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Asymptotic expansions for a renewal-reward process with Weibull distributed interference of chance

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Abstract. In this study, a renewal-reward process ($X(t)$) with a Weibull distributed interference of chance is investigated. Under the assumption that the process $X(t)$ is ergodic, two-term asymptotic expansion is obtained for the ergodic distiribution of the process $X(t)$, as $\lambda \to 0$. Also, the weak convergence theorem is proved for the ergodic distribution of the process $X(t)$, as $\lambda \to 0$. Moreover, two-term asymptotic expansions are derived for n^{th}-order moments $n = 1, 2, ...$ of the process $X(t)$, as $\lambda \to 0$. Based on these results, the asymptotic expansions are obtained for the skewness and kurtosis of the process $X(t)$, as $\lambda \to 0$.

Key words. Stochastic processes, probability theory, control theory.

1 Introduction

Many interesting problems resulting from the theories of stock control, reliability, queuing, mathematical biology, stochastic finance, mathematical insurance and so on can be expressed by the renewal process, renewal-reward process, random walk and other processes or by the help of the modification of these processes. There are many valuable studies on these processes in the literature (for example, [1,8,12,14,19–23]). However, the existing studies are generally theoretical and they are not exactly helpful to solve concrete problems in practice due to the complexity of their mathematical formulas. Therefore, in addition to exact formulas, several approximated formulas are offered for these kinds of problems in the literature (for example, [7–9, 12, 19–23]). For example, Brown and Solomon (1975) considered the following renewal-reward process with absolutely continuous component:

$$C(t) = \begin{cases} 0 & , \quad t < X_0 \\ \sum\limits_{k=0}^{N(t)-1} Y_k , & t \geq X_0; \end{cases} , \quad N(t) = min\{k : S_k > t\}; \quad S_n = \sum_{i=0}^{n} X_i, \quad n = 0, 1, 2, ...$$

Here $\{X_i,\ i = 0, 1, 2, ...\}$ is a renewal sequence and $\{(X_i, Y_i),\ i = 0, 1, 2, ...\}$ is a sequence of independent and identically distributed random vectors. Brown and Solomon obtained the second-order asymptotic expansions for the first and second moments of the renewal-reward process $C(t)$.

Note that the renewal-reward process $C(t)$ occurs in various stochastic optimization models, particularly in Markov and semi-Markov decision models. In these models, Y_i represents the reward or cost associated with a given policy over the renewal interval $(S_{i-1}, S_i]$.

Another important problem in this area was considered by Alsmeyer (1988). He considered the following extended renewal process $\{S_n, U_n\}_{n\geq 0}$, where $S_n = \sum\limits_{i=0}^{n} X_i$, $U_n = \sum\limits_{i=0}^{n} Y_i$. Under appropriate conditions on two dimensional random vectors (X_i, Y_i), $i \geq 0$, he obtained asymptotic expansions for $EU_{T(t)}$, $VarU_{T(t)}$ and $Cov(U_{T(t)}, T(t))$, as $t \to \infty$, where $T(t) = inf\{n \geq 0 : S_n > t\}$. Corresponding results for $EU_{N(t)}$, $VarU_{N(t)}$ and $Cov(U_{N(t)}, N(t))$ are obtained, when X_0, X_1 are both almost surely non-negative and $N(t) = sup\{n \geq 0 : S_n \leq t\}$.

One of the most common application areas of the above mentioned renewal-reward process $\{S_n, U_n\}_{n\geq 0}$ is insurance theory. In collective risk problems the random variables X_1, X_2,... are interpreted as the time between claims; Y_1, Y_2,... are interpreted as the corresponding claim amounts; $N(t)$, $t \geq 0$ denotes the number of claims up to time t, and $U_{N(t)}$ denotes the total value of claims made until the time t by the insurance company (see, for example, Ross, 1996).

