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Global existence and asymptotic behavior of a solution of Cauchy problem for a viscous Cahn–Hilliard type equation

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Abstract. We consider the existence, both locally and globally in time and the asymptotic behavior of solution of Cauchy problem for a viscous Cahn-Hilliard type equation. Under rather mild conditions on nonlinear term and initial data we prove that the above-mentioned problem admits a unique local solution, which can be continued to a global solution, and the problem is globally well-posed. Finally, under certain conditions, we prove that the global solution decays exponentially to zero as tends to infinity.

Key words. Cauchy problem, Cahn-Hilliard equation, local solution, global solution, asymptotic behavior.

1 Introduction

In this paper, we investigate the following Cauchy problem for the viscous Cahn–Hilliard type equation

$$u_t - u_{xx} + u_{xxxxt} + u_{xxxx} = g(u)_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{1.1}$$

$$u(x,0) = \varphi(x), \tag{1.2}$$

where $u(x,t)$ denotes the unknown function, $g(s)$ is the given nonlinear function, $\varphi(x)$ is the given initial value function, the subscripts x and t indicate the partial derivative with respect to x and t.

The standard viscous Cahn-Hilliard equation is as follows:

$$u_t - \alpha u_{xxt} + u_{xxxx} = \gamma(u)_{xx},$$

where $\alpha > 0$ is the viscosity coefficient, $\gamma(u)$ is the intrinsic chemical potential with a typical form $\gamma(u) = -u + \delta u^3$ for a constant $\delta \neq 0$.

The viscous and non-viscous (in absence of u_{xxxxt} term) Cahn–Hilliard equations are very important in materials science which describe spinodal decomposition, in the absence of mechanical stresses, of binary mixtures which appears, for example, in cooling processes of alloys, glasses or polymer mixtures (see [1, 2] and the references therein).

There is a considerable mathematical interest in Cahn–Hilliard equations which have been studied from various aspects (see [1-14] and references therein).

Miranville presented some models of Cahn-Hilliard equations based on a microforce balance proposed by M. Gurtin. He then studied some existence and uniqueness results for initial and boundary value problems [9, 10]. Gal and Miranville [2] were concerned with non-isothermal viscous and non-viscous Cahn–Hilliard equations with initial and dynamic boundary conditions. Changchun et al. [1] studied large time behavior of solutions for viscous Cahn–Hilliard equation with initial and boundary value conditions. Yin et al. [8] investigated the existence, uniqueness and asymptotic estimates of solutions to the Cahn–Hilliard type equations with time periodic potentials and sources.

Carvelho and Dlotko [3] were concerned with dynamics of the viscous Cahn–Hilliard equation with initial and boundary conditions. Liu et al. [11] studied the global existence, optimal temporal decay estimates and asymptotics of solution to the Cauchy problem of the Cahn–Hilliard equation.

The purpose of this paper is to give the existence, both locally and globally in time and the asymptotic behavior of solution of (1.1)-(1.2). The equation (1.1) is a typical higher order equation. Therefore, we need some a priori estimates of higher order terms.

The outline of the paper is as follows. In section 2, we study the existence and uniqueness of the local solution of problem (1.1)-(1.2). First, we solve the linear version of problem (1.1)-(1.2) and obtain estimates for the solution. Then we are going to give the proof of existence and uniqueness of the local solution of the Cauchy problem (1.1)-(1.2) by contraction mapping principle. In section 3, we prove the existence and uniqueness of the global solution of the problem, namely, the solution is extended to all times. In Section 4, we show that the solution depends continuously on the given initial data, so the problem is well-posed. The proof of the asymptotic behavior of the global solution is given in Section 5.

Throughout this paper, we use the following notations and lemmas.

$H^s(\mathbb{R}) = H^s$ will denote the L^2 Sobolev space on $\mathbb{R}$ for the H^s norm, we use the Fourier transform representation $\|u\|_{H^s}^2 = \int_{\mathbb{R}} \left(1+\xi^2\right)^s |\widehat{u}(\xi)|^2 d\xi$. We use $\|u\|_\infty$ and $\|u\|$ to denote L^∞ and L^2 norm, respectively.

