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Determination of a control parameter for the Schrödinger equation

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Abstract. In the present paper, the boundary value problem for the differential equation with parameter p

$$\begin{cases} i\frac{du(t)}{dt}+Au(t)+iu(t)=f(t)+p,\ 0<t<T, \\ u(0)=\varphi,\ u(T)=\psi \end{cases}$$

in a Hilbert space H with self-adjoint positive definite operator A is investigated. The well-posedness of this problem is established. The stability inequalities for the solutions of two different type of determination of a control parameter problems for the Schrödinger equation are obtained.

Key words. Determination of a control parameter problem, Schrödinger equation, stability, well-posedness.

1 Introduction. Formulation of the problem

The differential equations with parameters play a very important role in many branches of science and engineering. These kind of equations arise for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion and control theory (see [1-7]). The differential equations with parameters have been studied extensively by many researchers (see, [8-22] and the references therein). However, such problems were not well-investigated in general.

Our goal in this paper is to investigate Schrödinger equations with parameter. It is known that various boundary value problems for Schrödinger equations with parameter can be reduced to the boundary value problem for the differential equation with parameter p

$$\begin{cases} i\frac{du(t)}{dt}+Au(t)+iu(t)=f(t)+p,\ 0<t<T, \\ u(0)=\varphi,\ u(T)=\psi \end{cases} \tag{1.1}$$

in a Hilbert space H with self-adjoint positive definite operator A. In the present paper, the well-posedness of boundary value problem (1.1) for the differential equation with parameter p is investigated. It is clear that

$$u(t) = v(t) + (A + iI)^{-1} p, \tag{1.2}$$

and

$$p = (A + iI) (\psi - v(T)) , \tag{1.3}$$

where $v(t)$ is the solution of the following nonlocal boundary value problem for the differential equation with parameter p

$$\begin{cases} i\frac{dv(t)}{dt} + Av(t) + iv(t) = f(t),\ 0 < t < T, \\ v(0) - v(T) = \varphi - \psi. \end{cases} \tag{1.4}$$

The well-posedness of problem (1.1) is established. In applications, the stability inequalities for the solution of two determination of a control parameter problems for the Schrödinger equation are obtained. The paper is organized as follows. Section 1 is introduction. In Section 2, the main theorem on stability of problem (1.1) is established. In Section 3, theorems on the stability inequalities for the solutions of two determination of a control parameter problems for the Schrödinger equation are obtained. Finally, Section 4 is conclusion.

2 The main theorem on stability

Throughout this work, H is a Hilbert space, A is a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > 0$. We have that

$$\| exp\{-itA\} \|_{H \to H} \leq 1. \tag{2.1}$$

From (2.1) it follows that

$$\| (I - exp\{iA - I\})^{-1} \|_{H \to H} \leq \frac{1}{1 - e^{-1}} = M_1. \tag{2.2}$$

In this section, the solvability of problem (1.1) in the space $C(H)$ of the continuous H-valued functions $\varphi(t)$ defined on $[0, 1]$, equipped with the norm

$$\| \varphi \|_{C(H)} = \max_{0 \leq t \leq T} ||\varphi(t)||_H$$

is investigated. We will prove the following main theorem on continuously dependents of the solution on the given data.

Theorem 2.1 *Let φ, $\psi \in D(A)$ and $f(t) \in C^{(1)}(H)$. Then, for the solution $(v(t), p)$ of problem (1.1) in $C(H) \times H$ the estimates*

$$\| p \|_H \leq M \left[\|\psi\|_H + \|A\psi\|_H + \|\varphi\|_H + \|A\varphi\|_H + \| f \|_{C^{(1)}(H)} \right], \tag{2.3}$$

$$\| v \|_{C(H)} \leq M \left[\|\varphi\|_H + \|\psi\|_H + \| f \|_{C(H)} \right] \tag{2.4}$$

hold. Here, M does not depend on φ, ψ and $f(t)$ and $C^{(1)}(H)$ is the space obtained by completion of the space of all smooth H-valued functions $\varphi(t)$ on $[0,1]$ with the norm

$$\| \varphi \|_{C^{\beta}(H)} = \| \varphi \|_{C(H)} + \sup_{0 \leq t \leq T} \|f'(t)\|_H .$$

