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SOLUTION OF CONTACT PROBLEM USING "MIXED" MLPG FINITE VOLUME METHOD WITH MLS APPROXIMATIONS

Cengiz ERDÖNMEZ

Turkish Naval Academy, Tuzla, Istanbul, Turkiye
cerdonmez@dho.edu.tr

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ABSTRACT

Meshless methods are became an alternative to most popular numerical methods used to solve engineering problems such as Finite Difference and Finite Element Methods. Because of element free nature, problems are solved using meshless methods depending on the general geometry and conditions of the problem. Mixed Meshless Local Petrov-Galerkin (MLPG) approach is based on writing the local weak forms of PDEs. Moving least squares (MLS) is used as the interpolation schemes. In this study contact analysis problem is modelled using Meshless Finite Volume Method (MFVM) with MLS interpolation and solved for beam contact problem. Meshless discretization and linear complementary equation of the 2-D frictionless contact problems are described first. Then the problem is converted to a linear complementary problem (LCP) and solved using Lemke's algorithm. An elastic cantilever beam contact to a rigid foundation is considered as an example problem.

"KARIŞIK" MLPG SONLU HACİMLER YÖNTEMİ İLE MLS YAKLAŞTIRMASI KULLANILARAK TEMAS PROBLEMİNİN ÇÖZÜMÜ

ÖZ

Ağsız yöntemler son yıllarda Sonlu Farklar ve Sonlu Elemanlar Yöntemleri gibi mühendislik problemlerini çözmek için kullanılan en popüler sayısal yöntemlere alternatif haline gelmiş durumdadır. Ağsız yöntemlerin eleman bağımsız yapısı gereği problemlerin çözümleri yalnızca çözümün yapılacağı geometri ve problemin koşullarına bağlıdır. Karışık Ağsız Yerel Petrov-Galerkin (MLPG) yaklaşımı Kısmi Diferansiyel Denklemlerin (PDEs) yerel zayıf formlarının yazılması temeline dayanmaktadır. Hareketli En Küçük Kareler (MLS) yöntemi interpolasyon şeması olarak kullanılmaktadır. Bu çalışmada Ağsız Sonlu Hacimler Yöntemi (MFVM) ile MLS interpolasyon şeması birlikte kullanılarak temas analizi problemi modellenmiş ve kiriş temas problemi için çözülmüştür. Ağsız ayrıklaştırma ve 2-D sürtünmesiz temas problemlerinin doğrusal tamamlayıcı denklemleri ilk olarak açıklanmıştır. Daha sonra problem doğrusal tamamlayıcı probleme (LCP) dönüştürülmüş ve Lemke algoritması kullanılarak çözülmüştür. Örnek problem olarak rijit bir temele temas halindeki elastik bir konsol kiriş problemi ele alınmıştır.

Keywords*: Mixed Meshless Local Petrov-Galerkin, Moving least squares, Meshless Finite Volume Method, Linear Complimentary Problem, Cantilever beam contact.*

Anahtar Kelimeler: *Karışık Ağsız Yerel Petrov-Galerkin, Hareketli En Küçük Kareler, Ağsız Sonlu Hacimler Yöntemi, Doğrusal Tamamlayıcı Problem, Konsol kiriş teması.*

1. INTRODUCTION

Due to the complex nature of the engineering problems, scientists are interested in numerical methods to model and find solutions for engineering problems from different disciplines. In computational mechanics both Finite Difference and Finite Element Method (FEM) found a very important place to solve problems. Mostly FEM is preferred and many commercial programs are based on the theory of FEM to solve wide variety of engineering problems. But due to the element based nature, solution of the problems brings some difficulties and researches are made researches on new alternatives from the beginning of the 1990's. As a result number of mesh free methods are developed and still there are a lot of studies are continuing on this area.

The meshless methods overcoming the drawback of mesh-based methods, such as mesh-generation and poor solutions. Many mesh free methods are developed based on global weak forms, such as Smooth Particle Hydrodynamics (SPH) and the element-free methods which require certain meshes or background cells. Contrary to these methods, the meshless local Petrov Galerkin (MLPG) approach pioneered by Atluri (2004) [1] and his colleagues is based on writing the local weak forms of partial differential equations (PDEs), on overlapping local subdomains. The integration of the weak-form is also performed within the local sub-domains; thus negating any need for any kind of meshes or background cells, making the MLPG approach a truly meshless method. MLPG method is used in many areas such as elasto-statics [2-5], elasto-dynamics [6], fracture mechanics [7], fluid mechanics and etc.