Lemma 1.1 ([15]) *Assume that $f(u) \in C^k(\mathbb{R})$, $f(0)=0$, $u \in H^s \cap L^\infty$ and $k=[s]+1$, where $s \geq 0$. Then we have*

$$\|f(u)\|_{H^s} \leq K_1(M) \|u\|_{H^s}$$

if $\|u\|_\infty \leq M$, where $K_1(M)$ is a constant dependent on M.

Lemma 1.2 ([15]) *Assume that $f(u) \in C^k(\mathbb{R})$, $u, v \in H^s \cap L^\infty$ and $k=[s]+1$, where $s \geq 0$. Then we have*

$$\|f(u) - f(v)\|_{H^s} \leq K_2(M) \|u-v\|_{H^s}$$

if $\|u\|_\infty \leq M$, $\|v\|_\infty \leq M$, where $K_2(M)$ is a constant dependent on M.

Lemma 1.3 (Minkowski's inequality for integrals) ([16]) *If $1 \leq p \leq \infty$, $u(x,t) \in L^p(\mathbb{R}^n)$ for a.e. t, and function $t \to \|u(.,t)\|_p$ is in $L^1(I)$, where $I \subset [0,\infty)$ is an interval, then*

$$\left\| \int_I u(.,t)\, dt \right\|_p \leq \int_I \|u(.,t)\|_p \, dt.$$

2 Existence and uniqueness of local solution

In this section, we prove the existence and the uniqueness of the local solution for problem (1.1)-(1.2) by the contraction mapping principle. For this, we construct the solution of the problem as a fixed point of the solution operator associated with related family of Cauchy problem for linear equation.

Lemma 2.1 *Let $s \in \mathbb{R}$, for any $T > 0$, $\varphi \in H^s$ and $h \in L^1([0,T]; H^s)$, then the Cauchy problem for the linear equation*

$$u_t - u_{xx} + u_{xxxxt} + u_{xxxx} = (h(x,t))_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{2.1}$$

$$u(x,0) = \varphi(x) \tag{2.2}$$

has a unique solution in $C([0,T]; H^s)$ and the following estimation holds

$$\|u(t)\|_{H^s} \leq \|\varphi\|_{H^s} + \int_0^t \|h(\tau)\|_{H^s} d\tau, \qquad 0 \leq t \leq T. \tag{2.3}$$

Proof. Applying Fourier transform in (2.1) and (2.2) with respect to x, we obtain

$$\widehat{u}_t + \left(\frac{\xi^4+\xi^2}{1+\xi^4}\right)\widehat{u} = -\frac{\xi^2}{1+\xi^4}\widehat{h}\left(\xi,t\right), \tag{2.4}$$

$$\widehat{u}\left(\xi,0\right) = \widehat{\varphi}\left(\xi\right). \tag{2.5}$$

Then, it yields the solution formula

$$\widehat{u}\left(\xi,t\right) = e^{-\left(\frac{\xi^4+\xi^2}{1+\xi^4}\right)t}\widehat{\varphi}\left(\xi\right) - e^{-\left(\frac{\xi^4+\xi^2}{1+\xi^4}\right)t}\left(\int_0^t \frac{\xi^2 e^{\left(\frac{\xi^4+\xi^2}{1+\xi^4}\right)\tau}}{1+\xi^4}\widehat{h}\left(\xi,\tau\right)d\tau\right). \tag{2.6}$$

We obtain the estimate (2.3) from which the proof follows. ■

Let us define the function space

$$X\left(T\right) = \left\{u \in C\left(\left[0,T\right];H^s\right) : u\left(x,0\right) = \varphi\left(x\right)\right\}$$

which is endowed with the norm

$$\left\|u\right\|_{X(T)} = \max_{0\le t\le T}\left\|u\left(t\right)\right\|_{H^s}, \qquad \forall u \in X\left(T\right).$$

It is easy to see that $X\left(T\right)$ is a Banach space. For $s > \frac{1}{2}$, and any initial value $\varphi \in H^s$, let $a = \left\|\varphi\right\|_{H^s}$. Take the set

$$Y\left(A,T\right) = \left\{u : u \in X\left(T\right),\ \left\|u\right\|_{X(T)} \le A\right\}.$$

Obviously, $Y\left(A,T\right)$ is a nonempty bounded closed convex subset of $X\left(T\right)$ for any fixed $A > 0$ and $T > 0$. Here A is related to a. This means that the setting of $Y\left(A,T\right)$ depends on $\left\|\varphi\right\|_{H^s}$.

