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The space of initial data for the Robin boundary-value problem for parabolic differential-difference equations

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Abstract. In this paper, the third boundary-value problem for parabolic differential-difference equation is considered in Lipschitz spatial domain Q. It is proved that the space of initial data (i.e., the space of initial functions for which the strong solution exists) coincides with Sobolev space $H^1(Q)$.

Key words. Differential-difference equations, the space of initial data, Kato's conjecture.

1 Introduction

The space of initial data, i.e., the space of initial functions for which the strong solution exists, for the second boundary-value problem for parabolic differential-difference equation in the case of smooth spatial domain Q was described in terms of the Sobolev spaces in [7]. For the history of the problem and its connection with the Kato square root problem see [7] and literature therein.

In this paper, we spread this result to the Robin boundary conditions and we use the method developed in [1] to cover the case of Lipschitz spatial domain Q.

2 Statement of the problem

Let $Q \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with Lipschitz boundary ∂Q, i.e., locally it is a graph of function $x_n = \phi(x')$, satisfying the Lipschitz condition $|\phi(x') - \phi(y')| \leq K|x' - y'|$ with

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$K > 0$.

Introduce bounded difference operators $R_{ij},\ R_i\colon L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ by the formulas

$$\begin{aligned}(R_{ij}u)(x) &= \sum_{h\in M} a_{ijh}u(x+h) \quad (i,\, j = 1, 2, \cdots, n),\\ (R_i u)(x) &= \sum_{h\in M} a_{ih}u(x+h) \quad (i = 0,\, 1,\, 2, \cdots, n).\end{aligned} \tag{2.1}$$

Here, $a_{ijh},\ a_{ih}$ are complex numbers, the set M consists of a finite number of vectors $h \in \mathbb{R}^n$ with integer coordinates.

We introduce the following linear operators: $I_Q\colon L_2(Q) \to L_2(\mathbb{R}^n)$ is the operator of extension of a function by zero outside Q; $P_Q\colon L_2(\mathbb{R}^n) \to L_2(Q)$ is the projection operator of a function onto Q; and the operators $R_{ijQ},\, R_{iQ}\colon L_2(Q) \to L_2(Q)$ defined by the formulas $R_{ijQ} = P_Q R_{ij} I_Q,\ R_{iQ} = P_Q R_i I_Q$.

We consider the following differential-difference equation

$$u_t - \sum_{i,j=1}^{n} \left(R_{ijQ}u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} R_{iQ}u_{x_i} + R_{0Q}u = f(x,t) \quad ((x,t) \in Q_T) \tag{2.2}$$

with the boundary condition

$$\sum_{i,j=1}^{n} R_{ijQ}u_{x_j}\cos(\nu,\, x_i) + \sigma(x)u = 0 \quad ((x,t) \in \Gamma_T)\,, \tag{2.3}$$

and the initial condition

$$u|_{t=0} = \varphi(x) \quad (x \in Q)\,, \tag{2.4}$$

where $Q_T = Q \times (0,\, T)$, $0 < T < \infty$, $\Gamma_T = \partial Q \times (0,\, T)$, ν is the external unit normal to Γ_T (it exists at almost every point of Γ_T), $\sigma \in L_2(\partial Q)$ and $\sigma \geq 0$ on ∂Q, $f \in L_2(Q_T)$, and $\varphi \in L_2(Q)$.

Let $H^k(Q)$ be the Sobolev space of complex-valued functions from $L_2(Q)$ having all generalized derivatives up to the k-order from $L_2(Q)$.

Introduce the sesquilinear form $a[v,\, w]$ in $L_2(Q)$ with domain $H^1(Q)$ by the formula

$$\begin{aligned}a[v,\, w] = &\sum_{i,j=1}^{n} \left(R_{ijQ}v_{x_j},\, w_{x_i}\right)_{L_2(Q)}\\ &+ \sum_{i=1}^{n} (R_{iQ}v_{x_i},\, w)_{L_2(Q)} + (R_{0Q}v,\, w)_{L_2(Q)} + (\sigma v, w)_{L_2(\partial Q)}.\end{aligned} \tag{2.5}$$