The MLPG approach gives oppurtunity to select trial and test functions, define the size and shape of local sub-domains, and has the ability to use various unsymmetric and symmetric weak forms of the PDEs. As a test function Heaviside function can be used in symmetric-weak forms to eliminat domain integrals.

Meshless Finite Volume Method (MFVM), using the Meshless Local Petrov-Galerkin (MLPG) “Mixed” approach, is developed for solving elastostatic problems and labeled as MLPG-5 in [8]. In MLPG mixed approach MLS are used as the interpolation schemes. The MLPG local weak form is written for the equilibrium equations over the local sub-domains, and the Heaviside test function is used. Xiao et.al. solved the 2D contact problem using MLPG with RBF [9]. They implement the meshless linear complementary formulation which can be solved by using Lemke's algorithm.

In this study solution of the contact problem of elastic cantilever beam using Mixed MLPG Finite Volume Method (FVM) and Linear Complementary Problem (LCP) is modeled and solved. Meshless discretization and linear complementary equation of the 2-D frictionless contact problems is described. The problem is converted to LCP and solved using Lemke's algorithm. An elastic cantilever beam contacting a rigid foundation is considered as an example problem. The beam is modeled as a thin body in the plane stress state. The result found by using MFVM of beam contact problem is compared with the results available in the literature.

2. MESHLESS APPROXIMATIONS

Radial Basis Functions are proposed to interpolate large sets of multivariate data before by many researchers. But lately it has been shown that a fast and accurate approximation method for large sets of multivariate data can be accomplished. An alternative to radial basis function interpolation and approximation is the so-called moving least squares method (MLS). In the traditional moving least-squares (MLS) method the amount of work is shifted. There is no large system to solve. Instead, for every evaluation one needs to solve a small linear system to find the coefficients of the moving local approximant, and then evaluate a summation [10].

MLS is generally considered to be one of the best methods to interpolate random data with a reasonable accuracy, because of its completeness, robustness and continuity. A function $u(\mathbf{x})$ can be approximated over a number of scattered local points $\{\mathrm{x}_i\}$, (i = 1, 2,...,n) as,

$$u(\mathrm{x}) = \mathrm{p}^T(\mathrm{x})\mathrm{a}(\mathrm{x}), \quad \forall \mathrm{x} \in \Omega_s \tag{1}$$

where $\mathrm{p}^T(\mathrm{x}) = [p_1(\mathrm{x}), p_2(\mathrm{x}), ..., p_n(\mathrm{x})]$ is a monomial basis of order n and a(x) is a vector containing coefficients, which are functions of the global Cartesian coordinates $[x_1, x_2, x_3]$, depending on the monomial basis. They are determined by minimizing a weighted discrete L_2 norm, defined, as:

$$\begin{aligned} J(\mathrm{x}) &= \sum_{i=1}^{n} w_i(\mathrm{x}) \left[\mathrm{p}^T(\mathrm{x}_i)\mathrm{a}(\mathrm{x}) - \hat{u}_i \right]^2 \\ &\equiv \left[\mathrm{P} \cdot \mathrm{a(x)} - \hat{\mathrm{u}} \right]^T \mathrm{W} \left[\mathrm{P} \cdot \mathrm{a(x)} - \hat{\mathrm{u}} \right] \end{aligned} \tag{2}$$

where $w_i(\mathrm{x})$ are the weight functions and $\hat{u}_i$ are the fictitious nodal values.

One may obtain the shape function as,

$$u(\mathrm{x}) = \mathrm{p}^T(\mathrm{x})\mathrm{A}^{-1}(\mathrm{x})\mathrm{B}(\mathrm{x})\hat{\mathrm{u}} \equiv \Phi^T(\mathrm{x})\hat{\mathrm{u}}, \quad \forall \mathrm{x} \in \partial\Omega_x \tag{3}$$

where matrices A(x) and B(x) are defined as,

$$\mathrm{A(x)} = \mathrm{P}^T \mathrm{WP}, \quad \mathrm{B(x)} = \mathrm{P}^T \mathrm{W}, \quad \forall \mathrm{x} \in \partial\Omega_x . \tag{4}$$