Proof. It is known that for smooth data of the problem

$$iv'(t) + Av(t) + iv(t) = f(t) + p, (0 \leq t \leq T),\ v(0) = \varphi, \tag{2.5}$$

there exists a unique solution of problem (2.5), and the following formula holds:

$$v(t) = e^{i(A+i)t}\varphi + \int_0^t e^{-i(A+i)(t-s)}(-i)f(s)ds + (A+i)^{-1}[I - e^{-i(A+i)t}]p. \tag{2.6}$$

Actually, we have that

$$e^{-i(A+i)t}\left[v'(t) - (A+i)v(t)\right] = e^{-i(A+i)t}\left[-if(t) - ip\right].$$

Taking the integral, we get

$$\int_0^t \left[v(s)e^{-i(A+i)s}\right]' ds = \int_0^t e^{-i(A+i)s}(-i)f(s)ds + \int_0^t e^{-i(A+i)s}(-i)pds.$$

Then

$$v(t) - e^{i(A+i)t}v(0) = \int_0^t e^{-i(A+i)(t-s)}(-i)f(s)ds + \int_0^t e^{-i(A+i)(t-s)}(-i)pds.$$

Since $v(0) = \varphi$ and

$$\int_0^t e^{-i(A+i)(t-s)}(-i)ds = (A+i)^{-1}[I - e^{-i(A+i)t}],$$

we have that

$$v(t) = e^{i(A+i)t}\varphi + \int_0^t e^{-i(A+i)(t-s)}(-i)f(s)ds + (A+i)^{-1}[I - e^{-i(A+i)t}]p.$$

Using the condition $v(T) = \psi$, we get

$$\psi = v(T) = e^{i(A+i)T}\varphi + \int\limits_0^T e^{-i(A+i)(T-s)}(-i)f(s)ds + (A+i)^{-1}[I - e^{-i(A+i)T}]p.$$

From that it follows that

$$(A+i)^{-1}[I - e^{-i(A+i)T}]p = \psi - e^{i(A+i)T}\varphi - \int\limits_0^T e^{-i(A+i)(T-s)}(-i)f(s)ds.$$

Then

$$(A+i)^{-1}p = [I - e^{-i(A+i)T}]^{-1}\left[\psi - e^{i(A+i)T}\varphi - \int\limits_0^T e^{-i(A+i)(T-s)}(-i)f(s)ds\right] \tag{2.7}$$

or

$$p = [I - e^{-i(A+i)T}]^{-1}(A+i)\left[\psi - e^{i(A+i)T}\varphi - \int\limits_0^T e^{-i(A+i)(T-s)}(-i)f(s)ds\right].$$

We have that

$$-(A+i)\int\limits_0^T e^{-i(A+i)(T-s)}(-i)f(s)ds = f(T) - e^{-i(A+i)T}f(0) - \int\limits_0^T e^{-i(A+i)(T-s)}f'(s)ds.$$

Then,

$$p = [I - e^{-i(A+i)T}]^{-1}$$

$$\times\left\{(A+i)\left[\psi - e^{i(A+i)T}\varphi\right] + f(T) - e^{-i(A+i)T}f(0) - \int\limits_0^T e^{-i(A+i)(T-s)}f'(s)ds\right\}. \tag{2.8}$$

So, there exists a unique solution of problem (1.1), and for the solution we have formulas (2.6) and (2.8). Now, we establish estimates (2.3) and (2.4). Using formula (2.6), the triangle inequality and estimates (2.1) and (2.2), we get

$$\|p\|_H \le \left\|[I - e^{-i(A+i)T}]^{-1}\right\|_{H\to H}$$

$$\times\left\{\|(A+i)\psi\|_H + \left\|e^{i(A+i)T}\right\|_{H\to H}\|(A+i)\varphi\|_H + \|f(T)\|_H\right.$$

$$\left. + \left\|e^{i(A+i)T}\right\|_{H\to H}\|f(0)\|_H + \int\limits_0^T \left\|e^{-i(A+i)(T-s)}\right\|_{H\to H}\|f'(s)\|_H\, ds\right\}$$

$$\le M_1\left\{\|\psi\|_H + \|A\psi\|_H + \|\varphi\|_H + \|A\varphi\|_H + \|f(T)\|_H\right.$$

$$+\left\|f(0)\right\|_H+\int\limits_0^T\left\|f'(s)\right\|_H ds\Bigg\}$$

$$\leq M_2\left[\|\psi\|_H+\|A\psi\|_H+\|\varphi\|_H+\|A\varphi\|_H+\|\ f\ \|_{C^{(1)}(H)}\right].$$

The estimate (2.3) is established.