Moreover, Csenki (2000) derived an asymptotic representation for the expected value of renewal-reward processes with retrospective reward structure. Levy and Taqqu (2000) investigated renewal-reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards. In their paper, both the inter-renewal times and the rewards were allowed to have infinite variance.

In recent years, the inventory models have been extensively considered and some of their characteristics are investigated in the literature (for example, [2, 4–6, 16, 17]). In most of these studies, clear analytical solutions could not be obtained. Instead of analytical methods, heuristic approaches, dynamic programming, etc., are used in these studies. In addition, in most of these studies both the demand quantity (η_n) and the time (ξ_n) between sequential demands are assumed to have exponential distributions. However, in this study, η_n and ξ_n are assumed to be arbitrary random variables. Likewise, Bekar and et al. (2012) studied on three-term asymptotic expansion for the moments of a renewal-reward process with gamma distributed interference of chance (see, [13]). As distinct from the study [13], the random variables representing the interference have Weibull distribution in the present study. Gamma and Weibull distributions are the different classes. Therefore, they express different situations.

The approximation results are generally simpler and easier for application. On the other hand, it is desirable that approximation results should be reasonably close to exact expressions. One of the most effective methods to obtain this kind of approximation is the asymptotic expansion method. In many cases, it is possible to obtain such approximations, which are closer to the exact expressions, by increasing the number of terms in the asymptotic expansions. However, when the number of terms in the asymptotic expansions is increased, the approximations start to lose their simplicity and meaning. For this reason, we believe that in many cases, the two- or three-term asymptotic expansions are sufficient to obtain the convenient approximation formulas. Because of that, in this study, an inventory model is considered and two-term asymptotic expansions are obtained for the ergodic distribution and ergodic moments of this model.

Let us consider the following inventory model before expressing the problem mathematically.

The inventory model. Assume that the stock level in a depot at the initial time ($t = 0$) is equal to $X(0) \equiv X_0 \equiv s + v$, where $0 < s < \infty$ represents the stock control level and $v > 0$. In addition, it is assumed that, at random times T_1, T_2,..., T_n,..., the stock level ($X(t)$) in the depot decreases by η_1, η_2,..., η_n,...., respectively, until the stock level $X(t)$ falls below the predetermined control level s. Thus, the stock level in the depot changes as follows:

$$X(T_1) \equiv X_1 = s + v - \eta_1, \quad X(T_2) \equiv X_2 = s + v - (\eta_1 + \eta_2), ...,$$

$$X(T_n) \equiv X_n = s + v - \sum_{i=1}^{n} \eta_i, \tag{1.1}$$

where η_n represents the quantity of the n^{th} demand, $n = 1, 2, 3, ...$

In other words, demands are inserted to the system at the random times $T_n = \sum\limits_{i=1}^{n} \xi_i$, where ξ_n represents the time between $(n-1)^{th}$ and n^{th} demand, $n = 1, 2, 3, ...$ The system passes from one state to another by jumping at time T_n, according to the quantities of demand $\{\eta_n\}$, $n \geqslant 1$ as shown in equation (1.1). This variation of the system continues up to a certain random time τ_1, where τ_1 is the first time that the stock level $X(t)$ drops below the control level $s > 0$.

When this occurs, the system is immediately brought to the state ζ_1. Thus, the first period completes, and the second one starts. Afterwards, the process $X(t)$ continues to change from the new initial state ζ_1, similar to the way it changed in the first period. When the stock level $X(t)$ falls below s, for the second time, by an interference to the system, the stock level is brought to the random level ζ_2, similar to the preceding period, and so on. Here, ζ_1, ζ_2,... are independent and identically distributed positive-valued random variables.

