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FREE DAMPED VIBRATIONS OF VISCOELASTIC MATERIALS

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Abstract

Free damper vibrations of viscoelastic rod,beam,plate and shell reducible to the solution of a certain integro-differential equation. Full solution of this equation for the kernel of relaxation in the form of sum of N exponential functions with different negative indexes is constructed in the present article.Iteration processes for calculating frequency and damping coefficient, which are the real and imaginary parts of two complex-conjugated roots of frequency equation, are given.In the case of positive relaxed module, the fact that the frequency equation has N futher real negative poles, in addition to the two complex poles obtained above, is proved. Analysis of obtained solutions and their comparisons with results available in literature are performed.

Key words: viscoelastic material damped vibrations exponential kernel.
AMS subject classification: 45J05; 74H45.

1.Introduction

The theory of linear viscoelasticity finds numerous technical applications, connected with studies of the creeps of metals, plastics, concrete, rock, polymers, composites and other solids. This theory has extensively developed in the last half-century.The original methods of solutions are worked out to solve quasistatic and dynamic problems. In [1] method of averaging that belongs to Bogoliubov is applied to vibration problems of viscoelasticity. According to method of averaging,the viscous strength of material is small enough in comparison with elastic strength. The result obtained in [1] is found in [2] by Laplace transform method.The problem of free vibrations of viscoelastic system,with single degree of freedom has been analysed in[3] by method of complex modules.Here the ratio of imaginary part of the complex module to its real part is considered to be small enough and beginning from the second, all powers of this ratio are neglected. Free vibrations of plates are investigated in [4]. There are many works devoted to study of vibrations of viscoelastic bodies with specific kernels and models. The Voigt,Kelvin, Maxwell models and standard model of linear viscoelastic material are used in [5-8]. Kernels in the form of the sum of exponential functions with negative indices are often used.The problems are solved by the method of Laplace integral transform,but the inverses are found by using Mellin's formula. The knowledge of poles of integrands is assumed.Using this method Struik [9] studied a problem of free damped vibrations of linear viscoelastic materials and used the result for the determination of mechanical properties of materials.

2.Statement of Problem

Equations of transient vibrations of flexible string or longitudinal vibrations of homogeneous rod, transient vibrations of beam and plate of viscoelastic material are

$$\frac{\partial^2 w}{\partial x^2} - \varepsilon \int_0^t \Gamma(t-\tau) \frac{\partial^2 w}{\partial x^2} d\tau + \frac{Q}{E} = \frac{m}{E} \frac{\partial^2 w}{\partial t^2}, \tag{2.1}$$

$$\frac{\partial^4 w}{\partial x^4} - \varepsilon \int_0^t \Gamma(t-\tau) \frac{\partial^4 w}{\partial x^4} d\tau + \frac{m}{b} \frac{\partial^2 w}{\partial t^2} = \frac{Q}{b}, \tag{2.2}$$

$$\Delta^2 w - \varepsilon \int_0^t \Gamma(t-\tau) \Delta^2 w d\tau + \frac{m}{D} \frac{\partial^2 w}{\partial t^2} = \frac{Q}{D} \tag{2.3}$$

where w is displacement, E is the modulus of instantaneous elasticity, Q is the transverse load, m is the mass, b and D are flexural rigidites, $\varepsilon\Gamma(t)$ is the kernel of relaxation, ε is a positive parameter which we may put equal to one at the end of the operation.

We will consider a viscoelastic solid for which the kernel of relaxation $\varepsilon\Gamma(t)$ is a positive function which satisfies the condition [1]

$$\int_0^t \varepsilon\Gamma(\tau) d\tau << 1 \tag{2.4}$$

for any t.For this reason we will assume ε to be a small positive parameter.

To the equations (2.1)-(2.3) it is necessary to connect appropriate boundary and initial conditions.The initial conditions, appropriate to an initial stress-free state of rest, may be given by

$$w = W_0(x)\,, \qquad \frac{\partial w}{\partial t} = W_1(x) \quad \text{for } t = 0\,, \tag{2.5}$$

where $W_0(x)$ and $W_1(x)$ are given functions.As the boundary conditions, for example,we may put

$$w = 0\,, \quad \frac{\partial w}{\partial x} = 0\,,$$

for the clamped edge, and

$$w = 0\,, \quad \frac{\partial^2 w}{\partial x^2} = 0\,,$$

for the simply supported edge.