The weight function in Eq. (2) defines the range of influence of node I. Normally it has a compact support. A fourth order spline weight function is used which is defined as,

$$w_I(\mathrm{x}) = \begin{cases} 1 - 6\left(\dfrac{d_I}{r_I}\right)^2 + 8\left(\dfrac{d_I}{r_I}\right)^3 - 3\left(\dfrac{d_I}{r_I}\right)^4, & 0 \le d_I \le r_{I,} \\ 0 & , \quad d_I \ge r_I , \end{cases} \tag{5}$$

where $d_I = |\mathrm{x} - \mathrm{x}_I|$ is the Eucludian distance from node x_I to point x, r_I is the size of the support for the weight function w_I and thus determines support of the node x_I.

3. MLPG FINITE VOLUME METHOD

The equations of balance of linear and angular momentum can be written as:

$$\sigma_{ij,j} + f_i = 0; \quad \sigma_{ij} = \sigma_{ji}; \quad (\ )_{,i} \equiv \frac{\partial}{\partial \xi_i} \tag{6}$$

where σ_{ij} is the stress tensor corresponding to the displacement field u_i, f_i is the body force. The boundary conditions are given by,

$$\begin{aligned} u_i &= \overline{u}_i, \quad on \quad \Gamma_u \\ t_i &\equiv \sigma_{ij} n_j = \overline{t}_i, \quad on \quad \Gamma_t \end{aligned} \tag{7}$$

where $\overline{u}_i$ and $\overline{t}_i$ are the prescribed displacements and tractions respectively on the corresponding parts of the boundary and n_i is the unit outward normal to the boundary on the correponding points of Γ. A generalized local weak form of the differential equation (6) over a local sub-domain Ω_s, can be written as:

$$\int_{\Omega_s} \left(\sigma_{ij,j} + f_i\right) v_i d\Omega = 0 \tag{8}$$

where u_i and v_i are the trial and test functions, respectively. By applying the divergence theorem, imposing the traction b.c. and using the Heaviside function as the test function, the local symmetric weak form of Eq. (8) can be found as in [11],

$$-\int_{L_s} t_i d\Gamma - \int_{\Gamma_{su}} t_i d\Gamma = \int_{\Gamma_{st}} \overline{t}_i d\Gamma + \int_{\Omega_s} f_i d\Omega\,. \tag{9}$$

With the constitutive relations of an isotropic linear elastic homogeneous solid, the tractions in Eq. (9) can be written in term of the strains:

$$t_i = \sigma_{ij} n_i = E_{ijkl} \varepsilon_{kl} n_j \tag{10}$$

where, $E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ with λ and μ being the Lame's constants. The strains are independently interpolated as,

$$\varepsilon_{kl}(\mathrm{x})=\sum_{K=1}^{N}\Phi^{(K)}(\mathrm{x})\varepsilon_{kl}^{(K)}. \tag{11}$$

With Eqs. (10) and (11), one may discretize the local symmetric weak-form of Eq. (9), as,

$$-\sum_{K=1}^{N}\left[\int_{L_s}\Phi^{(K)}(\mathrm{x})E_{ijkl}n_j d\Gamma\right]\varepsilon_{kl}^{(K)} - \sum_{K=1}^{N}\left[\int_{L_{su}}\Phi^{(K)}(\mathrm{x})E_{ijkl}n_j d\Gamma\right]\varepsilon_{kl}^{(K)} = \int_{\Gamma_{st}}\bar{t}_i d\Gamma + \int_{\Omega_s} f_i d\Omega. \tag{12}$$

The number of the variables reduced by transforming the strain variables back to the displacement variables via the collocation methods. The interpolation of displacements can also be accomplished by using the same shape function, for the nodal displacement variables, and written as,

$$u_i(\mathrm{x})=\sum_{J=1}^{N}\Phi^{(J)}(\mathrm{x})u_i^{(J)}. \tag{13}$$

The strain-displacement relations are given by,

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right). \tag{14}$$

With the displacement approximation in Eq. (13), the two sets of nodal variables can be transformed through a linear algebraic matrix:

$$\varepsilon_{kl}^{(I)} = H_{klm}^{(I)(J)}u_m^{(J)} \tag{15}$$

which is reduced to the same number of nodal displacement variables by using the transformation. Data preparation, quadrature techniques and post processing issues are detailed in reference [11].