Weibull distribution is usually used for construction of the probability models of the situations in which the behavior of object is defined by "the weakest link". "The weakest link" property can be explained as follows. Let $X_1, X_2, ..., X_n$ are independent and identically distributed random variables and $X(1) = \min\{X_1, X_2, ..., X_n\}$,$X(n) = \min\{X_1, X_2, ..., X_n\}$. In this case, the random variables $X(1)$ or $X(n)$ express the property so-called "the weakest link". The basic and initial studies dedicated to the investigation of the distributions of the random variables $X(1)$ and $X(n)$ were made by B.V. Gnedenko and etc. It is shown that distributions of $X(1)$ and $X(n)$, as a rule, are well described by Weibull distribution, for the large n. In other hands, it is known that a discrete interference of chance usually arises as a result of numerous random factors. Therefore, we can accept it as a maximum (or a minimum) of the identically distributed random variables. For this reason, in this study we describe a discrete interference of chance by the Weibull distribution with parameters (α, λ), $\alpha > 1$, $\lambda > 0$. Under this assumption, we aim to investigate the asymptotic behavior of the process $X(t)$, as $\lambda \to 0$.

We now proceed to a mathematical construction of the process $X(t)$.

2 Mathematical construction of the process $X(t)$

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$ are two independent sequences of random variables defined on any probability space $(\Omega, \Im, P)$, such that variables in each sequence are independent and identically distributed. Introduce also, sequence of random variables $\{\zeta_n\}$, $n \geq 1$ which describes the discrete interference of chance and form an ergodic Markov chain with the stationary Weibull distribution with parameters (α, λ), $\alpha > 1$, $\lambda > 0$. Suppose that ξ_i's and η_i's take only positive values and their distribution functions be denoted by $\Phi(t)$ and $F(x)$, respectively. So,

$$\Phi(t) = P\{\xi_1 \leq t\},\ t > 0; \quad F(x) = P\{\eta_1 \leq x\},\ x > 0.$$

Using the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$, define the renewal sequences $\{T_n\}$, $\{S_n\}$ and their distribution functions as follows:

$$T_n = \sum_{i=1}^{n} \xi_i, \quad S_n = \sum_{i=1}^{n} \eta_i, \quad T_0 = S_0 = 0, \;\; n \geq 1,$$

$$\Phi_n(t) = P\{T_n \leq t\}, \quad F_n(x) = P\{S_n \leq x\}.$$

Moreover, we define the sequence of integer-valued random variables $\{N_n\}$, $n \geq 0$, as follows:

$$N_0 = 0, \quad N_1 = N(v) = inf\{k \geq 1 : S_k > v\}, \;\; v \geq 0;$$

$$\begin{aligned} N_{n+1} \equiv N_{n+1}(\zeta_n) &= inf\{k \geq N_n + 1 : s + \zeta_n - S_k + S_{N_n} < s\} \\ &= inf\{k \geq N_n + 1 : S_k - S_{N_n} > \zeta_n\}, \;\; n \geq 1, \end{aligned}$$

where $inf(\emptyset) = +\infty$ is stipulated. Put

$$\tau_0 = 0; \quad \tau_1 = T_{N_1} = \sum_{i=1}^{N(v)} \xi_i; \quad \tau_{n+1} \equiv \tau_{n+1}(\zeta_n) = T_{N_{n+1}} = \sum_{i=1}^{N_{n+1}} \xi_i, \;\; n \geq 1,$$

and define $\nu(t)$ as:

$$\nu(t) = max\{n \geq 0 : T_n \leq t\}, \;\; t > 0.$$

We can now construct desired stochastic process $X(t)$ as follows:

$$X(t) = s + \zeta_n - S_{\nu(t)} + S_{N_n}, \quad if \;\; \tau_n \leq t < \tau_{n+1}, \;\; n \geq 0,$$

where $\zeta_0 = s + v$ and $S_{\nu(\tau_n+0)} = S_{N_n}$.

We call the process $X(t)$ as "a renewal-reward process with Weibull distributed interference of chance".

Figure 1 gives a trajectory of the process $X(t)$.