Using formula (2.8), the triangle inequality and estimates (2.1) and (2.2), we get

$$\|v(t)\|_H\leq\left\|e^{i(A+i)t}\right\|_{H\to H}\|\varphi\|_H+\int\limits_0^t\left\|e^{-i(A+i)(t-s)}\right\|_{H\to H}\|f(s)\|_H ds$$

$$+\left[1+\left\|e^{-i(A+i)t}\right\|_{H\to H}\right]\left\|(A+i)^{-1}p\right\|_H$$

$$\leq\|\varphi\|_H+\int\limits_0^t\|f(s)\|_H ds+2\left\|(A+i)^{-1}p\right\|_H. \tag{2.9}$$

Using formula (2.7), the triangle inequality and estimates (2.1) and (2.2), we get

$$\left\|(A+i)^{-1}p\right\|_H\leq\left\|[I-e^{-i(A+i)}]^{-1}\right\|_{H\to H}$$

$$\times\left[\|\psi\|_H+\left\|e^{i(A+i)T}\right\|_{H\to H}\|\varphi\|_H+\int\limits_0^T\left\|e^{-i(A+i)(T-s)}\right\|_{H\to H}\|f(s)\|_H ds\right]$$

$$\leq M_1\left[\|\psi\|_H+\|\varphi\|_H+\int\limits_0^T\|f(s)\|_H ds\right]. \tag{2.10}$$

From estimates (2.9) and (2.10) it follows estimate (2.4). Theorem 1.1 is proved. ■

3 Applications

In this section, we consider the applications of main Theorem 1.1.

First, the nonlocal boundary value problem for the Schrödinger equation

$$\begin{cases} iu_t-(a(x)u_x)_x+\delta u+iu=p(x)+f(t,x),\ 0<t<T,\ 0<x<1,\\ u(0,x)=\varphi(x),u(T,x)=\psi(x),0\leq x\leq 1,\\ u(t,0)=u(t,1),\ u_x(t,0)=u_x(t,1),\ 0\leq t\leq T\end{cases} \tag{3.1}$$

is considered. Problem (3.1) has a unique smooth solution $(u(t,x),p(x))$ for the smooth functions $a(x)\geq a>0$, $x\in(0,1)$, $\delta>0$, $a(1)=a(0)$, $\varphi(x)$, $\psi(x)$ $(x\in[0,1])$ and $f(t,x)$ $\ (t\in(0,T),x\in$

$(0,1)$). This allows us to reduce nonlocal boundary value problem (3.1) to the nonlocal boundary value problem (1.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by the formula

$$A^x u(x) = -(a(x)u_x)_x + \delta u \tag{3.2}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[0,1], u(1) = u(0), u_x(1) = u_x(0)\right\}.$$

Theorem 3.1 *For solutions of problem (3.1), we have the following stability inequalities*

$$\| p \|_{L_2[0,1]} \leq M \left[\| \varphi \|_{W_2^2[0,1]} + \| \psi \|_{W_2^2[0,1]} + \| f(0,\cdot) \|_{L_2[0,1]} + \max_{0\leq t\leq T} \| f'(t,\cdot) \|_{L_2[0,1]} \right],$$

$$\| u \|_{C(L_2[0,1])} \leq M \left[\| \varphi \|_{L_2[0,1]} + \| \psi \|_{L_2[0,1]} + \| f \|_{C(L_2[0,1])} \right],$$

where M is independent of $\varphi(x)$, $\psi(x)$ and $f(t,x)$. The Sobolev space $W_2^2[0,1]$ is defined to be the set of all functions $u \in L_2[0,1]$ such that for every multi-index α with $|\alpha| \leq 2$, the weak partial derivative $D^\alpha u$ belongs to $L_2[0,1]$.

The proof of Theorem 2.1 is based on Theorem 1.1 and the symmetry properties of the space operator A^x defined by formula (3.2).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n (x = (x_1, \cdots, x_n) : 0 < x_k < 1, k = 1, \cdots, n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, the boundary value problem for the multi-dimensional Schrödinger equation

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + iu = p(x) + f(t,x), \\ x = (x_1, \ldots, x_n) \in \Omega,\ 0 < t < T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), x \in \overline{\Omega}, \\ u(t,x) = 0,\ x \in S,\ 0 \leq t \leq T \end{cases} \tag{3.3}$$

is considered. Here $a_r(x) \geq a > 0$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T),\ x \in \Omega)$ are given smooth functions.

