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TRANSFORMATION OPERATORS FOR STURM-LIOUVILLE EQUATION WITH THE POTENTIAL INVOLVING NEGATIVE DEGREES OF THE SPECTRAL PARAMETER

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Abstract: The existence of transformation operators with the condition at infinity which tarnsform the solution of the equation $y''+\lambda^{2n}y=0$ to the solution of the equation

$$-y''+q_0(x)+\frac{1}{\lambda}q_1(x)+...+\frac{1}{\lambda^{n-1}}q_{n-1}(x)y=\lambda^{2n}y$$

is proved. Some properties of kernels of transformation operators are investigated.

Keywords: Sturm-Liouville operators, Transformation operators, Integral representations, Fractional integrals and derivatives, Direct and inverse problems.

Let us consider the differential equation on the right halfline $(0 \le x < \infty)$

$$-y''+\sum_{k=0}^{n-1}\lambda^{-k}q_k(x)y=\lambda^{2n}y, \tag{1}$$

where λ is a complex parameter, $y=y(x,\lambda)$ is the required function. Suppose that the following conditions are satisfied:

(a) $q_1(x), q_2(x), ..., q_{n-1}(x)$ are differentiable functions on $(0;\infty)$;

(b) The integrals

$$\int_0^\infty (1+s)^{1+\frac{k}{n}}\left|q_k'(s)\right|ds, \qquad k=1,2,...,n-1$$

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are convergent, and $\sigma(0) < \infty$ where

$$\sigma(x) = \int\limits_x^{+\infty} (1+s)\,|q_0(s)|\,ds + \sum_{k=1}^{n-1} c_k \int\limits_x^{+\infty} (1+s)^{1+\frac{k}{n}}\,|q_k(s)|\,ds,$$

$$c_k = 2^{\frac{k}{n}} \left[\frac{1}{\Gamma\left(2+\frac{k}{n}\right)} + \Gamma\left(1-\frac{k}{n}\right) \right], k = 1, 2, ..., n-1$$

$\Gamma(.)$ gamma function.

The paper proves that if the conditions (a) and (b) are satisfied then there exists the franfromtation operator (see [1], [2] for general definition of transformation operators) which transforms the solution $\exp(i\lambda^n x)$ of the simple equation $y'' = \lambda^{2n} y$ to the solution of the equation (1) with condition at infinity. Note that in case $n = 1$, the transformation operators are completely investigated in [3], [4]. So, later it will be assumed $n > 1$.

For Sturm-Liouville equation with the potential involving positive degrees of the spectral parameter, exactly for the equation

$$-y'' + \sum_{k=0}^{n-1} \lambda^k q_k(x)\, y = \lambda^{2n} y,$$

the transformation operators were considered in [5], [6]. First, I.M. Guseinov ([5]) applied the notations Riemann-Liouville fractional integrals and derivatives to investigate the properties of the kernel of the transformation operator with condition at infinity. In [6] this idea was developed to transformation operators with initial conditions. In this paper, the transformation operators for the equation (1) with condition at infinity are constructed and the properties of the kernels of transformation operators (which strictly distinct from results in [5] as expected) are investigated by using Riemann-Liouville fractional integro-differentiation..

To formulate the main result of this paper, introduce the following notations:

$$\tau(x)=\int_x^{+\infty}|q_0(s)|\,ds+\sum_{k=1}^{n-1}c_k\int_x^{+\infty}(1+s)^{1+\frac{k}{n}}\left|q_k'(s)\right|ds,$$

$$\theta(x)=\int_x^{+\infty}|q_0(s)|\,ds+\sum_{k=1}^{n-1}\frac{2^{2-\frac{k}{n}}}{\Gamma\left(2-\frac{k}{n}\right)}\int_x^{+\infty}(1+s)^{1-\frac{k}{n}}|q_{n-k}(s)|\,ds$$

$$\|\varphi\|(x)=\int_0^{\infty}|\varphi(x,t)|\,dt,\quad D_x=\frac{\partial}{\partial x},D_t=\frac{\partial}{\partial t}.$$

Let I_t^{α} and $D_t^{\alpha}\,(0<\alpha<1)$ denote the operators of fractional integration and differentiation with respect to t, respectively (see [7]):

$$I_t^{\alpha}\varphi(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(x,s)\,ds,\quad D_t^{\alpha}\varphi(x,t)=\frac{\partial}{\partial t}I_t^{1-\alpha}\varphi(x,t)\,.$$