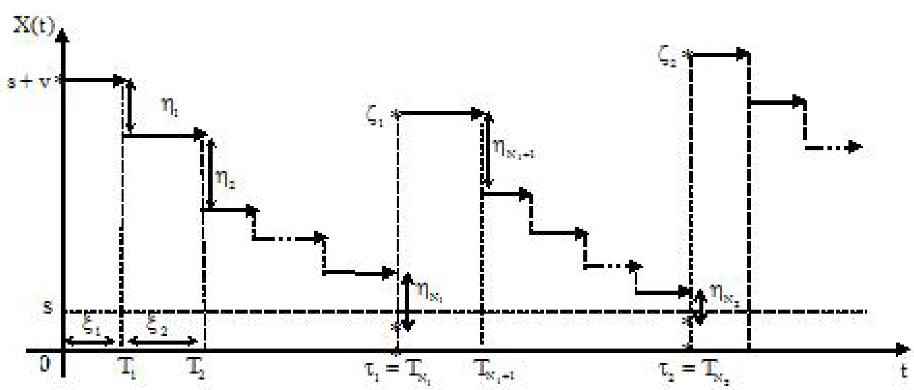

Figure 1: A trajectory of the process $X(t)$.

3 Preliminary discussions

First, we state the following proposition on the ergodicity of the process $X(t)$.

Proposition 3.1 *Let the initial sequences of the random variables* $\{\xi_n\}$, $\{\eta_n\}$ *and* $\{\zeta_n\}$, $n \geq 1$ *satisfy the following additional conditions:*

1) $E\xi_1 < \infty$*;*

2) $m_2 = E(\eta_1^2) < \infty$*;*

3) η_1 *is non-arithmetic random variable;*

4) *The sequence of the random variables* $\{\zeta_n\}$, $n \geq 1$ *which describes the discrete interference of chance forms an ergodic Markov chain having Weibull distribution.*

Then the process $X(t)$ *is ergodic (see,* [15]*).*

4 Asymptotic expansion for the ergodic distribution of the process $X(t)$

For a detailed study, let X_λ be a stochastic process with ergodic distribution of $X(t) - s$. Then

$$Q_\lambda(x) \equiv \lim_{t\to\infty} P\{X_\lambda \leq x\} = 1 - \frac{EU(\zeta_1 - x)}{EU(\zeta_1)}, \tag{4.1}$$

where $U(x) = \sum\limits_{n=0}^{\infty} F_n(x)$ is a renewal function generated by the sequence $\{\eta_n\}$, $n \geq 1$ (see, [15]).

Remark 4.1 *Obtaining exact and evident formulas for the renewal function* $U(x)$ *is very difficult, when the random variable* η_1 *has a general distribution. Therefore, the use of equation (4.1) is not advisable to obtain an exact expression for the process* X_λ*. So, it has to obtain asymptotic results for the process* X_λ*, when the distribution of random variable* η_1 *is from a general class.*

For this reason, let's give the following auxiliary lemma:

Lemma 4.2 $G_n(x) \equiv x^n g(x)$*,* $n \in \mathbb{N}$ *, where* $g(x)$ ($g : \mathbb{R}^+ \to \mathbb{R}$) *be a bounded function and* $\lim\limits_{x\to\infty} g(x) = 0$*. Then the following asymptotic relation is true for each* $\alpha > 1$*, as* $\lambda \to 0$*:*

$$\int_0^\infty e^{-t} G_n\left(\frac{t^{1\backslash\alpha}}{\lambda}\right) dt = o\left(\frac{1}{\lambda^n}\right).$$

This lemma is proved similar to Lemma 4.1 in the study [15].

Let's now investigate the asymptotic behavior of the ergodic distribution function $(Q_{W_\lambda}(x))$ of the process $W_\lambda(t) \equiv \lambda(X(t)-s)$, as $\lambda \to 0$.