We consider the Hilbert space $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \left(\int \cdots \int_{x\in\overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right)^{\frac{1}{2}}.$$

Problem (3.3) has a unique smooth solution $(u(t,x), p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce problem (3.3) to nonlocal boundary value problem

(1.1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by the formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} \tag{3.4}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, u(x) = 0, x \in S \right\}.$$

Theorem 3.2 *For the solutions of problem (3.3) the following stability inequalities*

$$\| p \|_{L_2(\overline{\Omega})} \leq M \left[\| \varphi \|_{W_2^2(\overline{\Omega})} + \| \psi \|_{W_2^2(\overline{\Omega})} + \| f(0,\cdot) \|_{L_2(\overline{\Omega})} + \max_{0 \leq t \leq T} \| f^{'}(t,\cdot) \|_{L_2(\overline{\Omega})} \right],$$

$$\| u \|_{C(L_2(\overline{\Omega}))} \leq M \left[\| \varphi \|_{L_2(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} + \| f \|_{C(L_2(\overline{\Omega}))} \right]$$

hold, where M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$. Here and in future, The Sobolev space $W_2^2(\overline{\Omega})$ is defined to be the set of all functions $u \in L_2(\overline{\Omega})$ such that for every multi-index α with $|\alpha| \leq 2$, the weak partial derivative $D^{\alpha} u$ belongs to $L_2(\overline{\Omega})$.

The proof of Theorem 2.2 is based on Theorem 2.1, the symmetry properties of the operator A^x defined by formula (3.4) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.3 ([23]) *For the solutions of the elliptic differential problem*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ u(x) = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M_1 ||\omega||_{L_2(\overline{\Omega})}.$$

Here M_1 does not depend on $\omega(x)$.

Third, in $[0,T] \times \Omega$, the boundary value problem for the multi-dimensional Schrödinger equation

$$\begin{cases} i\frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u + iu = p(x) + f(t,x), \\ x = (x_1, \ldots, x_n) \in \Omega, \ 0 < t < T, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial \vec{n}} = 0, \ x \in S, \ 0 \leq t \leq T \end{cases} \tag{3.5}$$

with the Neumann condition is considered. Here, $\vec{n}$ is the normal vector to S, $a_r(x) \geq a > 0$, $(x \in \Omega)$, $\varphi(x)$, $\psi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T),\ x \in \Omega)$ are given smooth functions and $\delta > 0$.

Problem (3.5) has a unique smooth solution $(u(t,x),\ p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce problem (3.5) to nonlocal boundary value problem (1.1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u \tag{3.6}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \leq r \leq n, \frac{\partial u\,(x)}{\partial \vec{n}} = 0, x \in S \right\}.$$

Theorem 3.4 *For the solutions of problem (3.5) the following stability inequalities*

$$\parallel p \parallel_{L_2(\overline{\Omega})} \leq M \left[\parallel \varphi \parallel_{W_2^2(\overline{\Omega})} + \parallel \psi \parallel_{W_2^2(\overline{\Omega})} + \parallel f(0,\cdot) \parallel_{L_2(\overline{\Omega})} + \max_{0 \leq t \leq T} \parallel f^{'}(t,\cdot) \parallel_{L_2(\overline{\Omega})} \right],$$

$$\parallel u \parallel_{C(L_2(\overline{\Omega}))} \leq M \left[\parallel \varphi \parallel_{L_2(\overline{\Omega})} + \parallel \psi \parallel_{L_2(\overline{\Omega})} + \parallel f \parallel_{C(L_2(\overline{\Omega}))} \right]$$

hold. Here, M does not depend on $\varphi(x)$, $\psi(x)$ and $f(t,x)$.

The proof of Theorem 2.4 is based on Theorem 2.1, the symmetry properties of the operator A^x defined by formula (3.5) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 3.5 ([23]) *For the solutions of the elliptic differential problem*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ \frac{\partial u(x)}{\partial \vec{n}} = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M_1 ||\omega||_{L_2(\overline{\Omega})}.$$

Here M_1 does not depend on $\omega(x)$.

4 Conclusion

In the present paper, the well-posedness of the boundary value problem (1.1) is established. In applications, the stability inequalities for the solution of determination of a control parameter problems for the Schrödinger equation are obtained. Moreover, applying the result of paper [17] the single-step difference schemes for the numerical solution of boundary value problem (1.1) can be presented. Of course, such type stability estimates results hold for the solution of these difference schemes.

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