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A Criterion for Linear Independence of Special Series *

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Abstract

The main result of this paper is a criterion for linear independence of special infinite series which consist of rational numbers and converge very fast.

1 Introduction

Mahler's method is a very powerful tool used in proving irrationality, linear independence or algebraic independence of infinite series. A nice survey of this type of result we can find in the book of Nishioka [5].

Algebraic independence is a special case of linear independence. There are several results in this field. Among them we mention Töpfer [7], Loxton and Poorten [4] or Kubota [3].

Another type of proof is the linear independence of logarithms of special rational numbers which can be found in Sorokin [6] or Bezivin in [1] which proves linear independence of roots of special functional equations.

If the series tends to infinity very fast then we can defined what we call linearly unrelated sequences.

Definition 1.1 *Let* $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, ..., K)$ *be sequences of positive real numbers. If for every sequence* $\{c_n\}_{n+1}^{\infty}$ *of positive integers the numbers* $\sum_{n=1}^{\infty} \frac{1}{a_{1,n}c_n}$, $\sum_{n=1}^{\infty} \frac{1}{a_{2,n}c_n}$, ..., $\sum_{n=1}^{\infty} \frac{1}{a_{K,n}c_n}$, *and 1 are linearly independent, then the sequences* $\{a_{i,n}\}_{n=1}^{\infty}$ $(i = 1, ..., K)$ *are said to be linearly unrelated.*

This definition is taken from [2], where one also finds the following theorem.

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Theorem 1.1 *Let ϵ be a positive real number and let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, ..., K-1)$ be sequences of positive integers such that*

$$\frac{a_{1,n+1}}{a_{1,n}} \geq 2^{K^{n-1}}, \qquad a_{1,n}/a_{1,n+1} \quad (a_{1,n} \ \text{divides} \ a_{1,n+1}),$$

$$b_{i,n} < 2^{K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}} \qquad i = 1, ..., K-1,$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0 \qquad \text{for all} \quad j, i \in \{1, ..., K-1\}, i > j$$

and

$$a_{i,n}2^{-K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}} < a_{1,n} < a_{i,n}2^{K^{n-(\sqrt{2}+\epsilon)\sqrt{n}}} \qquad i = 1, ..., K-1,$$

for every sufficiently large natural number n. Then the sequences $\{\frac{a_{i,n}}{b_{i,n}}\}_{n=1}^{\infty}$ $(i = 1, ..., K-1)$ are linearly unrelated.

2 Main result

The main result of this paper is the following criterion for the linear independence of special series of rational numbers and the number 1.

Theorem 2.1 *Let K be a positive integer and A be a real number with $A > 1$. Assume that $\{d_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers greater than 1. Let $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, ..., K)$ be sequences of positive integers such that*

$$\lim_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} = A, \tag{1}$$

$$\frac{A}{a_{1,n}^{\frac{1}{(K+1)^n}}} \geq \prod_{j=n}^{\infty} d_j, \tag{2}$$

$$b_{i,n} \leq d_n^{(K+1)^{n-2}} \qquad i = 1, ..., K, \tag{3}$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0, \qquad \text{for all} \quad j, i \in \{1, ..., K\}, i > j, \tag{4}$$

$$a_{i,n}d_n^{-(K+1)^{n-2}} < a_{1,n} < a_{i,n}d_n^{(K+1)^{n-2}} \qquad i = 2, ..., K \tag{5}$$

and

$$\lim_{n\to\infty} d_{n+1}^{(-K^2-2K+1)(K+1)^{n-1}} \prod_{j=1}^{n} d_j^{(K-1)(K+1)^{j-2}} = 0, \tag{6}$$

for every large positive integer n. Then the series $\sum_{n=1}^{\infty}\frac{b_{1,n}}{a_{1,n}},\dots,\sum_{n=1}^{\infty}\frac{b_{K,n}}{a_{K,n}}$ *and the number* 1 *are linearly independent over the rational numbers.*