The difference operators $R_{ijQ},\, R_{iQ},\, R_{0Q}\colon L_2(Q) \to L_2(Q)$ are bounded. Therefore, it follows that there exists a constant $c_0 > 0$ such that

$$|a[v,\, w]| \leq c_0\|v\|_{H^1(Q)}\|w\|_{H^1(Q)}, \quad (v, w \in H^1(Q)). \tag{2.6}$$

Since the sesquilinear form $a[v,\, w]$ is continuous with respect to w in $H^1(Q)$, there exists a linear bounded operator $A\colon H^1(Q) \to \left[H^1(Q)\right]'$ such that

$$\langle Av,\, \overline{w}\rangle = a[v,\, w] \quad (v, w \in H^1(Q)), \tag{2.7}$$

where $\left[H^1(Q)\right]'$ is a dual space to $H^1(Q)$.

Definition 2.1 *The form $a[v,\, w]$ is said to be* coercive *if there exist numbers $c_1 > 0$ and $c_2 \ge 0$ such that*

$$\operatorname{Re}\, a[v,\, v] \ge c_1 \|v\|^2_{H^1(Q)} - c_2 \|v\|^2_{L_2(Q)} \qquad (v \in H^1(Q)). \tag{2.8}$$

For the necessary and sufficient conditions of coercivity in algebraic form see [8, Lemma 2.2 and Lemma 2.3] (they are true also in case ∂Q is Lipschitz and $\sigma \in L_2(Q)$). Further we shall assume that the form $a[v,\, w]$ is coercive.

3 Strong solutions and the spaces of initial data

1. The sesquilinear form $a[v,\, w]$ is continuous with respect to w in $L_2(Q)$, therefore it defines an operator $\mathcal{A}\colon D(\mathcal{A}) \subset L_2(Q) \to L_2(Q)$ such that

$$\mathcal{A}v = Av \quad (v \in D(\mathcal{A})). \tag{3.1}$$

Here $\|v\|_{D(\mathcal{A})} = \|\mathcal{A}v\|_{L_2(Q)} + \|v\|_{L_2(Q)}$. Since the operator $\mathcal{A}$ is closed, the space $D(\mathcal{A})$ is a Hilbert space, moreover $D(\mathcal{A})$ is dense in $H^1(Q)$. It is well known that operator $-\mathcal{A}$ is a generator of an analytic contractive semigroup (cf. [4, Chapter IX, Section 1, Theorem 1.24]).

Introduce the Hilbert space

$$\mathcal{W} = \{w \in L_2(0,\, T;\, D(\mathcal{A})) : w_t \in L_2(Q_T)\}$$

with the norm

$$\|u\|^2_{\mathcal{W}} = \int_0^T \|\mathcal{A}u\|^2_{L_2(Q)}\, dt + \int_0^T \|u\|^2_{L_2(Q)}\, dt + \int_0^T \|u_t\|^2_{L_2(Q)}\, dt.$$

Here, the derivatives are considered in the sense of distributions on Q_T.

Define the bounded operator $L\colon \mathcal{W} \to L_2\,(0,\, T;\; L_2(Q))$ by the formula $Lv(\cdot,\, t) = \mathcal{A}v(\cdot,\, t)$ for almost all $t \in (0,\, T)$.

Definition 3.1 *A function $u \in \mathcal{W}$ is called a* strong solution *of problem (2.2)–(2.4) if it satisfies the equation*

$$\frac{du}{dt} + Lu = f \quad \text{for almost all } t \in (0,\,T) \tag{3.2}$$

and the initial condition

$$u|_{t=0} = \varphi. \tag{3.3}$$

We can assume that $c_2 = 0$ in inequality (2.8). In opposite case, we set $u = z\,e^{c_2 t}$. Then, problem (3.2)-(3.3) will be equivalent to the problem $\dfrac{dz}{dt} + (L + c_2 I)z = e^{-c_2 t} f$, $z|_{t=0} = \varphi$.

Theorem 3.1 *Let the form $a[v,\,w]$ be coercive and $c_2 = 0$. Then, problem (2.2)–(2.4) for any $f \in L_2(Q_T)$ and $\varphi \in [L_2(Q),\,D(\mathcal{A})]_{1/2}$ has a unique strong solution given by the formula*

$$u(x,\,t) = T_t\varphi(x) + \int_0^t T_{t-s} f(x,\,s)\,ds, \tag{3.4}$$

where $\{T_t\}$ $(t \geq 0)$ is the analytic semigroup generated by the operator $-\mathcal{A}$.