Theorem : If the conditions (a) and (b) are satisfied, then for all

$$\lambda\in\Delta_m=\left\{\lambda:\frac{2m\pi}{n}\leq\arg\lambda\leq\frac{\pi}{n}+\frac{2m\pi}{n}\right\}(m=0,1,...,n-1)$$

the solution $y(x,\lambda)$ of the equation (1) satisfying the condition

$$\lim_{x\to+\infty}y(x,\lambda)\,e^{-i\lambda^n x}=1 \tag{2}$$

can be represented in the form of

$$y(x,\lambda)=e^{i\lambda^n x}\left(1+\int_0^{\infty}A_m(x,t)\,e^{2i\lambda^n t}dt\right),\quad\lambda\in\Delta_m. \tag{3}$$

Where the kernels $A_m(x,t)$ have the following properties :

1) the inequality

$$\|A_m\|(x)\leq\exp\{\sigma(x)\}-1 \tag{4}$$

is satisfied;

2) there are summable with respect to t derivatives on $(0;+\infty)$

$$D_x A_m(x,t), D_t^{\frac{1}{n}} A_m(x,t), ..., \left(D_t^{\frac{1}{n}}\right)^{n-1} A_m(x,t),$$
$$\left(D_t^{\frac{1}{n}}\right)^{n} A_m(x,t) = D_t A_m(x,t)$$

and

$$\|D_x A_m\|(x) \le \tau(x)(1+\sigma(x))\exp\{\sigma(x)\} \tag{5}$$

$$\|D_t A_m\|(x) \le \sigma(x)(1+\theta(x)) \tag{5'}$$

3) The functions $H_m(x,t) = D_t A_m(x,t) - D_x A_m(x,t)$, $D_t^{\frac{1}{n}} H_m(x,t), ..., \left(D_t^{\frac{1}{n}}\right)^{n-1} H_m(x,t)$ are summable on $(0;+\infty)$ with respect to t and

$$D_x \left(D_t^{\frac{1}{n}}\right)^{n-1} H_m(x,t) + \sum_{k=0}^{n-1} \gamma_{n-k}^{(m)}(x)\, q_{n-k}(x) \left(D_t^{\frac{1}{n}}\right)^{k} A_m(x,t) = 0 \tag{6}$$

4)

$$I_t^{1-\frac{1}{n}} \left(D_t^{\frac{1}{n}}\right)^{j} A_m(x,t)\,|_{t=0} = 0, j = 0,1,2,...,n-2, \tag{7}$$

$$I_t^{1-\frac{1}{n}} \left(D_t^{\frac{1}{n}}\right)^{n-1} A_m(x,t)\,|_{t=0} = A_m(x,0) = \int_x^{+\infty} q_0(s)\,ds \tag{8}$$

$$I_t^{1-\frac{1}{n}} \left(D_t^{\frac{1}{n}}\right)^{j-1} \left(D_t A_m(x,t) - D_x A_m(x,t)\right)|_{t=0} = \gamma_j^{(m)} \int_0^{+\infty} q_j(s)\,ds, j = \overline{1, n-1}. \tag{9}$$

Proof of the theorem. Let us prove the theorem step by step.

1. First, we prove the existence of the solution of equation (1) satisfying the condition (2) with the properties (3) and (4).

Consider the integral equation

$$y(x,\lambda) = e^{i\lambda^n x} + \int_x^{+\infty} \frac{\sin \lambda^n (t-x)}{\lambda^n} \sum_{k=0}^{n-1} \lambda^{-k} q_k(t)\, y(t,\lambda)\, dt, \quad \lambda \in \Delta_m, \tag{10}$$

which is equivalent to the problem (1),(2).

Search the solution of (10) in the form (3). In order the function of type (3) ($\lambda \in \Delta_m$) be the solution of (10), the equality

$$\int_0^\infty A_m(x,t)\, e^{2i\lambda^n t} dt = \int_x^{+\infty} \sum_{k=0}^{n-1} q_k(t) \frac{e^{2i\lambda^n (t-x)}}{2i\lambda^{n+k}} \left[1 + \int_0^\infty A_m(x,s)\, e^{2i\lambda^n s} ds\right] dt \tag{11}$$

must be satisfied and conversely, if the function $A_m(x,t)$ satisfies the equality (11) for all $\lambda \in \Delta_m$, then (3) is the solution of the integral equation (10), as well as of the problem (1),(2). Using the formulae (see [8])

$$\frac{e^{2i\lambda^n t}}{2i\lambda^{n+k}} = -\frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \int_t^\infty (s-t)^{\frac{k}{n}} e^{2i\lambda^n s} ds, \lambda \in \Delta_m, k = 1,2,\ldots,n-1 \tag{12}$$