Theorem 4.3 *Let the conditions of Proposition 3.1 be satisfied. Then the following asymptotic expansion for the ergodic distribution of the process $W_\lambda(t)$ can be written, as $\lambda \to 0$:*

$$Q_{W_\lambda}(x) = R_\alpha(x) + \frac{m_2}{2m_1 c_1(\alpha)} \left(\pi_{\alpha,1}(x) - R_\alpha(x)\right)\lambda + o(\lambda), \tag{4.2}$$

where

$$m_k = E(\eta_1^k),\;\; k=1,2;\;\; R_\alpha(x) = \frac{1}{c_1(\alpha)}\int_0^x (1-\pi_{\alpha,1}(t))dt;\;\; c_1(\alpha) = \Gamma(1+1\setminus\alpha);$$

$$\pi_{\alpha,1} = 1 - e^{-x^\alpha} \;\; is\ the\ Weibull\ distribution\ function\ with\ parameters\ (\alpha,1).$$

Proof. The exact formula for the ergodic distribution function of the process $W_\lambda(t) \equiv \lambda(X(t)-s)$ can be written as follows:

$$Q_{W_\lambda}(x) \equiv \lim_{t\to\infty} P\left\{W_\lambda(t) \le x\right\} = 1 - EU(\zeta_1 - \frac{x}{\lambda})\,\left(EU(\zeta_1)\right)^{-1}. \tag{4.3}$$

If the condition $m_2 = E(\eta_1^2) < +\infty$ is satisfied, then the following asymptotic expansion is correct (see, [22, p.366]):

$$U(z) = \frac{z}{m_1} + \frac{m_2}{2m_1^2} + G_1(z),\;\; as\;\; z \to \infty, \tag{4.4}$$

where $G_1(z) = g(z),\;\; g(z) = o(1)$.

Then using equation(4.4), the following equation is obtained , as $\lambda \to 0$:

$$EU(\zeta_1) = \frac{1}{m_1}E(\zeta_1) + \frac{m_2}{2m_1^2} + E(G_1(\zeta_1)), \tag{4.5}$$

where

$$E(G_1(\zeta_1)) = \int_0^\infty G_1(z)d\pi(z) = \int_0^\infty e^{-t} g(\frac{t^{1\setminus\alpha}}{\lambda})dt. \tag{4.6}$$

Using Lemma 4.2 for equation (4.6), we have for each $\alpha > 1$, as $\lambda \to 0$, $E(G_1(\zeta_1)) = o(1)$, then regarding the random variable ζ_1 has the Weibull distribution

$$EU(\zeta_1) = \frac{c_1(\alpha)}{\lambda m_1} + \frac{m_2}{2m_1^2} + o(1)\;\; such\ that\;\; c_1(\alpha) = \Gamma(1+1\setminus\alpha). \tag{4.7}$$

Analogically, we get:

$$EU(\zeta_1 - \frac{x}{\lambda}) = \frac{1}{\lambda m_1}\left(c_1(\alpha) - \int_0^{x^\alpha} e^{-t}t^{1\setminus\alpha}dt\right) + \left(\frac{m_2}{2m_1^2} - \frac{x}{\lambda m_1}\right)e^{-x^\alpha} + o(\lambda). \tag{4.8}$$

Substituting equation (4.7) and (4.8) into equation (4.3) and carrying out the corresponding calculations, we get equation (4.2), as $\lambda \to 0$. ∎

Now, we can prove the weak convergence theorem for the ergodic distribution function of the process $W_\lambda(t)$, as $\lambda \to 0$.

Theorem 4.4 (Weak convergence theorem) *Under the conditions of Theorem 4.3, for each* $\alpha > 1$ *we have:*

$$\lim_{\lambda\to 0} Q_{W_\lambda}(x) = R_\alpha(x). \tag{4.9}$$

Proof. Since $R_\alpha(x)$ and $\pi_{\alpha,1}(x)$ are distribution functions, then we have $0 \le R_\alpha(x) \le 1$ and $0 \le \pi_{\alpha,1}(x) \le 1$ for each $x \ge 0$. Therefore

$$\max_x |\pi_{\alpha,1}(x) - R_\alpha(x)| \le 1. \tag{4.10}$$

Since $m_2 = E(\eta_1^2) < \infty$, by using equation (4.10), then we have for each $\alpha > 1$:

$$\frac{m_2}{2m_1c_1(\alpha)} |\pi_{\alpha,1}(x) - R_\alpha(x)| \le \frac{m_2}{2m_1c_1(\alpha)} < \infty. \tag{4.11}$$