Let the load $Q(x,t)$ for simply supported beam be represented by the Fourier series on x

$$Q(x,t) = m\sum_{k=1}^{\infty}\varphi_k(t)\sin\frac{k\pi x}{l},$$

where l is the length of the beam.A solution of equation (2.2) is assumed to be in the form

$$w(x,t) = \sum_{k=1}^{\infty} T_k(t)\sin\frac{k\pi x}{l}. \tag{2.6}$$

For the unknown functions $T_k(t)$ we obtain an integro-differential equation

$$T_n'' + \lambda_n^2 T_n = \varepsilon\lambda_n^2\int_o^t \Gamma(t-\tau)T_n(\tau)d\tau + \varphi_n(t)\,, \tag{2.7}$$

where

$$\lambda_n = (\frac{n\pi}{l})^2\sqrt{\frac{b}{m}}(n = 1,2,\ldots)$$

are the frequencies of elastic vibrations.

Let the functions $W_0(x)$ and $W_1(x)$ be represented by the Fourier series

$$W_0(x)=\sum_{k=1}^{\infty} T_{ko}\sin\frac{k\pi x}{l},\quad W_1(x)=\sum_{k=1}^{\infty} T_{k1}\sin\frac{k\pi x}{l}.$$

Using (2.5) and (2.6) we find the initial conditions for the equation (2.7)

$$T_k(0)=T_{ko},\quad T_k'(0)=T_{k1}.$$

Exactly the same method without any alterations applies to vibrations of plates, shells,and arbitrary three-dimensional bodies if the eigenfunctions and eigenvalues of the elasticity problem are known.Using the method of separation of variables or Bubnov-Galerkin method,replacing differential operators with respect to space coordinates with finite differences and many other methods, the dynamical system of viscoelasticity can be reduced to the equations of form (2.7).

3.Solution by Laplace Transform.Determination of Complex Roots

The equation

$$T'' + \lambda^2 T = \varepsilon\lambda^2 \int_o^t \Gamma(t-\tau)T(\tau)d\tau\ ,\quad t>0 \tag{3.1}$$

will be solved for the folloving initial conditions

$$T(0)=T_o,\quad T'(0)=T_1. \tag{3.2}$$

Using the Laplace transform we obtain the following image of the solution of this problem

$$\overline{T}(p)=\frac{pT_o+T_1}{p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}(p)} \tag{3.3}$$

where p is the complex parameter of transformation.The function $\overline{T}(p)$ and the image of kernel of relaxation $\overline{\Gamma}(p)$ are analytic in the right half-plane Re $p>0$.

Assume that the Laplace transform $\overline{T}(p)$ is an analytic function in the whole of the complex p-plane except at isolated singular points.The inverse transformation of function (3.3) can be found by using the well-known Mellin formula

$$T(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(pT_o+T_1)e^{pt}}{p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}(p)}dp \tag{3.4}$$

here the integration is carried out in the plane of complex variable p along an infinite straight line parallel to the imaginary axis and situated so that all singular points of the function $\overline{T}(p)$ are located to the left of this straight line.The calculation of this integral is usually accomplished through the use of the residue theory.For this reason it is necessary to know the poles and the branch points of integrand considered before being analytically continued to the left-half p-plane.Poles are roots of the equation

$$p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}(p)=0. \tag{3.5}$$

For $\varepsilon=0$ the equation (3.5) has two solutions $p_1=i\lambda$ and $p_2=-i\lambda$.
Let $-\alpha\pm i\beta$ be the roots of the equation (3.5). Substituting $p=-\alpha+i\beta$ to (3.5) and splitting in real and imaginary parts,gives

$$\int_o^\infty e^{\alpha\tau}\Gamma(\tau)\cos\beta\tau d\tau=\frac{\alpha^2+\lambda^2-\beta^2}{\varepsilon\lambda^2},\quad \int_o^\infty e^{\alpha\tau}\Gamma(\tau)\sin\beta\tau d\tau=\frac{2\alpha\beta}{\varepsilon\lambda^2}. \tag{3.6}$$

Thus the equation (3.5) is equivalent to the system of two equations (3.6).