where $\gamma_k^{(m)} = 2^{\frac{k}{n}} e^{i\frac{\pi k}{2n}(4m-1)}$, we transform the right side of (11) similar to the left side. Since

$$\frac{e^{2i\lambda^n (t-x)} - 1}{2i\lambda^{n+k}} = \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left\{ \int_0^\infty (s)^{+\frac{k}{n}} e^{2i\lambda^n s} ds - \int_{t-x}^\infty (s-t+x)^{\frac{k}{n}} e^{2i\lambda^n s} ds \right\},$$

we have

$$\begin{aligned} \int_x^{+\infty} \sum_{k=0}^{n-1} q_n(t) \frac{e^{2i\lambda^n (t-x)} - 1}{2i\lambda^{n+k}} dt \; = \; & \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left[\int_0^{+\infty} \left(t^{\frac{k}{n}} \int_x^{+\infty} q_k(s)\, ds \right.\right. \\ & \left.\left. - \int_x^{x+t} (x+t-s)^{\frac{k}{n}} q_n(s)\, ds \right) e^{2i\lambda^n t} dt \right]. \end{aligned} \tag{13}$$

In the same way it can be shown that

$$\int_x^{+\infty} \sum_{k=0}^{n-1} q_n(t) \frac{e^{2i\lambda^n (t-x)} - 1}{2i\lambda^{n+k}} dt \int_0^\infty A_m(t,s)\, e^{2i\lambda^n s} ds$$

$$= \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \int_0^{\infty} \left(\int_x^{+\infty} q_k(s)\, ds \int_0^t (t-\xi)^{\frac{k}{n}} A_m(s,\xi)\, d\xi - \right. \tag{14}$$

$$\left. - \int_x^{x+t} q_k(s)\, ds \int_0^{x+t-s} (x+t-s-\xi)^{\frac{k}{n}} A_m(s,\xi)\, d\xi \right) e^{2i\lambda^n t} dt$$

It follows from (13) and (14) that equality (11) holds for all $\lambda \in \Delta_m$ if the function $A_m(x,t)$ satisfies the integral equation

$$A_m(x,t) = \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left[t^{\frac{k}{n}} \int_x^{+\infty} q_k(s)\, ds - \int_x^{x+t} (x+t-s)^{\frac{k}{n}} q_k(s)\, ds \right. \tag{15}$$

$$+ \int_x^{+\infty} q_k(s)\, ds \int_0^t (t-\xi)^{\frac{k}{n}} A_m(s,\xi)\, d\xi$$

$$\left. - \int_x^{x+t} q_k(s)\, ds \int_0^{x+t-s} (x+t-s-\xi)^{\frac{k}{n}} A_m(s,\xi)\, d\xi \right]$$

Hence, to prove the first part of the theorem, it is sufficient to show that the integral equation (15) has the solution satisfying inequality (4). Apply to integral equation (15) the method of successive approximation by putting

$$A_m^{(0)}(x,t) = \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left[t^{\frac{k}{n}} \int_{x+t}^{+\infty} q_k(s)\, ds + \int_x^{x+t} \left[t^{\frac{k}{n}} - (t+x-s)^{\frac{k}{n}} \right] q_k(s)\, ds \right],$$

$$A_m^{(p)}(x,t) = \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left[\int_0^t (t-\xi)^{\frac{k}{n}} d\xi \int_{x+t-\xi}^{+\infty} q_k(s) A_m^{(p-1)}(s,\xi)\, ds \right. \tag{16}$$

$$\left. + \int_0^t d\xi \int_x^{x+t-\xi} \left[(t-\xi)^{\frac{k}{n}} - (x+t-s-\xi)^{\frac{k}{n}} \right] q_k(s) A_m^{(p-1)}(s,\xi)\, ds \right]$$

$$p = 1, 2, \ldots .$$

Since $\left| t^{\frac{k}{n}} - (x+t-s)^{\frac{k}{n}} \right| \le \frac{k}{n} (x+t-s)^{\frac{k}{n}-1}$ for $x < s < x+t$ and $\max\limits_{0 \le \eta \le 1} \eta^{1+\frac{k}{n}} |q_k(x+\eta t)| = |q_k(x+t)|$ for $q_k(x) \in L_1(0;\infty)$ $(x > 0, t > 0, k =$