Proof. We will prove that for every K-tuple of integers $\alpha_1, \alpha_2,\dots,\alpha_K$ (not all equal to zero) the sum

$$\alpha = \sum_{j=1}^{K-1}\alpha_j\sum_{n=1}^{\infty}\frac{b_{j,n}}{a_{j,n}c_n}$$

is an irrational number. Suppose that there exist $\alpha_1, \alpha_2,\dots,\alpha_K$ such that α is a rational number. Thus there are integers p and q with $q > 0$ such that $\alpha = \frac{p}{q}$. Let R be a maximal index such that $\alpha_R \neq 0$. Then we have

$$\alpha = \sum_{j=1}^{K-1}\alpha_j\sum_{n=1}^{\infty}\frac{b_{j,n}}{a_{j,n}c_n} = \sum_{n=1}^{\infty}\sum_{j=1}^{R}\alpha_j\frac{b_{j,n}}{a_{j,n}c_n} =$$

$$= \sum_{n=1}^{\infty}\frac{b_{R,n}}{a_{R,n}c_n}\left(\sum_{j=1}^{R-1}\alpha_j\frac{b_{j,n}a_{R,n}}{a_{j,n}b_{R,n}} + \alpha_R\right). \tag{7}$$

From this and (4) we obtain that there is a natural number N such that for every $n \geq N$ the number

$$\sum_{j=1}^{R-1}\alpha_j\frac{b_{j,n}a_{R,n}}{a_{j,n}b_{R,n}} + \alpha_R$$

and the number α_R have the same sign. This implies that without loss of generality we may assume that $\alpha_R > 0$ and (1)-(5) hold for every $n \geq N$. This and (7) imply that for every $M \geq N$

$$T_M = \left(\prod_{j=1}^{K}\prod_{i=1}^{M}a_{j,i}\right)\left(p - q\sum_{j=1}^{K}\sum_{i=1}^{M}\alpha_j\frac{b_{j,i}}{a_{j,i}}\right) = q\left(\prod_{j=1}^{K}\prod_{i=1}^{M}a_{j,i}\right)\left(\sum_{j=1}^{K}\sum_{i=M+1}^{\infty}\alpha_j\frac{b_{j,i}}{a_{j,i}}\right) =$$

$$q\left(\prod_{j=1}^{K}\prod_{i=1}^{M}a_{j,i}\right)\left(\sum_{i=M+1}^{\infty}\frac{b_{R,n}}{a_{R,n}}\left(\sum_{j=1}^{R-1}\alpha_j\frac{b_{j,n}a_{R,n}}{a_{j,n}b_{R,n}} + \alpha_R\right)\right) \tag{8}$$

is a positive integer. To complete the proof of Theorem 2.1 it suffices to find positive integer $k \geq N$ such that $T_k < 1$. From (3), (5) and (8) we obtain for every sufficiently large M

$$T_M \leq q\left(\prod_{j=1}^{K}\prod_{i=1}^{M}a_{j,i}\right)\left(\sum_{j=1}^{K}\sum_{i=M+1}^{\infty}\alpha_j\frac{b_{j,i}}{a_{j,i}}\right) \leq$$

$$q(\prod_{j=1}^{K}\prod_{i=1}^{M}a_{j,i})(\sum_{i=M+1}^{\infty}\sum_{j=1}^{K}|\alpha_j|\frac{b_{j,i}}{a_{j,i}})\leq$$

$$Q(\prod_{i=1}^{M}(a_{1,i}^{K}d_i^{(K-1)(K+1)^{i-2}}))(\sum_{j=1}^{K}|\alpha_j|)(\sum_{i=M+1}^{\infty}\frac{d_i^{2(K+1)^{i-2}}}{a_{1,i}})\leq$$

$$P(\prod_{i=1}^{M}(a_{1,i}^{K}d_i^{(K-1)(K+1)^{i-2}}))\frac{1}{a_{1,M+1}}d_{M+1}^{2(K+1)^{M-1}} \tag{9}$$

where Q and P are constants which do not depend on M. Inequalities (2) and (1) imply that for infinitely many n

$$a_{1,n+1}^{\frac{1}{(K+1)^{n+1}}}\geq d_{n+1}\max_{j=N,\ldots,n}a_{1,j}^{\frac{1}{(K+1)^j}}. \tag{10}$$