For the proof see, e.g., the proof of [8, Theorem 4.2].

2. Theorem 3.1 is connected with the problem on the space of initial data. In [2], it was proved that condition $\varphi \in [L_2(Q),\,D(\mathcal{A})]_{1/2}$ is necessary and sufficient for existence of strong solutions of problem (2.2)–(2.4). Thus, there arises the problem of description of the interpolational space $[L_2(Q),\,D(\mathcal{A})]_{1/2}$.

Theorem 3.2 *Let the form $a[v,\,w]$ be coercive with $c_2 = 0$. Then,*

$$[L_2(Q),\,D(\mathcal{A})]_{1/2} = H^1(Q). \tag{3.5}$$

Proof. By [5, Theorem 3.1], $[L_2(Q),\,D(\mathcal{A})]_{1/2} = D(\mathcal{A}^{1/2})$. Therefore, we can use [1, Theorem 3.2] to prove equality (3.5).

It is enough to show that operator $A\colon H^{1+s}(Q) \to \widetilde{H}^{-1+s}(Q)$ is bounded, when $|s| < 1/2$. But for convenience of the reader we give the complete proof. For $s \geq 0$ we denote by $H^s(Q)$ the Sobolev—Slobodetsky space $W_2^s(Q)$, and for $s < 0$ we denote by $\widetilde{H}^s(Q)$ the dual space.

I. Consider the form $a[v,w]$ on $H^{1+s}(Q) \times H^{1-s}(Q)$:

$$a[v,w] = \sum_{i,j=1}^{n} \bigl\langle R_{ij,Q} v_{x_j}, \overline{w}_{x_i} \bigr\rangle + \sum_{i=1}^{n} \bigl\langle R_{i,Q} v_{x_i}, \overline{w} \bigr\rangle + (R_{0,Q} v, w)_{L_2(Q)} + (\sigma v, w)_{L_2(\partial Q)}.$$

If $v \in H^{1+s}(Q)$ and $w \in H^{1-s}(Q)$, then $v_{x_j} \in H^s(Q)$ and $w_{x_i} \in H^{-s}(Q)$. Recall that for $-1/2 < s < 0$, H^s coincides with $\widetilde{H}^s$. The operators $R_{ij,Q}$, $R_{i,Q}$ are bounded in $H^s(Q)$, when $|s| < 1/2$, because the extension of a function by zero outside Q is a bounded operator from $H^s(Q)$ to $H^s(\mathbb{R}^n)$ (cf. [6, Theorem 3.33] for $s \geq 0$. For $s < 0$ it follows from equality $(\chi_Q\varphi, \psi)_{L_2(\mathbb{R}^n)} = (\varphi, \chi_Q\psi)_{L_2(\mathbb{R}^n)}$, where $\chi_Q(x) = 1$ if $x \in \overline{Q}$ and $\chi_Q(x) = 0$ otherwise and $\varphi,\ \psi \in C_0^\infty(\mathbb{R}^n)$). Thereby, for a fixed $|s| < 1/2$ there exists a constant $C_s > 0$ such that

$$|a[v, w]| \leq C_s \|v\|_{H^{1+s}(Q)} \|w\|_{H^{1-s}(Q)}.$$

Then the operator A is defined by the form:

$$\langle Av, \overline{w}\rangle = a[v, w], \qquad v \in H^{1+s}(Q), w \in H^{1-s}(Q),$$

where $f = Av$ is from $\widetilde{H}^{-1+s}(Q)$.

II. The form $\overline{a[w, v]}$ defines operators A^* and $\mathcal{A}^*$. From [10, Theorem 2.8], it follows that there exists $0 < \varepsilon < 1/2$ such that operators A and A^* on $H^{1+s}(Q)$ have bounded inverse, when $|s| < \varepsilon$. Let $0 < s < \varepsilon$, then by [3, Theorem 1.1] and reiteration theorem (cf. [9])

$$D(\mathcal{A}^{(1-s)/2}) = D(\mathcal{A}^{*(1-s)/2}) = D(\mathcal{S}^{(1-s)/2}) = [L_2(Q), D(\mathcal{S}^{1/2})]_{1-s} = H^{1-s}(Q). \tag{3.6}$$