$1,2,...,n-1)$, from (16) by using some simple transformations, we obtain

$$\left\|A_m^{(0)}\right\|(x) \leq \sum_{k=0}^{n-1} c_k \int_x^{+\infty} s^{1+\frac{k}{n}} |q_k(s)|\, ds \leq \sigma(x),$$

$$\left\|A_m^{(p)}\right\|(x) \leq \sum_{k=0}^{n-1} c_k \int_x^{+\infty} s^{1+\frac{k}{n}} |q_k(s)| \left\|A_m^{(p-1)}\right\|(s)\, ds, \quad p=1,2,...$$

where $c_k = 2^{\frac{k}{n}} \left[\frac{1}{\Gamma(2+\frac{k}{n})} + \Gamma\left(1-\frac{k}{n}\right)\right], k = 0,1,2,...$.

From this, by induction we conclude that

$$\left\|A_m^{(p)}\right\|(x) \leq \frac{[\sigma(x)]^{p+1}}{(p+1)!}, p = 0,1,2,.... \tag{17}$$

Therefore, the series

$$A_m(x,.) = \sum_{p=0}^{\infty} A_m^{(p)}(x,.) \tag{18}$$

converges uniformly by $x \in [0,\infty)$ on the space $L_1(0,\infty)$ and the sum of this series, which is the solution of the integral equation (15), satisfies inequality (4).

2. Now we prove the properties 2) of the function $A_m(x,t)$. From (16) we conclude that the functions $A_m^{(p)}(x,t), p = 0,1,2,...$ have derivatives $D_x A_m^{(p)}(x,t)$ with respect to x:

$$D_x A_m^{(0)}(x,t) = \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left\{-t^{\frac{k}{n}} q_k(x+t) + \int_0^t \left[t^{\frac{k}{n}} - (t-s)^{\frac{k}{n}}\right] q_k'(x+s)\, ds\right\},$$

$$\begin{aligned} D_x A_m^{(p)}(x,t) &= -\int_x^{x+t} q_0(s) A_m(s, x+t-s)\, ds \\ &+ \sum_{k=0}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma(1+\frac{k}{n})} \left\{\int_0^t (t-\xi)^{\frac{k}{n}} d\xi \int_{x+t-\xi}^{+\infty} D_s\left[q_k(s) A_m^{(p-1)}(s,\xi)\right] ds \right. \\ &\left. + \int_0^t d\xi \int_x^{x+t-\xi} \left[(t-\xi)^{\frac{k}{n}} - (x+t-\xi-s)^{\frac{k}{n}}\right] D_s\left[q_k(s) A_m^{(p-1)}(s,\xi)\right] ds \right\} \end{aligned}$$

In analogy with estimations for $A_m^{(p)}(x,t)$, we obtain

$$\left\|D_x A_m^{(0)}\right\|(x) \leq \int_x^{+\infty} |q_0(s)|\, ds + \sum_{k=0}^{n-1} c_k \int_x^{+\infty} s^{1+\frac{k}{n}} |q_k'(s)|\, ds \leq \tau(x),$$

$$\left\|D_x A_m^{(p)}\right\|(x) \le \int_x^{+\infty} |q_0(s)| \left\|A_m^{(p-1)}\right\|(s)\,ds + \sum_{k=0}^{n-1} c_k \int_x^{+\infty} s^{1+\frac{k}{n}} \left(|q_k'(s)| \left\|A_m^{(p-1)}\right\|(s) + |q_k(s)| \left\|D_s A_m^{(p-1)}\right\|(s)\right) ds$$

By virtue of (17), inductively we get

$$\|D_x A_m\|(x) \le \tau(x) \frac{[\sigma(x)]^p}{p!}, (p+1), \ \ p = 0, 1, 2, \dots.$$

Where

$$\begin{aligned} D_x A_m(x,t) &= -q_0(x+t) - \int_x^{x+t} q_0(s) A_m(s, x+t-s)\,ds \\ &+ \sum_{k=1}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left\{ t^{\frac{k}{n}} \int_x^{+\infty} q_k'(s)\,ds - \int_x^{x+t} (x+t-s)^{\frac{k}{n}} q_k'(s)\,ds \right. \\ &+ \int_x^{+\infty} ds \int_0^t (t-\xi)^{\frac{k}{n}} D_s[q_k(s) A_m(s,\xi)]\,d\xi - \\ &\left. \int_x^{x+t} ds \int_0^{x+t-s} (x+t-s-\xi)^{\frac{k}{n}} D_s[q_k(s) A_m(s,\xi)]\,d\xi \right\} \end{aligned} \tag{15'}$$

Hence, the series (18) can be differentiated with respect to x and inequality (5) is satisfied.

