231.