For $w \in Y\left(A,T\right)$, we consider the problem

$$u_t - u_{xx} + u_{xxxxt} + u_{xxxx} = g\left(w\right)_{xx}, \tag{2.7}$$

$$u\left(x,0\right) = \varphi\left(x\right). \tag{2.8}$$

We observe that with $g\left(w\left(x,t\right)\right) = h\left(x,t\right)$, this problem is reduced to the problem given by (2.1) and (2.2). Thus, Lemma 2.1 can be applied. We let $S\left(w\right) = u\left(x,t\right)$, where $u\left(x,t\right)$ is the unique solution of problem (2.7)-(2.8). Here S denotes the map which carries w into the unique solution of problem (2.7)-(2.8). Our aim is to show that S has a unique fixed point in $Y\left(A,T\right)$ for appropriately chosen T. For this purpose we shall employ the contraction mapping principle and Lemma 2.1. First, we prove the following lemma.

Lemma 2.2 *Assume that* $s > \frac{1}{2}$, $\varphi \in H^s$ *and* $g \in C^{[s]+1}(\mathbb{R})$, *then* S *is a contractive mapping from* $Y(A,T)$ *into itself for* T *sufficiently small relative to* A.

Proof. Let $w \in Y(A,T)$ be given. Then, for $t \in [0,T]$, from Sobolev imbedding theorem and Lemma 1.1, we have

$$\|w(t)\|_\infty \leq d\|w(t)\|_{H^s} \leq d\|w(t)\|_{X(T)} \leq dA,$$

$$\|g(w(t))\|_{H^s} \leq K_1(dA)\|w(t)\|_{H^s},$$

where $K_1(dA)$ is a constant which depends on d and A. We get from (2.3) that

$$\begin{aligned}
\|S(w)\|_{X(T)} &= \max_{0\leq t\leq T}\|u(t)\|_{H^s} \\
&\leq \|\varphi\|_{H^s} + \max_{0\leq t\leq T}\int_0^t \|g(w(\tau))\|_{H^s}\,d\tau \\
&\leq \|\varphi\|_{H^s} + T\left(\max_{0\leq t\leq T}\|g(w(t))\|_{H^s}\right) \\
&\leq \|\varphi\|_{H^s} + TK_1(dA)\|w\|_{X(T)} \\
&\leq a + TK_1(dA)A.
\end{aligned}$$

To prove the lemma, we need to show that $\|S(w)\|_{X(T)} \leq A$, so that $S(Y(A,T)) \subset Y(A,T)$, i.e., $a + TK_1(dA)A \leq A$. Let $A = ka$, $k > 0$. Thus we obtain

$$a + TK_1(dka)ka \leq ka.$$

By choosing $k = 2$ and T small enough to have $TK_1(dka)k \leq 1$, we get $\|S(w)\|_{X(T)} \leq 2a = A$. Therefore, $S(Y(A,T)) \subset Y(A,T)$.

Now we are going to prove that the map S is strictly contractive. Let $w, \widetilde{w} \in Y(A,T)$ and $u = S(w)$, $\widetilde{u} = S(\widetilde{w})$. Set $V = u - \widetilde{u}$, $W = w - \widetilde{w}$. Then V satisfies

$$V_t - V_{xx} + V_{xxxxt} + V_{xxxx} = (g(w) - g(\widetilde{w}))_{xx}, \tag{2.9}$$

$$V(x,0) = 0. \tag{2.10}$$

Making use of Lemma 1.2 and Lemma 2.1, we get

$$\begin{aligned}
\|V(t)\|_{H^s} &\leq \int_0^t \|g(w(\tau)) - g(\widetilde{w}(\tau))\|_{H^s}\,d\tau \\
&\leq \int_0^t K_2(dA)\|w(\tau) - \widetilde{w}(\tau)\|_{H^s}\,d\tau \\
&\leq TK_2(dA)\max_{0\leq t\leq T}\|W(t)\|_{H^s}.
\end{aligned}$$