Now we show that the function $D_t A_m(x,t)$ is summable on $[0,\infty)$ with respect to t. After differentiating the both sides of (15) by t, we get

$$\begin{aligned} D_t A_m(x,t) &= -q_0(x+t) + \int_x^{+\infty} q_0(s) A_m(s,t) \\ &- \int_x^{x+t} q_0(s) A_m(s, x+t-s)\,ds \\ &+ \sum_{k=0}^{n-1} \frac{\gamma_{n-k}^{(m)}}{\Gamma\left(1-\frac{k}{n}\right)} \left\{ t^{-\frac{k}{h}} \int_x^{+\infty} q_{n-k}(s)\,ds - \right. \end{aligned}$$

$$\int_{x}^{x+t} (x+t-s)^{-\frac{k}{n}} q_{n-k}(s)\,ds$$

$$+\int_{x}^{+\infty} q_{n-k}(s)\,ds \int_{0}^{t} (t-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi - \qquad (19)$$

$$\left.\int_{x}^{x+t} q_{n-k}(s)\,ds \int_{0}^{x+t-s} (x+t-s-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi \right\}$$

Rewriting equality (19) in the form

$$\begin{aligned} D_t A_m(x,t) &= -q_0(x+t) \\ &+ \int_{x}^{+\infty} q_0(s) A_m(s,t) - \int_{x}^{x+t} q_0(s) A_m(s,x+t-s)\,ds \\ &+ \sum_{k=1}^{n-1} \frac{\gamma_{n-k}^{(m)}}{\Gamma\left(1-\frac{k}{n}\right)} \left\{ t^{-\frac{k}{h}} \int_{x}^{+\infty} q_{n-k}(s)\,ds + \right. \\ &\int_{x}^{x+t} \left[t^{-\frac{k}{h}} - (x+t-s)^{-\frac{k}{n}}\right] q_{n-k}(s)\,ds \\ &+ \int_{0}^{t} (t-\xi)^{-\frac{k}{n}}\,d\xi \int_{x+t-\xi}^{+\infty} q_{n-k}(s) A_m(s,\xi)\,ds \\ &+ \int_{0}^{t} d\xi \int_{x}^{x+t-\xi} \left[(t-\xi)^{-\frac{k}{h}} \right. \\ &\left.\left. - (x+t-\xi-s)^{-\frac{k}{n}}\right] q_{n-k}(s) A_m(s,\xi)\,ds \right\}, \end{aligned}$$

we have

$$\begin{aligned} \|D_t A_m\|(x) &\leq \int_{x}^{+\infty} |q_0(s)|\,ds \\ &+ \sum_{k=1}^{n-1} \frac{2^{2-\frac{k}{n}}}{\Gamma\left(2-\frac{k}{n}\right)} \int_{x}^{+\infty} (s-x)^{1-\frac{k}{n}} |q_{n-k}(s)|\,ds \\ &+2 \int_{x}^{+\infty} |q_0(s)|\,\|A_m\|(s)\,ds + \end{aligned}$$

$$\sum_{k=1}^{n-1} \frac{2^{2-\frac{k}{n}}}{\Gamma\left(2-\frac{k}{n}\right)} \int_{x}^{+\infty} (s-x)^{1-\frac{k}{n}} \left|q_{n-k}(s)\right| \|A_m\|(s)\, ds$$

$$\leq \sigma(x) \left(1 + \int_{x}^{+\infty} |q_0(s)|\, ds \right.$$

$$\left. + \sum_{k=1}^{n-1} \frac{2^{2-\frac{k}{n}}}{\Gamma\left(2-\frac{k}{n}\right)} \int_{x}^{+\infty} (s-x)^{1-\frac{k}{n}} \left|q_{n-k}(s)\right| ds \right)$$

Therefore, the function $D_t A_m(x,t)$ is summable on the interval $[0;+\infty)$ with respect to t and inequality $(5')$ is valid.