4. CONTACT PROBLEM OF A CANTILEVER ON RIGID FOUNDATION USING LCP

An elastic cantilever beam, in the plane stress state, contacting a rigid foundation is considered. The boundary and constraint conditions are illustrated in Fig. (1). Assume the beam, with a constant rigidity EI, is subjected to a uniform loading *f*. If the initial gap between the beam and the rigid foundation is given as δ^0, then the location of the contact interface computed from thin beam theory without shear deformation effect [9] is;

$$l_c = \sqrt[4]{\frac{72EI\delta^0}{f}} . \tag{16}$$

In order to verify the prediction by the present method, the thick beam theory [12] should be used as a reference.

In this paper the MLPG FVM is used with the MLS approximation and the problem is modeled using the LCP and solved. The beam is placed over a foundation with an initial gap g_i is configured in Figure (1). The boundary conditions can be easily seen from the Figure (1).

The beam parameters are: E=30,000, v=0.3, D=6, L=48, f=1, δ^0=0.01. The cross-section of the beam is assumed as rectangular with k=0.85, $G = \left(E/2(1+v)\right)$ and A=6. The location of contact by thin beam theory is l_c=24.97 and by thick beam theory [9] is l_c=21.37.

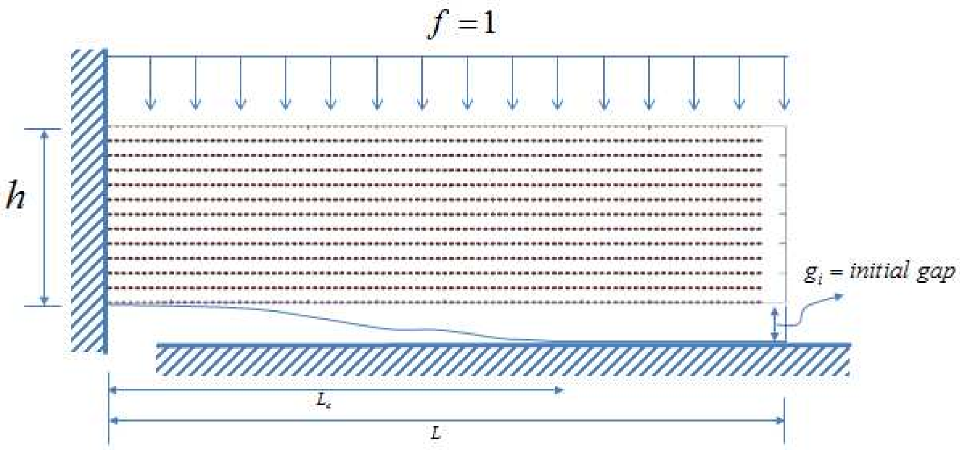

Figure 1: Contact problem of a cantilever on rigid foundation.

The contact problem is modeled using the MLPG-FVM. For each body the following equation has to be satisfied:

$$\sigma_{ij,j}(u^e) + b_i^e = 0 \text{ in } \Omega^e \ \text{(e=1,2)}. \quad (17)$$

The weak form of the equation (17) is found using the MLPG mixed method which gives the following equation for two body;

$$-\int_{\Gamma_{si}^e} t_i^e v^e d\Gamma - \int_{\Gamma_{st}^e} \overline{t_i}^e v^e d\Gamma \geq 0, \ \text{(e=1,2)}. \quad (18)$$

By applying meshless method to the equation (18), the following discrete equation for each node is found;

$$\sum_{j=1}^{n} K_{ij} u_j - f_i \geq 0 \quad (19)$$

where,

$$K_{ij} = -\int_{\Gamma_{si}^e} t_i^e v^e d\Gamma, \quad f_i = \int_{\Gamma_{st}^e} \overline{t_i}^e v^e d\Gamma. \quad (20)$$

Finally the following matrix system is obtained for two contacted elastic body by collecting the equations obtained from each local sub-domain $\Omega_s^{(i)}$ and without any element assembly one can write;

$$\sum_{e=1}^{2}\left(K^e u^e - f^e\right) \geq 0 . \tag{21}$$

In order to obtain Linear Complementary Equation (LCE), we can write the stiffness matrix in partitioned matrix form as:

$$K^e = \begin{bmatrix} K_{ii}^e & K_{ic}^e \\ K_{ci}^e & K_{cc}^e \end{bmatrix} \tag{22}$$