Here operators S and $\mathcal{S}$ are defined by the form $\frac{1}{2}\left(a[v, w] + \overline{a[w, v]}\right)$. Therefore, operator $\mathcal{A}^{*-(1-s)/2}$ is bounded from $L_2(Q)$ to $H^{1-s}(Q)$ and there exists $C_1 > 0$ such that

$$\|S^{1/2}\mathcal{A}^{*-(1-s)/2} w\|_{H^{-s}(Q)} \leq C_1 \|w\|_{L_2(Q)} \quad (w \in L_2(Q)). \tag{3.7}$$

III. Set

$$B = S^{-1/2} A S^{-1/2}. \tag{3.8}$$

Operator B is a bounded operator in $H^{\pm s}(Q)$. Then

$$A = S^{1/2} B S^{1/2} \tag{3.9}$$

and there exists $C_2 > 0$ such that

$$\|BS^{1/2}v\|_{H^s(Q)} = \|S^{-1/2}Av\|_{H^s(Q)} \leq C_2 \|v\|_{H^{1+s}(Q)} \quad (v \in H^{1+s}(Q)). \tag{3.10}$$

IV. Operator A is one-to-one and bounded from $D(\mathcal{A})$ and $H^{1+s}(Q)$ to $L_2(Q)$ and $\widetilde{H}^{-1+s}(Q)$, respectively. Since $L_2(Q) \subset \widetilde{H}^{-1+s}(Q)$ densely and continuously, then $D(\mathcal{A}) \subset H^{1+s}(Q)$ densely and continuously.

For functions $v \in D(\mathcal{A})$ and $w \in D(\mathcal{A}^*)$, we have

$$(v, \mathcal{A}^{*(1+s)/2} w)_{L_2(Q)} = (v, \mathcal{A}^* \mathcal{A}^{*-(1-s)/2} w)_{L_2(Q)} = (\mathcal{A}v, \mathcal{A}^{*-(1-s)/2} w)_{L_2(Q)} \tag{3.11}$$

or

$$(\mathcal{A}^{(1+s)/2} v, w)_{L_2(Q)} = (Av, \mathcal{A}^{*-(1-s)/2} w)_{L_2(Q)} = (BS^{1/2} v, S^{1/2} \mathcal{A}^{*-(1-s)/2} w)_{L_2(Q)}. \tag{3.12}$$

From equations (3.7), (3.10), and Schwarz inequality, it follows that

$$\left|(\mathcal{A}^{(1+s)/2} v, w)_{L_2(Q)}\right| \le C_3 \|BS^{1/2} v\|_{H^s(Q)} \|S^{1/2} \mathcal{A}^{*-(1-s)/2} w\|_{H^{-s}(Q)} \tag{3.13}$$

$$\le C_4 \|v\|_{H^{1+s}(Q)} \|w\|_{L_2(Q)}. \tag{3.14}$$

Since $D(\mathcal{A}^*)$ is dense in $L_2(Q)$,

$$\|\mathcal{A}^{(1+s)/2} v\|_{L_2(Q)} \le C_4 \|v\|_{H^{1+s}(Q)}. \tag{3.15}$$

The last inequality is true for $v \in H^{1+s}(Q)$, because $D(\mathcal{A})$ is dense in $H^{1+s}(Q)$. Therefore, $H^{1+s}(Q) \subset D(\mathcal{A}^{(1+s)/2})$.

V. By reiteration theorem (cf. [9])

$$H^1(Q) = [L_2(Q), H^{1+s}(Q)]_{1/(1+s)} \subset [L_2(Q), D(\mathcal{A}^{(1+s)/2})]_{1/(1+s)} = D(\mathcal{A}^{1/2}). \tag{3.16}$$

The same is true for $\mathcal{A}^*$. Then equality (3.5) follows from [5, Corollary 5.1]. ∎

Corollary 3.3 *If the form $a[v, w]$ is coercive, then $D(\mathcal{A}^{1/2}) = D(\mathcal{A}^{*1/2}) = H^1(Q)$.*

Remark 3.4 *The boundary condition (2.3) satisfies under additional smoothness conditions (see* [8, Corollary 5.1]*). In general case it must be understood in the sense of the Green inequality.*

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