In order to prove the summability of the functions $\left(D_t^{\frac{1}{n}}\right)^{\nu} A_m(x,t)$ on the interval $[0;+\infty)$ with respect to t, apply the operator of fractional derivative $D_t^{\frac{1}{n}}$ to both sides of equality (15) ν times successively and use the formula

$$D_t^{\frac{1}{n}} \left(\frac{t^{\pm\frac{k}{n}}}{\Gamma\left(1 \pm \frac{k}{n}\right)} \right) = \frac{t^{\pm\frac{k\mp 1}{n}}}{\Gamma\left(1 \pm \frac{k\mp 1}{n}\right)},$$

$$D_t^{\frac{1}{n}} \left(\frac{1}{\Gamma\left(1 \pm \frac{k}{n}\right)} \int_{x}^{x+t} (x+t-s)^{\pm\frac{k}{n}} q_k(s)\, ds \right)$$

$$= \frac{1}{\Gamma\left(1 \pm \frac{k\mp 1}{n}\right)} \int_{x}^{x+t} (x+t-s)^{\pm\frac{k\mp 1}{n}} q_k(s)\, ds$$

(see [7],p.47). Then we get for $\nu = \overline{1, n-1}$

$$\left(D_t^{\frac{1}{n}}\right)^{\nu} A_m(x,t) = \sum_{k=1}^{\nu} \frac{\gamma_{\nu-k}^{(m)}}{\Gamma\left(1-\frac{k}{n}\right)} \left\{ t^{-\frac{k}{n}} \int_{x}^{+\infty} q_{\nu-k}(s)\, ds \right.$$

$$- \int_{x}^{x+t} (x+t-s)^{-\frac{k}{n}} q_{\nu-k}(s)\, ds$$

$$+ \int_{x}^{+\infty} q_{\nu-k}(s)\, ds \int_{0}^{t} (t-\xi)^{-\frac{k}{n}} A_m(s,\xi)\, d\xi$$

$$- \int_{x}^{x+t} q_{\nu-k}(s)\, ds \int_{x}^{x+t-s} (x+ \qquad (20)$$

$$t-s-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi\Big\}$$

$$+\sum_{k=\nu}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k-\nu}{n}\right)} \left\{ t^{\frac{k-\nu}{n}} \int_x^{+\infty} q_k(s)\,ds \right.$$

$$-\int_x^{x+t} (x+t-s)^{\frac{k-\nu}{n}} q_k(s)\,ds$$

$$+\int_x^{+\infty} q_k(s)\,ds \int_0^t (t-\xi)^{\frac{k-\nu}{n}} A_m(s,\xi)\,d\xi$$

$$\left. -\int_x^{x+t} q_k(s)\,ds \int_0^{x+t-s} (x+t-s-\xi)^{\frac{k-\nu}{n}} A_m(s,\xi)\,d\xi \right\}.$$

Analogous to estimation for $D_t A_m(x,t)$, from (20), we conclude that the functions $\left(D_t^{\frac{1}{n}}\right)^{\nu} A_m(x,t)\,, \nu = 1,2,...,n-1$ are summable on $[0;+\infty)$ with respect to t.

3. Let $H_m(x,t) = D_t A_m(x,t) - D_x A_m(x,t)$. Taking into account formulae (15) and (19), we obtain

$$\begin{aligned} H_m(x,t) \quad = \quad & \sum_{k=1}^{n-1} \frac{\gamma_{n-k}^{(m)}}{\Gamma\left(1-\frac{k}{n}\right)} \left\{ t^{-\frac{k}{n}} \int_x^{+\infty} q_{n-k}(s)\,ds - \int_x^{x+t} (x+t-s)^{-\frac{k}{n}} q_{n-k}(s)\,ds \right. \\ & + \int_x^{+\infty} q_{n-k}(s)\,ds \int_0^t (t-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi \\ & \left. - \int_x^{x+t} q_{n-k}(s)\,ds \int_x^{x+t-s} (x+t-s-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi \right\} \qquad (21) \\ & + \int_x^{+\infty} q_0(s) A_m(s,t)\,ds - \sum_{k=1}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k}{n}\right)} \left\{ t^{\frac{k}{h}} \int_x^{+\infty} q_k'(s)\,ds \right. \\ & + \int_x^{+\infty} ds \int_0^t (t-\xi)^{-\frac{k}{n}} D_s\left[q_k(s) A_m(s,\xi)\right] d\xi \\ & \left. - \int_x^{x+t} ds \int_x^{x+t-s} (x+t-s-\xi)^{\frac{k}{n}} D_s\left[q_k(s) A_m(s,\xi)\right] d\xi \right\}. \end{aligned}$$