Then second term in asymptotic expansion (4.2) tends to zero, as $\lambda \to 0$. In other words, the ergodic distribution of the process $W_\lambda(t)$ weakly converges to the limit distribution $R_\alpha(x)$, as $\lambda \to 0$. ∎

5 Asymptotic expansion for the ergodic moments of the process $X(t)$

To shorten the notations, let $EX_\lambda^n = \lim_{t\to\infty} E(X_\lambda^n(t))$, $n = 1, 2, ...$

Theorem 5.1 *Let the conditions of Proposition 3.1 be satisfied. Then the ergodic moments* EX_λ^n*,* $n = 1, 2, ...$ *can be written as follows:*

$$EX_\lambda^n = nEU_n(\zeta_1)[EU(\zeta_1)]^{-1}, \tag{5.1}$$

where

$$U_n(x) \equiv x^{n-1} * U(x) \equiv \int_0^x (x-t)^{n-1} U(t)dt, \ \ n \ge 1.$$

Proof. Using equation (4.1), we can obtain:

$$EX_\lambda^n = n\int_0^\infty x^{n-1}(1 - Q_\lambda(x))dx = \frac{n}{EU(\zeta_1)} E\int_0^{\zeta_1} x^{n-1} U(\zeta_1 - x)dx = \frac{nEU_n(\zeta_1)}{EU(\zeta_1)}. \tag{5.2}$$

■

Using this formula, it is possible to obtain exact formulas for the moments of the process $X(t)$ such as in following example.

Example 5.1 *Let the conditions of Theorem 5.1 be satisfied. Assume that the random variable η_1 has the exponential distribution with parameter $\mu > 0$. Then n^{th}-order $(n = 1, 2, ...)$ ergodic moments of the process $X(t)$ can be written, as follows:*

$$EX_\lambda^n = \frac{\mu}{\lambda + \mu c_1(\alpha)} \frac{c_{(n+1)}(\alpha)}{(n+1)} \frac{1}{\lambda^n} + \frac{c_n(\alpha)}{\lambda + \mu c_1(\alpha)} \frac{1}{\lambda^{n-1}}, \tag{5.3}$$

where

$$c_{n1}(\alpha) = \frac{c_n(\alpha)}{nc_1(\alpha)}; \quad c_n(\alpha) = \Gamma(1 + n \setminus \alpha).$$

Proof. Since the random variable η_1 has the exponential distribution with parameter $\mu > 0$, then it is known that the renewal function $U(z) = \mu z + 1$ is generated by distribution of η_1. Then for $z - x \geq 0$

$$\int_0^\infty x^{n-1} EU(\zeta_1 - x)dx = \frac{\mu}{n(n+1)} E(\zeta_1^{n+1}) + \frac{1}{n} E(\zeta_1^n) \tag{5.4}$$

and

$$EU(\zeta_1) = \int_0^\infty U(z)d\pi(z) = \mu E(\zeta_1) + 1. \tag{5.5}$$

As a result, regarding the random variable ζ_1 has the Weibull distribution, substituting equations (5.4), (5.5) into equation (5.1), we get the exact formula for EX_λ^n $(n = 1, 2, ...)$, as equation (5.3). ■

Now we aim to obtain two-term asymptotic expansion for the moments EX_λ^n, when the distribution of random variable η_1 is from a general class, using the asymptotic properties of the renewal function $U(z)$.