Let us represent (3.3) in the form

$$\overline{T}(p)=\frac{pT_o+T_1}{p^2+\lambda^2-\varepsilon\lambda^2\overline{\Gamma}}=\frac{pT_o+T_1}{(p+\alpha)^2+\beta^2-(\varepsilon\lambda^2\overline{\Gamma}+2\alpha p+\alpha^2-2\lambda\gamma+\gamma^2)}=$$

$$=\frac{pT_o+T_1}{(p+\alpha)^2+\beta^2}\frac{1}{1-\overline{B}(p)}, \tag{3.7}$$

where

$$\overline{B}(p)=\frac{\varepsilon\lambda^2\overline{\Gamma}+2\alpha p+\alpha^2+\beta^2-\lambda^2}{(p+\alpha)^2+\beta^2}.$$

In the half-plane $\operatorname{Re}p>0$ we have $|\overline{B}(p)|<1$, thus (3.7) may be expanded into geometrical series

$$\overline{T}(p)=\overline{A}(p)(1+\overline{B}+\overline{B}^2+\ldots), \tag{3.8}$$

where

$$\overline{A}(p) = \frac{pT_o + T_1}{(p+\alpha)^2 + \beta^2}.$$

The inverse transforms of $\overline{A}(p)$ is

$$A(t) = e^{-\alpha t}[T_o \cos\beta t + \frac{T_1 - \alpha T_o}{\beta}\sin\beta t] = A_0 e^{-\alpha t}\cos(\beta t - \psi). \qquad (3.9)$$

Using (3.6) the function $B(t)$ obtained as

$$B(t) = \frac{\varepsilon\lambda^2}{\beta}\int_0^\infty \Gamma(t+s)e^{\alpha s}\sin\beta s ds \qquad (3.10)$$

Remark. *If we know the original*

$$\frac{\overline{B}(p)}{1-\overline{B}(p)} \doteq \Phi(t), \qquad (3.11)$$

from (3.8) we find required solution of the problem (3.1),(3.2) as

$$T(t) = A(t) + \int_0^t A(t-\tau)\Phi(\tau)d\tau. \qquad (3.12)$$

According to (3.7) equation (3.5) may be written as

$$p^2 + \lambda^2 - \varepsilon\lambda^2\overline{\Gamma}(p) = [(p+\alpha)^2 + \beta^2][1-\overline{B}(p)] = 0 \qquad (3.13)$$

The equation $1-\overline{B}(p) = 0$ has only real roots and may be solved easily by comparison with (3.5).

Consider the kernel

$$\Gamma(t) = \sum_{k=1}^{N} q_k e^{-\eta_k t}, \qquad (3.14)$$

where $q_k > 0, \eta_k > 0$ $(k = 1,2,\ldots,N)$ and $\eta_1 < \eta_2 < \ldots < \eta_N$. The equations (3.6) for obtaining the values α and β gives us

$$\sum_{k=1}^{N}\frac{q_k}{(\eta_k-\alpha)^2+\beta^2} = \frac{2\alpha}{\varepsilon\lambda^2}, \quad \sum_{k=1}^{N}\frac{q_k(\eta_k-\alpha)}{(\eta_k-\alpha)^2+\beta^2} = \frac{\alpha^2+\lambda^2-\beta^2}{\varepsilon\lambda^2} \qquad (3.15)$$

The iterations

$$\alpha_{n+1} = \frac{\varepsilon\lambda^2}{2}\sum_{k=1}^{N}\frac{q_k}{(\eta_k-\alpha_n)^2+\beta_n^2},$$

$$\beta_{n+1}^2 = \lambda^2 + 3\alpha^2{}_{n+1} - \varepsilon\lambda^2\sum_{k=1}^{N}\frac{q_k\eta_k}{(\eta_k-\alpha_{n+1})^2+\beta_n^2},$$