Applying the operators $\left(D_t^{\frac{1}{n}}\right)^{\nu}$, $\nu = 1, 2, ..., n-1$, to both sides of (21) we obtain:

$$\left(D_t^{\frac{1}{n}}\right)^{\nu} H_m(x,t) = \sum_{k=1}^{n-\nu-1} \frac{\gamma_{n-k}^{(m)}}{\Gamma\left(1-\frac{k+\nu}{n}\right)} \left\{ t^{-\frac{k+\nu}{n}} \int_x^{+\infty} q_{n-k}(s)\,ds \right.$$
$$- \int_x^{x+t} (x+t-s)^{-\frac{k+\nu}{n}} q_{n-k}(s)\,ds$$
$$+ \int_x^{+\infty} q_{n-k}(s)\,ds \int_0^t (t-\xi)^{-\frac{k+\nu}{n}} A_m(s,\xi)\,d\xi$$
$$- \int_x^{x+t} q_{n-k}(s)\,ds \int_x^{x+t-s} (x+t \tag{22}$$
$$\left. -s-\xi)^{-\frac{k+\nu}{n}} A_m(s,\xi)\,d\xi \right\}$$
$$- \sum_{k=\nu+1}^{n-1} \frac{\gamma_k^{(m)}}{\Gamma\left(1+\frac{k-\nu}{n}\right)} \left\{ t^{\frac{k-\nu}{n}} \int_x^{+\infty} q_k'(s)\,ds \right.$$
$$- \int_x^{x+t} (x+t-s)^{\frac{k-\nu}{n}} q_k'(s)\,ds +$$
$$\int_x^{+\infty} ds \int_0^t (t-\xi)^{\frac{k-\nu}{n}} D_s\left[q_k(s) A_m(s,\xi)\right]$$
$$- \int_x^{x+t} ds \int_0^{x+t-s} (x+t-s$$
$$\left. -\xi)^{\frac{k-\nu}{n}} D_s\left[q_k(s) A_m(s,\xi)\right] d\xi \right\}$$
$$+ \sum_{k=0}^{\nu} \gamma_k^{(m)} \int_x^{+\infty} q_k(s) \left(D_t^{\frac{1}{n}}\right)^{\nu-k} A_m(s,t)\,ds,$$

$$\left(D_t^{\frac{1}{n}}\right)^{n-1} H_m(x,t) = \sum_{k=0}^{n-1} \gamma_k^{(m)} \int_x^{+\infty} \left(D_t^{\frac{1}{n}}\right)^{n-k-1} A_m(s,t)\,ds, \nu = \overline{1, n-2} \tag{23}$$

From (22) and (23) by using estimations (4),(5),(5′) we have that the derivatives $\left(D_t^{\frac{1}{n}}\right)^\nu H_m(x,t)$, $\nu = 1, 2, ..., n-1$ are summable on $[0; +\infty)$ with respect to t. Now, formula (6) is immediately obtained from (23).

4. Finally, let us prove the properties 4) of the functions $A_m(x,t)$. Applying the operator $I_t^{1-\frac{1}{n}}$ to both sides of (20) and taking into account the formulae

$$I_t^{1-\frac{1}{n}}\left(\frac{t^{\pm\frac{k}{n}}}{\Gamma\left(1\pm\frac{k}{n}\right)}\right) = \frac{t^{1\pm\frac{k\mp1}{n}}}{\Gamma\left(2\pm\frac{k\mp1}{n}\right)}, \tag{24}$$

$$I_t^{1-\frac{1}{n}}\left(\frac{1}{\Gamma\left(1\pm\frac{k}{n}\right)}\int_x^{x+t}(x+t-s)^{\pm\frac{k}{n}}q_k(s)\,ds\right) = \frac{1}{\Gamma\left(2\pm\frac{k\mp1}{n}\right)}\int_x^{x+t}(x+t-s)^{1\pm\frac{k\mp1}{n}}q_k(s)\,ds$$

we have for $\nu = 0, 1, 2, ..., n-1$

$$\begin{aligned} I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^\nu A_m(x,t) &= \sum_{k=1}^{\nu}\frac{\gamma_{\nu-k}^{(m)}}{\Gamma\left(2-\frac{k+1}{n}\right)}\Bigg\{t^{1-\frac{k+1}{n}}\int_x^{+\infty}q_{\nu-k}(s)\,ds \\ &\quad -\int_x^{x+t}(x+t-s)^{1-\frac{k+1}{n}}q_{\nu-k}(s)\,ds \\ &\quad +\int_x^{+\infty}q_{\nu-k}(s)\,ds\int_0^t(t-\xi)^{1-\frac{k+1}{n}}A_m(s,\xi)\,d\xi \\ &\quad -\int_x^{x+t}q_{\nu-k}(s)\,ds\int_0^{x+t-s}(x+t \\ &\quad -s-\xi)^{1-\frac{k+1}{n}}A_m(s,\xi)\,d\xi\Bigg\} \\ &\quad +\sum_{k=\nu}^{n-1}\frac{\gamma_k^{(m)}}{\Gamma\left(2+\frac{k-\nu-1}{n}\right)}\Bigg\{t^{1+\frac{k-\nu-1}{n}}\int_x^{+\infty}q_k(s)\,ds \\ &\quad -\int_x^{x+t}(x+t-s)^{1+\frac{k-\nu-1}{n}}q_k(s)\,ds \end{aligned} \tag{25}$$