Hence,

$$\|V\|_{X(T)} \leq TK_2\,(dA)\,\|W\|_{X(T)}\,.$$

By choosing T small enough, so that $TK_2\,(dA) \leq \frac{1}{2}$, $S : Y\,(A,T) \to Y\,(A,T)$ becomes strictly contractive. Thus, the lemma is proved. ■

Theorem 2.3 *Suppose that* $s > \frac{1}{2}$, $\varphi \in H^s$, $g \in C^{[s]+1}\,(\mathbb{R})$. *Then the problem (1.1)-(1.2) admits a unique local solution* $u\,(x,t) \in C\,([0,T_0)\,;H^s)$, *where* $[0,T_0)$ *is the maximal interval. Moreover, if*

$$\lim_{t\uparrow T_0} \sup \|u\,(t)\|_{H^s} < \infty, \tag{2.11}$$

then $T_0 = \infty$.

Proof. From Lemma 2.2 and the contraction mapping principle, it follows that for appropriately chosen $T > 0$, S has a unique fixed point $u\,(x,t) \in Y\,(A,T)$, which is a strong solution of problem (1.1), (1.2). It is not difficult to prove the uniqueness of the solution which belongs to $X\,(T')$ for each $T' > 0$.

Let $u_1, u_2 \in X\,(T')$ be two solutions of problem (1.1)-(1.2). Let $u = u_1 - u_2$ then

$$u_t - u_{xx} + u_{xxxxt} + u_{xxxx} = (g\,(u_1) - g\,(u_2))_{xx}\,.$$

Multiplying the equation above by u and integrating the product with respect to x, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\|u\|^2 + \|u_{xx}\|^2\right] + \|u_x\|^2 + \|u_{xx}\|^2 &= \int_{\mathbb{R}} (g\,(u_1) - g\,(u_2))_{xx}\,u dx \\ &= \int_{\mathbb{R}} (g\,(u_1) - g\,(u_2))\,u_{xx} dx. \end{aligned} \tag{2.12}$$

From the definition of the space $X\,(T')$, $s > \frac{1}{2}$ and Sobolev imbedding theorem we have $\|u_i\,(t)\|_\infty \leq C_1\,(T')$ for $i = 1,2$ and $0 \leq t \leq T' < T$, where $C_1\,(T')$ is a constant depends on T'. Thus, we get from Cauchy inequality that

$$\left|\int_{\mathbb{R}} (g\,(u_1) - g\,(u_2))\,u_{xx} dx\right| \leq \|g\,(u_1) - g\,(u_2)\|\,\|u_{xx}\| \leq C_2\,(T')\,\|u\|\,\|u_{xx}\|\,,$$

where $C_2\,(T')$ is a constant depends on $C_1\,(T')$. From the above inequality and (2.12) we have

$$\|u\|^2 + \|u_{xx}\|^2 \leq C_2\,(T') \int_0^t \left(\|u\|^2 + \|u_{xx}\|^2\right) d\tau. \tag{2.13}$$

By the Gronwall inequality, we get from (2.13) that $\|u\|^2 + \|u_{xx}\|^2 = 0$ for $0 \leq t \leq T'$. Hence $u = u_1 - u_2 = 0$ for $0 \leq t \leq T'$, i.e., problem (1.1)-(1.2) has at most one solution which belongs to $X(T')$.

Now, let $[0,T_0)$ be the maximal time interval of existence for $u\in X(T_0)$. We want to show that if (2.11) is satisfied, then $T_0=\infty$.

Suppose that (2.11) holds and $T_0<\infty$. For each $T'\in[0,T_0)$, we consider the Cauchy problem

$$\begin{aligned} v_t - v_{xx} + v_{xxxxt} + v_{xxxx} &= g(v)_{xx}, \qquad (2.14)\\ v(x,0) &= u(x,T'). \qquad (2.15)\end{aligned}$$