$$n = 0,1,2,\ldots,\alpha_0 = 0, \beta_0^2 = \lambda^2, \qquad (3.16)$$

define α and β

$$\alpha = \frac{\varepsilon\lambda^2}{2}\sum_{k=1}^{N}\frac{q_k}{\psi_k} + \frac{\varepsilon^2\lambda^4}{2}\sum_{k=1}^{N}\frac{q_k}{\psi_k^2}\left(\sum_{j=1}^{N}\frac{q_j(\eta_k+\eta_j)}{\psi_j}\right)+\ldots,\ \psi_k = \lambda^2+\eta_k^2,$$

$$\beta^2 = \lambda^2 - \varepsilon\lambda^2\sum_{k=1}^{N}\frac{q_k\eta_k}{\psi_k} - \varepsilon^2\lambda^4\left[\sum_{k=1}^{N}\frac{q_k\eta_k}{\psi_k^2}\left(\sum_{j=1}^{N}\frac{q_j(\eta_k+\eta_j)}{\psi_j}\right) - \frac{3}{4}\left(\sum_{j=1}^{N}\frac{q_j}{\psi_j}\right)^2\right]+\ldots$$

Using (3.14) from (3.10) we found

$$B(t) = \sum_{k=1}^{N}\varepsilon c_k e^{-\eta_k t},$$

where

$$c_k = \frac{\lambda^2 q_k}{\beta^2+(\eta_k-\alpha)^2}.$$

From the first of the Eq.(3.15) we get

$$\sum_{k=1}^{N}\varepsilon c_k = 2\alpha. \qquad (3.17)$$

The row-sum norms of derivatives of iteration vector-function with respect to α and β for (3.16) are estimated as below

$$\max\left|\frac{\varepsilon}{2\beta}\sum_{k=1}^{N}c_k\frac{2\beta(\eta_k-\alpha)}{\beta^2+(\eta_k-\alpha)^2}\right| + \max\left|\frac{\varepsilon}{\beta}\sum_{k=1}^{N}c_k\frac{\beta^2}{\beta^2+(\eta_k-\alpha)^2}\right| \le \frac{3\varepsilon}{2\beta}\sum_{k=1}^{N}c_k = \frac{3\alpha}{\beta}$$

,

$$\max\left|\frac{\varepsilon}{2\beta}\sum_{k=1}^{N}c_k\frac{2\beta(\eta_k-\alpha)}{\beta^2+(\eta_k-\alpha)^2}\right| \le \frac{\varepsilon}{2\beta}\sum_{k=1}^{N}c_k = \frac{\alpha}{\beta}.$$

So the iteration process (3.16) is convergent to a unique limit if $3\alpha/\beta < 1$.

Lemma *.Let the kernel of relaxation be given as (3.14) and the inequality (2.4) be valid. Then the equation*

$$1-\bar{B}(p)\equiv 1-\sum_{k=1}^{N}\frac{\varepsilon c_k}{p+\eta_k}=0 \qquad (3.18)$$

has N real negative roots $p_k=-\rho_k(\rho_k>0,k=1,2,\ldots,N)$*, which may be calculated by the iteration procedure*

$$p_{kn+1}=-\eta_k+\varepsilon c_k+\sum_{j=1}^{N}{}'\frac{\varepsilon c_j}{p_{kn}+\eta_j}(p_{kn}+\eta_k),\ n=0,1,2,\ldots, \qquad (3.19)$$

$$p_{k0}=-\eta_k+\varepsilon c_k,\ \sum_{j=1}^{N}{}'=\sum_{\substack{j=1\\ j\neq k}}^{N}\quad \text{(k=1,2,...N)}.$$

Proof. The function $1-\bar{B}(p)$ is analytic in the entire $p-$plane, except at the simple poles $-\eta_k\,(k=1,2,\ldots,N)$ on the negative part of the real axis. The function $1-\bar{B}(p)$ tends to $+\infty$ as $p\to-\eta_k-0$ and tends to $-\infty$ for $p\to-\eta_k+0$. In the origin we have

$$1-\bar{B}(0)=1-\sum_{k=1}^{N}\frac{\varepsilon c_k}{\eta_k}>0.$$

Indeed, we have from (2.4)

$$\sum_{k=1}^{N}\frac{\varepsilon q_k}{\eta_k}<1. \qquad (3.20)$$

It is easy to see that $B(\infty)=0$. Thus $c_k\approx q_k$ and from (3.20)

$$\sum_{k=1}^{N}\frac{c_k\varepsilon}{\eta_k}<1 \qquad (3.21)$$

follows. The zeros of the function $1-\bar{B}(p)$ are located between the poles, hence

$$-\eta_k<-\rho_k<-\eta_{k-1},k=1,2,\ldots,N;\eta_0=0.$$

The equation (3.18) may be written as

$$1=\frac{c_k\varepsilon}{p+\eta_k}+\sum_{j=1}^{N}{}'\frac{c_j\varepsilon}{p+\eta_j} \qquad (3.22)$$