Theorem 5.2 *Let the conditions of Theorem 5.1 be satisfied and $E(\eta_1^3) < +\infty$. Then the following two-term asymptotic expansion for the n^{th}-order $(n = 1, 2, ...)$ moments of the process $X(t)$ can be written, for each $\alpha > 1$, as $\lambda \to 0$:*

$$EX_\lambda^n = \frac{A_n}{\lambda^n} + \frac{B_n}{\lambda^{n-1}} + o(\frac{1}{\lambda^{n-1}}), \tag{5.6}$$

where

$$m_k = E(\eta_1^k), \;\; k = 1, 2; \;\; c_{n1}(\alpha) = \frac{c_n(\alpha)}{nc_1(\alpha)}; \;\; c_n(\alpha) = \Gamma(1 + n \setminus \alpha);$$

$$A_n = c_{(n+1)1}(\alpha); \;\; B_n = \left(nc_{n1}(\alpha) - \frac{c_{(n+1)1}(\alpha)}{c_1(\alpha)}\right) \frac{m_2}{2m_1}.$$

Proof. According to equation (5.1), it needs to use the asymptotic expansions of $EU_n(\zeta_1)$ and $EU(\zeta_1)$ to obtain two-term asymptotic expansion for the moments EX_λ^n, as $\lambda \to 0$. From equation (4.7), it is known that

$$EU(\zeta_1) = \frac{c_1(\alpha)}{\lambda m_1} + \frac{m_2}{2m_1^2} + o(1). \tag{5.7}$$

If $E(\eta_1^3) < +\infty$ is satisfied, then the following asymptotic expansion is correct

$$U_n(z) = \frac{z^{n+1}}{m_1 n(n+1)} + \frac{m_2 z^n}{2m_1^2 n} + G_n(z),\ \ as\ \ z \to \infty \tag{5.8}$$

where $G_n(z) = z^n g(z)$, $g(z) = o(1)$.

Then using Lemma 4.2, we can evidently obtain for each $\alpha > 1$, as $\lambda \to 0$:

$$EU_n(\zeta_1) = \frac{1}{m_1 n(n+1)} E(\zeta_1^{n+1}) + \frac{m_2}{2m_1^2 n} E(\zeta_1^n) + o(\frac{1}{\lambda^n}). \tag{5.9}$$

As a result, regarding the random variable ζ_1 has the Weibull distribution, substituting equation (5.7), (5.9) into equation (5.1), we get the asymptotic expansion for EX_λ^n such as equation (5.6). ∎

Corollary 5.3 *Let the conditions of Theorem 5.2 be satisfied. Then the following asymptotic expansions can be written for the skewness (γ_3) and kurtosis (γ_4) of the process $X(t)$, as $\lambda \to 0$:*

$$\gamma_3 \sim \frac{A_3 - 3A_1A_2 + 2A_1^3}{\sqrt{A_2^3 - 3A_1^2A_2^2 + 3A_1^4A_2 - A_1^6}},\ \gamma_4 \sim \frac{A_4 + 12A_1^2A_2 - 6A_1^4 - 3A_2^2}{A_2^2 - 2A_1^2A_2 + A_1^4}.$$

6 Conclusion

In this study, a semi-Markovian model which occurs frequently in engineering sciences (for example, in inventory, stock control, queuing and, etc., models) is considered. This model is expressed using of a renewal-reward process with a discrete interference of chance. An exact expression of the ergodic distribution function and ergodic moments of the process considered in this study is derived when the random variable ζ_1, which defines a discrete interference of chance, has Weibull distribution. However, it is very difficult to use this exact expression in solving concrete problems of inventory or queuing theory, because this exact expression has a very complex mathematical structure. In this study, an asymptotic method is used to overcome this mathematical difficulty. At the same time, the two-term asymptotic expansions for the ergodic distribution and ergodic moments are obtained. Moreover, weak convergence theorem for ergodic distribution is proved.

Note that it is important to obtain similar asymptotic results for the delayed (s, S) models by using the methods and approaches introduced in this paper. Applying this approach to the other distributions is another promising direction for future research.

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