Subscript '*c*' refers the contact surface, and subscript '*i*' refers to other nodes. If we use eqn. (22) in eqn. (21) and eliminate we can write eqn. (21) in a small form where we have just the unknown values of *u* along the contact surface. In this aspect we can write equation (21) as follows:

$$K_{ii}^e u_i^e + K_{ic}^e u_c^e = f_i^e , \tag{23}$$

$$\sum_{e=1}^{2}\left(K_{ci}^e u_i^e + K_{cc}^e u_c^e\right) \geq 0 . \tag{24}$$

We can find u_i from equation (23) and then put into equation (24) so we will find the following relation:

$$\sum_{e=1}^{2}\left(K_p^e u_c^e - f_p^e\right) \geq 0 \tag{25}$$

where,

$$K_p^e = K_{cc}^e - K_{ci}^e [K_{ii}^e]^{-1} K_{ic}^e , \tag{26}$$

$$f_p^e = -K_{ci}^e [K_{ii}^e]^{-1} f_i^e . \tag{27}$$

The initial gap is defined as $g_0 = u_{c1} + u_{c2}$ in Γ^c. So the following transformation based on this assumption, the gap is defined as;

$$\begin{Bmatrix} u_c^1 \\ u_c^2 \end{Bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{Bmatrix} u_c^1 \\ g \end{Bmatrix}. \tag{28}$$

If we write equation (25) in expanded form we found the following one;

$$K_p^1 u_c^1 - f_p^1 + K_p^2 u_c^2 - f_p^2 \geq 0. \tag{29}$$

It can be written in matrix form by making some arrangements;

$$\begin{bmatrix} K_p^1 + K_p^2 & 0 \\ -K_p^2 & K_p^2 \end{bmatrix} \begin{Bmatrix} u_c^1 \\ g \end{Bmatrix} \geq \begin{Bmatrix} f_p^1 - f_p^2 \\ f_p^2 \end{Bmatrix} \tag{30}$$

$$u_c^1 = (K_p^1 + K_p^2)^{-1} \left[K_p^2 g + f_p^1 - f_p^2 \right] \tag{31}$$

$$\left[K_p^2 - K_p^2 (K_p^1 + K_p^2)^{-1} K_p^2 \right] g + K_p^2 (K_p^1 + K_p^2)^{-1} \left[f_p^2 - f_p^1 \right] - f_p^2 \geq 0. \tag{32}$$

If we denote *K* and *f* as follows;

$$\begin{aligned} K &= \left[K_p^2 - K_p^2 (K_p^1 + K_p^2)^{-1} K_p^2 \right] \\ f &= K_p^2 (K_p^1 + K_p^2)^{-1} \left[f_p^2 - f_p^1 \right] - f_p^2 \end{aligned}, \tag{33}$$

equation (32) becomes to the following form:

$$Kg + f \geq 0, \tag{34}$$

which can be solved by using Lemke's algorithm as a LCE problem.

5. RESULTS OF THE BEAM CONTACT PROBLEM

The beam contact problem modeled using the MLPG Finite Volume Method as in equation (34) and solved using Lemke's algorithm as a LCE problem. Distribution of the displacement along the beam and the contact force is presented in Figure (2) and Figure (3) respectively.

As it can be seen from Figure (1), the line of contact of the surface is represented by L_c. Figure (2) shows the displacement of the beam and it is observed that the displacement distribution along the beam is compatible with the line of contact along the beam surface. Figure (3) shows the distribution of the contact force along the beam.