Otherwise there would exist $N_1\geq N$ such that for every $n\geq N_1$

$$a_{1,n+1}^{\frac{1}{(K+1)^{n+1}}}<d_{n+1}\max_{j=N,\ldots,n}a_{1,j}^{\frac{1}{(K+1)^j}}<$$

$$d_{n+1}d_n\max_{j=N,\ldots,n-1}a_{1,j}^{\frac{1}{(K+1)^j}}<\ldots<(\prod_{j=N_1+1}^{n+1}d_j)\max_{j=N,\ldots,N_1}a_{1,j}^{\frac{1}{(K+1)^j}}. \tag{11}$$

Let N_2 be a positive integer such that $N\leq N_2\leq N_1$ and

$$a_{1,N_2}^{\frac{1}{(K+1)^{N_2}}}=\max_{j=N,\ldots,N_1}a_j^{\frac{1}{(K+1)^j}}.$$

This and (11) imply that

$$a_{1,n+1}^{\frac{1}{(K+1)^{n+1}}}<(\prod_{j=N_1+1}^{n+1}d_j)\max_{j=N,\ldots N_1}a_{1,j}^{\frac{1}{(K+1)^j}}\leq(\prod_{j=N_2}^{n+1}d_j)a_{1,N_2}^{\frac{1}{(K+1)^{N_2}}}$$

which contradicts (1) and (2) for a sufficiently large n. Thus (10) holds. From (10) we obtain for infinitely many n

$$a_{1,n+1}\geq(d_{n+1}\max_{j=N,\ldots,n}a_{1,j}^{\frac{1}{(K+1)^j}})^{(K+1)^{n+1}}>$$

$$d_n^{(K+1)^{n+1}}(\max_{j=N,\ldots,n}a_{1,j}^{\frac{1}{(K+1)^j}})^{K((K+1)^n+(K+1)^{n-1}+\ldots+(K+1)^N)}\geq$$

$$d_{n+1}^{(K+1)^{n+1}}(\prod_{j=1}^{n} a_{1,j})^K(\prod_{j=1}^{N-1} a_{1,j})^{-K}. \tag{12}$$

Inequalities (9) and (12) imply

$$T_n \le P(\prod_{i=1}^{n} a_{1,i}^K d_i^{(K-1)(K+1)^{i-2}})d_{n+1}^{2(K+1)^{n-1}}\frac{1}{a_{1,n+1}} \le$$

$$P(\prod_{i=1}^{n} a_{1,i}^K d_i^{(K-1)(K+1)^{i-2}})d_{n+1}^{2(K+1)^{n-1}}\frac{(\prod_{j=1}^{n-1} a_{1,j})^K}{d_{n+1}^{(K+1)^{n+1}}\prod_{j=1}^{n} a_{1,j}^K} =$$

$$P(\prod_{j=1}^{N-1} a_{1,j})^K(\prod_{i=1}^{n} d_i^{(K-1)(K+1)^{i-2}})d_{n+1}^{2(K+1)^{n-1}-(K+1)^{n+1}}.$$

From this and (6) we obtain that $T_n < 1$ for infinitely many n and the proof of Theorem 2.1 is complete. □