By (2.11),

$$\|u(.,t)\|_{H^s}\le K,$$

where K is a positive constant independent of $T'\in[0,T_0)$. From Lemma 2.2 and the contraction mapping principle we see that there exists a constant $T_1\in(0,T_0)$ such that for each $T'\in[0,T_0)$ problem (2.14)-(2.15) has a unique solution $v(x,t)\in X(T_1)$. Take $T'=T_0-T_1/2$ and define

$$\widetilde{u}(x,t)=\begin{cases} u(x,t), & t\in[0,T'],\\ v(x,t-T'), & t\in[T',T_0+T_1/2],\end{cases}$$

then $\widetilde{u}(x,t)$ is a solution of equations (1.1), (1.2) on interval $[0,T_0+T_1/2]$, and by the uniqueness, $\widetilde{u}$ extends u, which violates the maximality of $[0,T_0)$. Therefore, if (2.11) holds $T_0=\infty$. Theorem 2.3 is proved. ■

3 Existence and uniqueness of global solution

In this section, we prove the existence and the uniqueness of the global solution for problem (1.1)-(1.2). For this purpose we are going to make a priori estimate of the local solution for problem (1.1)-(1.2).

Theorem 3.1 *Assume that $s\ge 2$, $g\in C^{[s]+1}(\mathbb{R})$, and for any $s\in\mathbb{R}$, $g'(s)\ge -1$. Then problem (1.1)-(1.2) admits a unique global solution $u\in C([0,\infty);H^s)$.*

Proof. We first prove the theorem for the case $s=2$. By virtue of Theorem 2.3 we only need to prove that condition (2.11) holds. Multiplying both sides of (1.1) by u and integrating with respect to x over $\mathbb{R}$, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|u\|^2+\|u_{xx}\|^2\right]+\|u_x\|^2+\|u_{xx}\|^2 &= \int_{\mathbb{R}} g(u)_{xx}\,u\,dx\\ &= -\int_{\mathbb{R}} g'(u)(u_x)^2\,dx\\ &\le \|u_x\|^2. \qquad (3.1)\end{aligned}$$

From the above inequality we get

$$\|u\|^2 + \|u_{xx}\|^2 \leq C_1(T) < \infty, \qquad \forall t \in [0,T]. \tag{3.2}$$

Furthermore, by the lemma on page 133 in [17], inequality (3.2) implies that

$$\|u\|_{H^2}^2 < \infty, \qquad \forall t \in [0,T]. \tag{3.3}$$

Now, we prove for the case $s > 2$. From the estimate (2.3) it follows that

$$\|u(t)\|_{H^s} \leq \|\varphi\|_{H^s} + \int_0^t \|g(u(\tau))\|_{H^s}\, d\tau \leq \|\varphi\|_{H^s} + \int_0^t C_2\left(\|u(\tau)\|_\infty\right)\|u(\tau)\|_{H^s}\, d\tau. \tag{3.4}$$

Taking into consideration that $\|u(t)\|_\infty < \infty$ in (3.3), we get from (3.4)

$$\|u(t)\|_{H^s} < C_3(T) < \infty, \qquad \forall t \in [0,T].$$

Consequently, there is a unique global solution $u \in C([0,\infty); H^s)$. ■

4 Continuous dependence on initial data

We now want to show that solution of problem (1.1)-(1.2) depends continuously on the initial data so that the problem is well-posed. For this purpose, we take two solutions u_1, u_2 of (1.1) with initial data φ_1, φ_2, respectively defined on some interval $[0,T]$.

Let $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} v_t - v_{xx} + v_{xxxxt} + v_{xxxx} &= \left(g(u_1) - g(u_2)\right)_{xx}, \\ v(x,0) &= \varphi_1(x) - \varphi_2(x). \end{aligned}$$

By Lemma 2.1, we have

$$\|u_1 - u_2\|_{H^s} \leq \|\varphi_1 - \varphi_2\|_{H^s} + \int_0^t \|g(u_1) - g(u_2)\|_{H^s}\, d\tau.$$

By the Sobolev imbedding theorem, u_1 and u_2 are in L^∞. Letting $M = \max\{\|u_1\|_\infty, \|u_2\|_\infty\}$, from Lemma 1.2, we get

$$\|u_1 - u_2\|_{H^s} \leq \|\varphi_1 - \varphi_2\|_{H^s} + K_2(M)\int_0^t \|u_1 - u_2\|_{H^s}\, d\tau.$$

The Gronwall inequality implies that

$$\|u_1 - u_2\|_{H^s} \leq \left(\|\varphi_1 - \varphi_2\|_{H^s}\right) e^{K_2(M)t}, \qquad \forall t \in [0,T].$$

Therefore, the solution depends continuously on the given initial data since it is bounded by a continuous function related with the difference of the initial data.