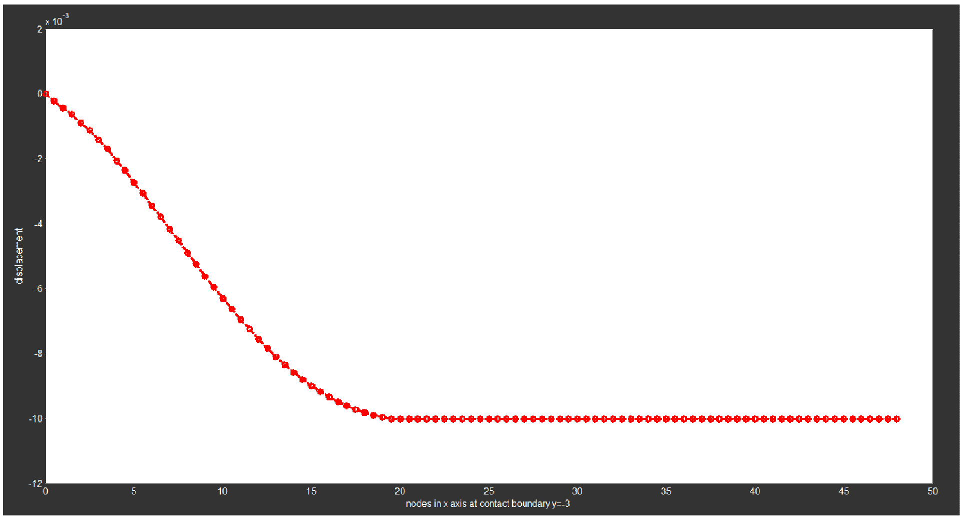

Figure 2: Distribution of the displacement along the beam.

In reference [9] it has been explained that, the location of contact (transition point) by thin beam theory is lc=24.97 and by thick beam theory is lc=21.37. Both MQ and TPS are studied in reference [9] and the calculated results with MQ shows closer to the thick beam result. The simulated contact force along the interface by using TPS with $\eta = 4$ gives a better result. In this study, it has been shown that the contact force along the interface is in good agreement with the thick beam theory. This result is found by using mixed MLPG FVM and using the LCP solution of the beam problem.

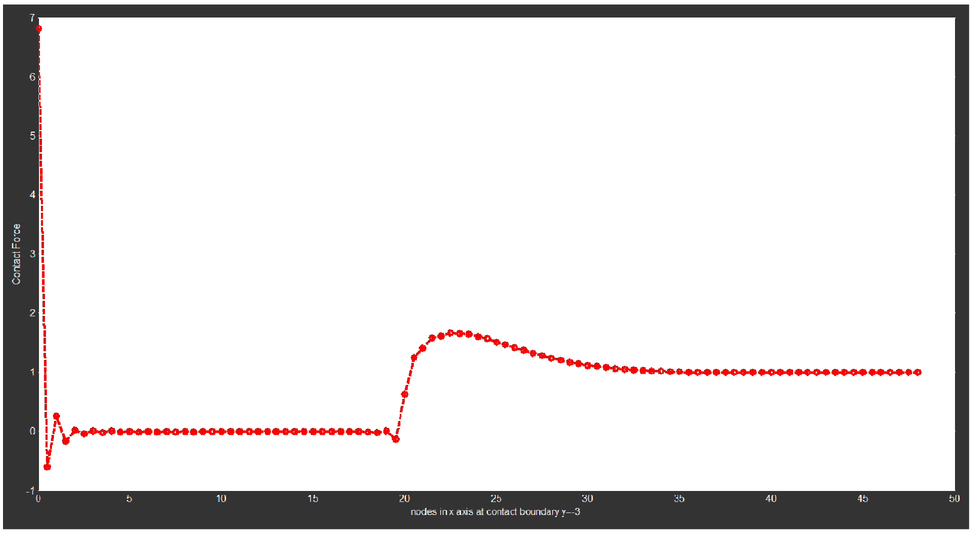

Figure 3: Contact force along the interface of a cantilever beam using MLS.

6. CONCLUSION

In this paper first of all the meshless approximations using MLS is mentioned. Then the general theory of the "mixed" MLPG Finite Volume Method is used to model the balance of linear and angular momentum. Contact problem of a cantilever beam on rigid foundation is modeled using the MLPG-FVM. For each body a contact equation is written and the inequality is solved by using Lemke's algorithm as a LCE problem.

Distribution of the displacement along the beam and the contact force in agreement with the theory given in literature. The earlier studies in the literature reports that, the location of contact (transition point) by thin beam theory is *lc*=24.97 and by thick beam theory is *lc*=21.37. Both MQ and TPS interpolating schemes are studied before by other researchers shows closer results to the thick beam theory. In this study, it has been shown that the contact force along the interface is in good agreement with the thick beam theory. This means that the proposed "mixed" MLPG FVM using LCP with MLS approximation gives good results and could be used to solve contact problems.

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