Here we define

$$p = -\eta_k + \varepsilon c_k + \sum_{j=1}^{N}{}' \frac{c_j \varepsilon}{p+\eta_j}(p+\eta_k) \qquad (3.23)$$

and we form the iteration process (3.19).From (3.22) we may write the equality

$$\sum_{j=1}^{N}{}' \frac{c_j \varepsilon}{\eta_j - \rho_k} = 1 - \frac{c_k \varepsilon}{\eta_k - \rho_k}.$$

Then the derivative of iteration function in (3.23) is estimated as below

$$\left| \sum_{j=1}^{N}{}' \frac{c_j \varepsilon(\eta_j - \eta_k)}{(\eta_j - \rho_k)^2} \right| = \left| \sum_{j=1}^{N}{}' \frac{c_j \varepsilon}{\eta_j - \rho_k} + \sum_{j=1}^{N}{}' c_j \varepsilon \frac{\rho_k - \eta_k}{(\eta_j - \rho_k)^2} \right| = \left| 1 - \frac{c_k \varepsilon}{\eta_k - \rho_k} - \sum_{j=1}^{N}{}' c_j \varepsilon \frac{\eta_k - \rho_k}{(\eta_j - \rho_k)^2} \right| =$$

$$= \left| 1 - (\eta_k - \rho_k) \sum_{j=1}^{N} \frac{c_j \varepsilon}{(\eta_j - \rho_k)^2} \right| < 1$$

Thus the iteration process (3.19) is convergent .The proof of the lemma is now complete.

Inverse transform of $\overline{B}(p)/(1-\overline{B}(p))$ is found by using the residues at poles $-\rho_k \,(k = 1,2,\ldots,N)$:

$$\Phi(t) = \sum_{k=1}^{N} \chi_k e^{-\rho_k t} \qquad (3.24)$$

where

$$\chi_k = \left[\sum_{j=1}^{N} \frac{c_j \varepsilon}{(\eta_j - \eta_k)^2} \right]^{-1}, k = 1,2,\ldots,N.$$

As we see $\chi_k > 0$ and from the condition

$$\Phi(0) = \lim_{p \to \infty} \frac{p\overline{B}}{1-\overline{B}} = B(0) = 2\alpha$$

we have

$$\Phi(0)=\sum_{k=1}^{N}\chi_k=2\alpha. \tag{3.25}$$

The number $-\rho_k$ is the real root of equation (3.5) for the kernel (3.14),thus

$$\rho_k^2+\lambda^2-\varepsilon\lambda^2\sum_{j=1}^{N}\frac{q_j}{\eta_j-\rho_k}=0; k=1,2,\ldots,N \tag{3.26}$$

are identities.Moreover,from equality

$$1+\overline{\Phi}=\frac{1}{1-\overline{B}}$$

the identity

$$1+\sum_{j=1}^{N}\frac{\chi_j}{\rho_j-\eta_k}=0; k=1,2,\ldots,N. \tag{3.27}$$

is obtained.Using (3.9) and (3.24) we find

$$T(t)=\left[T_o+\sum_{k=1}^{N}\frac{\chi_k(\rho_k T_o-T_1)}{(\rho_k-\alpha)^2+\beta^2}\right]e^{-\alpha t}\cos\beta t+$$
$$\left[\frac{T_1-\alpha T_o}{\beta}+\sum_{k=1}^{N}\frac{\chi_k}{(\rho_k-\alpha)^2+\beta^2}(T_o\beta+\frac{T_1-\alpha T_o}{\beta}(\rho_k-\alpha))\right]e^{-\alpha t}\sin\beta t+$$
$$+\sum_{k=1}^{N}\frac{\chi_k(T_1-\rho_k T_o)}{(\rho_k-\alpha)^2+\beta^2}e^{-\rho_k t} \tag{3.28}$$