5 Asymptotic behavior of solution

Theorem 5.1 *Assume that $g'(s) \geq 0$ and $E(0) = \frac{1}{2}\left(\|\varphi\|^2 + \|\varphi_{xx}\|^2\right) > 0$. Then for the global solution of problem (1.1)-(1.2) there exist a positive constant δ_1 such that*

$$E(t) = \frac{1}{2}\left(\|u\|^2 + \|u_{xx}\|^2\right) \leq E(0)\, e^{-\delta_1 t}, \qquad t > 0.$$

Proof. Let $u(x,t)$ be a global solution of problem (1.1)-(1.2). Taking the L^2 inner product of (1.1) with u it follows that

$$\frac{d}{dt}E(t) + \|u_x\|^2 + \|u_{xx}\|^2 - \int_{\mathbb{R}} g(u)_{xx}\, u dx = 0, \qquad t > 0. \tag{5.1}$$

Multiplying (5.1) by $e^{\delta t}$ gives

$$\frac{d}{dt}\left(e^{\delta t}E(t)\right) + e^{\delta t}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \int_{\mathbb{R}} g'(u)(u_x)^2\, dx\right) = \delta e^{\delta t}E(t). \tag{5.2}$$

Integrating (5.2) over $(0,t)$ we get

$$\begin{aligned}
& e^{\delta t}E(t) + \int_0^t e^{\delta\tau}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \int_{\mathbb{R}} g'(u)(u_x)^2\, dx\right) d\tau \\
= \;& E(0) + \delta\int_0^t e^{\delta\tau}E(\tau)\, d\tau \\
= \;& E(0) + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\|u\|^2 + \|u_{xx}\|^2\right) d\tau.
\end{aligned} \tag{5.3}$$

By using (5.1) and integration by parts, we will estimate the second term on the right-hand side of (5.3) as

$$\begin{aligned}
\frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\|u\|^2 + \|u_{xx}\|^2\right) d\tau \;\leq\; & \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\|u\|^2 + \|u_x\|^2 + \|u_{xx}\|^2\right) d\tau \\
= \; & -\frac{\delta}{4}e^{\delta t}\left(\|u\|^2 + \|u_{xx}\|^2\right) + \frac{\delta}{4}\left(\|\varphi\|^2 + \|\varphi_{xx}\|^2\right) \\
& + \frac{\delta^2}{4}\int_0^t e^{\delta\tau}\left(\|u\|^2 + \|u_{xx}\|^2\right) d\tau \\
& - \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\int_{\mathbb{R}} g'(u)(u_x)^2\, dx\right) d\tau + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\|u\|^2\right) d\tau.
\end{aligned}$$

Substituting the above inequality into (5.3)

$$\begin{aligned}
& e^{\delta t}E(t) + \int_0^t e^{\delta\tau}\left(\|u_x\|^2 + \|u_{xx}\|^2 + \int_{\mathbb{R}} g'(u)(u_x)^2\, dx\right) d\tau \\
\leq \;& E(0) - \frac{\delta}{2}e^{\delta t}E(t) + \frac{\delta}{2}E(0) + \frac{\delta^2}{2}\int_0^t e^{\delta\tau}E(\tau)\, d\tau \\
& - \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\int_{\mathbb{R}} g'(u)(u_x)^2\, dx\right) d\tau + \frac{\delta}{2}\int_0^t e^{\delta\tau}\left(\|u\|^2\right) d\tau.
\end{aligned}$$

Since $g'(s) \geq 0$, we obtain

$$e^{\delta t} E(t) \leq E(0) + \frac{\delta^2 + \delta}{\delta + 2} \int_0^t e^{\delta \tau} E(\tau)\, d\tau. \tag{5.4}$$

Applying the Gronwall inequality to (5.4) we obtain the result

$$E(t) \leq E(0)\, e^{-\frac{\delta}{\delta+2} t},$$

where $\frac{\delta}{\delta+2} = \delta_1 > 0$. ■

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