3 Comments and Examples

Corollary 3.1 *Let K be a positive integer and A be a real number with $A > 1$. Assume that $\{a_{j,n}\}_{n=1}^{\infty}$ and $\{b_{j,n}\}_{n=1}^{\infty}$ $(j = 1, ..., K)$ are sequences of positive integers such that*

$$\lim_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} = A,$$

$$a_{1,n}^{\frac{1}{(K+1)^n}}(1+\frac{1}{n}) \le A,$$

$$b_{j,n} \le 2^{\frac{1}{n^3}(K+1)^{n-2}} \qquad (j = 1, 2, ..., K),$$

$$\lim_{n\to\infty}\frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0, \qquad \text{for all} \quad j, i \in 1, ..., K, i > j$$

and

$$a_{j,n}2^{-\frac{1}{n^3}(K+1)^{n-2}} \le a_{1,n} \le a_{j,n}2^{\frac{1}{n^3}(K+1)^{n-2}} \qquad (j = 2, ..., K),$$

for every sufficiently large positive integer n. Then the series $\sum_{n=1}^{\infty}\frac{b_{j,n}}{a_{j,n}}$ $(j = 1, ..., K)$ and the number 1 are linearly independent over the rational numbers.

This is the immediate consequence of Theorem 2.1 if we put $d_n = 1 + \frac{1}{n^{\frac{5}{2}}}$.

Corollary 3.2 *Let K be a positive integer and A be a real number with $A > 1$. Assume that $\{a_{j,n}\}_{j=1}^{\infty}$ and $\{b_{j,n}\}_{j=1}^{\infty}$ $(j = 1, ..., K)$ are sequences of positive integers such that*

$$\lim_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} = A,$$

$$a_{1,n}^{\frac{1}{(K+1)^n}}(1 + \frac{1}{K^{n-2}}) \leq A,$$

$$b_{j,n} \leq 2^{\frac{1}{K^3}(1+\frac{1}{K})^{n-2}} \qquad j = 1, ..., K,$$

$$\lim_{n\to\infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0, \qquad \text{for all} \quad j, i \in \{1, ..., K\}, i > j$$

and

$$a_{j,n}2^{\frac{1}{K^3}(1+\frac{1}{K})^{n-2}} \leq a_{1,n} \leq a_{j,n}2^{\frac{1}{K^3}(1+\frac{1}{K})^{n-2}} \qquad j = 2, ..., K,$$

for every sufficiently large positive integer n. Then the series $\sum_{n=1}^{\infty} \frac{b_{j,n}}{a_{j,n}}$ $(j = 1, ..., K)$ and the number 1 are linearly independent over the rational numbers.

This is the immediate consequence of Theorem 2.1 if we put $d_n = 1 + \frac{1}{K^n}$.

Let $[x]$ be the greatest integer greater or equal x.

Open Problem 1 *Let K be a positive integer. Are the series*

$$\sum_{n=1}^{\infty} \frac{3^{(K+1)^n} + 2^{jn}}{[(2 - \frac{1}{n})^{(K+1)^n} j] + 2^{3n}}$$

$(j = 1, ..., K)$ and the number 1 linearly independent?

Example 1 *Let $\pi(x)$ be the number of primes less than or equal x and K be a positive integer greater than 1. Then the series*

$$\sum_{n=1}^{\infty} \frac{3^{(K+1)^{\pi(n)}+j^2n} + 2^n}{[j(2 - \frac{2}{n})^{(K+1)^n} + 5^{(K+1)^{\pi(n)}}]}$$

$(j = 1, ..., K)$ are linearly independent over the rational numbers.

This is the immediate consequence of Corollary 3.1.

Example 2 *Let K be a positive integer greater than* 1 *and $q(x)$ be the number of divisors of the number n. Then the series*

$$\sum_{n=1}^{\infty} \frac{4^{(K+1)^{q(n)}}(j!)^n + (j+1)3^{nj}}{[(3-\frac{3}{K^{n-2}})^{(K+1)^n+j!n^2} + 2^{(K+1)^{q(n)}}]}$$

$(j = 1, ..., K)$ are linearly independent over the rational numbers.

This is the immediate consequence of Corollary 3.2.

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