$$+\int_{x}^{+\infty} q_k(s)\,ds\int_0^t (t-\xi)^{1+\frac{k-\nu-1}{n}} A_m(s,\xi)\,d\xi$$
$$-\int_{x}^{x+t} q_k(s)\,ds\int_0^{x+t-s} (x+t$$
$$-s-\xi)^{1+\frac{k-\nu-1}{n}} A_m(s,\xi)\,d\xi$$

From (25) it is easy to see that

$$I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^{\nu} A_m(x,t)\,|_{t=0}=0,\qquad \nu=0,1,2,...,n-2 \text{ and}$$

$$I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^{n-1} A_m(x,t)=A_m(x,0)=\int_x^{+\infty} q_0(s)\,ds,$$

i.e., equalities (7),(8) are obtained. Further, from (21), after some simple transformations we have

$$\begin{aligned} H_m(x,t) &= \int_x^{+\infty} q_0(s)\,dsA_m(s,t)\,ds+\sum_{k=1}^{n-1}\frac{\gamma_{n-k}}{\Gamma\left(1-\frac{k}{n}\right)}\left\{t^{-\frac{k}{n}}\int_x^{+\infty} q_{n-k}(s)\,ds\right.\\ &\left.+\int_x^{+\infty} q_{n-k}(s)\,ds\int_0^t (t-\xi)^{-\frac{k}{n}} A_m(s,\xi)\,d\xi\right\}. \qquad (26)\end{aligned}$$

Similarly, applying the operator $I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^{\nu}, \nu=0,1,2,...,n-1$ to (26), we have

$$\begin{aligned} I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^{\nu} H_m(x,t) &= \sum_{k=1}^{n-\nu-1}\frac{\gamma_{n-k}^{(m)}}{\Gamma\left(2-\frac{k+\nu+1}{n}\right)}\left\{t^{1-\frac{k+\nu+1}{n}}\int_x^{+\infty} q_{n-k}(s)\,ds\right.\\ &+\int_x^{+\infty} q_{n-k}(s)\,ds\int_0^t (t-\xi)^{1-\frac{k+\nu+1}{n}}\\ &\left. A_m(s,\xi)\,d\xi\right\} \qquad (27)\\ &+\sum_{k=0}^{\nu}\gamma_{\nu-k}^{(m)}\int_x^{+\infty} q_{\nu-k}(s)\,I_t^{1-\frac{1}{n}}\left(D_t^{\frac{1}{n}}\right)^{k} A_m(s,t)\,ds,\\ \nu &= 0,1,2,...,n-2\end{aligned}$$

Now, equality (9) follows immediately from (27).

The proof of the theorem is completed.

Remark : It is clear from (15′) and (19) that if the functions $q_0(x), q_1'(x), ..., q_{n-1}'(x)$ are differentiable and the conditions

$$\int_0^{+\infty} (1+s)|q_0'(s)|\,ds < \infty, \qquad \int_0^{+\infty} (1+s)^{1+\frac{k}{n}}|q_k''(s)|\,ds < \infty$$

are valid, then the functions $D_{xx}^2 A_m(x,t), D_{xt}^2 A_m(x,t), \left(D_t^{\frac{1}{n}}\right)^n (D_t A_m(x,t))$ are summable with respect to t on the interval $(0;+\infty)$, and $A_m(x,t)$ satisfies the equation

$$\begin{aligned} D_{xx}^2 A_m(x,t) - D_{xt}^2 A_m(x,t) &= q_0(x) A_m(x,t) \\ &+ \sum_{k=0}^{n-2} \gamma_{n-k-1}^{(m)} q_{n-k-1}(x) I_t^{1-\frac{1}{n}} \left(D_t^{\frac{1}{n}}\right)^k A_m(x,t). \end{aligned}$$

Perspectives

The results of this paper, particularly the properties of the kernels $A_m(x,t)$, can be applied to investigate the direct and inverse scattering problems for equation (1) with boundary condition $y(0) = 0$, what needs independent investigation.

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