For the sake of the conciseness, let

$$E_1=T_o+\sum_{k=1}^{N}\frac{\chi_k(T_1-\rho_k T_o)}{\beta^2+(\rho_k-\alpha)^2},$$
$$E_2=\frac{T_1-\alpha T_o}{\beta}+\sum\frac{\chi_k}{\beta^2+(\rho_k-\alpha)^2}[T_o\beta+\frac{1}{\beta}(T_1-\alpha T_o)(\rho_k-\alpha)],$$
$$E_{3k}=\frac{\chi_k(T_1-\rho_k T_o)}{\beta^2+(\rho_k-\alpha)^2}.$$

Then (3.28) may be written as

$$T(t)=E_1e^{-\alpha t}\cos\beta t+E_2e^{-\alpha t}\sin\beta t+\sum_{k=1}^{N}E_{3k}e^{-\rho_k t} \tag{3.29}$$

Theorem *.Let the conditions of Lemma be valid.Then the function $T(t)$ defined by the formula (3.29) is a solution of the problem (3.1),(3.2).*

*Proof.*The derivatives of (3.29) are

$$T'(t) = E_1e^{-\alpha t}(-\alpha\cos\beta t - \beta\sin\beta t) + E_2e^{-\alpha t}(-\alpha\sin\beta t + \beta\cos\beta t) - \sum_{k=1}^{N} E_{3k}\rho_k e^{-\rho_k t}, \quad (3.30)$$

$$T''(t) = E_1e^{-\alpha t}(\alpha^2\cos\beta t + 2\alpha\beta\sin\beta t - \beta^2\cos\beta t) + E_2e^{-\alpha t}(\alpha^2\sin\beta t - 2\alpha\beta\cos\beta t -$$

$$-\beta^2\sin\beta t) + \sum E_{3k}\rho_k^2 e^{-\rho_k t}. \quad (3.31)$$

Now we calculate

$$\varepsilon\lambda^2\int_o^t \Gamma(t-\tau)T(\tau)d\tau = \varepsilon\lambda^2\sum_{k=1}^{N} q_k\left[\frac{E_1(\eta_k-\alpha)-\beta E_2}{\beta^2+(\eta_k-\alpha)^2}e^{-\alpha t}\cos\beta t + \frac{E_2(\eta_k-\alpha)+\beta E_1}{\beta^2+(\eta_k-\alpha)^2}\cdot\right.$$

$$\left. .e^{-\alpha t}\sin\beta t + \frac{\beta E_2-(\eta_k-\alpha)E_1}{\beta^2+(\eta_k-\alpha)^2}e^{-\eta_k t} + \sum_{j=1}^{N}\frac{E_{3j}}{\eta_k-\rho_j}\left(e^{-\rho_j t}-e^{-\eta_k t}\right)\right] \quad (3.32)$$

Substituting (3.29),(3.31) and (3.32) into equation (3.1)and using (3.15),(3.26) and (3.27) we see that the function (3.29) satisfies (3.1).From (3.28) it follows that $T(0) = T_o$,and from (3.30) we find $T'(0) = T_1$ using (3.25).

The first terms in (3.29) describes the damped vibrations process,and the last term shows the transient part of solution.As it follows from (3.29),the transient part of the solution is proportional to α (or to ε).

Solution for the exponential kernel (which describes various tree-parametric models,(particularly in Maxwell and Voigt models) is obtained from (3.29) when $N = 1$.This solutions is well-known in the literature [3,5-8].In all of these studies α and β were assumed to be determinable from (3.5).There exist no formulae obtained similar to (3.16).

If in the formulas for α and β we neglect all terms begining from the second and third respectively,i.e. take into account only the terms linear in ε ,we will get the result [1],for the kernel (3.14), obtained by Bogolyubov's averaging method.The approach [3,p.61] leads to the same result,obtained by the method of complex modules,where the ratio of the imaginary part of complex module to its real part is considered small enough and all of its powers over the first are neglected.

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