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ON THE POINTWISE GROWTH OF POLYNOMIALS IN UNBOUNDED REGIONS WITH QUASICONFORMAL BOUNDARY

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Abstract. In this present work, we continue studying the estimation of Bernstein-Walsh type for algebraic polynomials in the regions with quasiconformal boundary.

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1. Introduction and Main Results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $B := B(0,1) := \{w : |w| < 1\}$, $\Delta := \Delta(0,1) := \{w : |w| > 1\}$, $\Omega := ext\overline{G}$ (respect to $\overline{\mathbb{C}}$); $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty, \Phi^{'}(\infty) > 0$, and $\Psi := \Phi^{-1}$. Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let σ be the two-dimensional Lebesque measure and let $h(z)$ be a weight function defined in G.

Let $A_p(h, G),\ p > 0$ denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|^p_{A_p(h,G)} := \iint\limits_G h(z)\,|f(z)|^p\, d\sigma_z < \infty,$$

and $A_p(1, G) \equiv A_p(G)$.

In case of when L is rectifiable, let $\mathcal{L}_p(L),\ p > 0$, denote the class of functions f which are integrable on L and satisfy the condition

$$\|f\|^p_{\mathcal{L}_p(L)} := \int\limits_L |f(z)|^p\, |dz| < \infty.$$

For any $R > 1$, let us set $L_R := \{z : \ |\Phi(z)| = R\}, G_R := intL_R$, $\Omega_R := extL_R$. Well known Bernstein -Walsh lemma [12] says that:

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$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}. \tag{1.1}$$

Taking $R = 1 + \frac{1}{n}$, from (1.1) we see that the $C-$norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ is identical, i.e. the norm $\|P_n\|_{C(\overline{G})}$ increases with at most a constant.

Similar estimation to (1.1) in space $\mathcal{L}_p(L)$ was investigated in [11] and obtained as following:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0. \tag{1.2}$$

To formulate the corresponding result for the $A_p(h, G)$ space we give the following

Definition 1.1. *[8, p.97],[9] The Jordan arc (or curve) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).*

Let $F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. Then, L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

It is well know that there exists quasiconformal curve which is not rectifiable [8, p104].

Let L be a $K-$quasiconformal and $y(.)$ be a regular quasiconformal reflection across L (for detail see Part 2). For $R > 1$, let $L^* := y(L_R)$, $G^* := intL^*$, $\Omega^* := extL^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty, \Phi_R^{'}(\infty) > 0$, and $\Psi_R := \Phi_R^{-1}$; $d(\Gamma, L) := \inf \{|\zeta - z| : z \in \Gamma,\ \zeta \in L\}$.

The Bernstein-Walsh type estimation in the space $A_p(h, G)$, $p > 0$, for regions with quasiconformal boundary is contained in [3]. In particular, for $h(z) \equiv 1$ have

$$\|P_n\|_{A_p(G_R)} \leq c_1\ {R^*}^{n+\frac{1}{p}} \|P_n\|_{A_p(G)}, \quad p > 0. \tag{1.3}$$

where $R^* := 1 + c_2(R-1)$. Therefore, if we choose $R = 1 + \frac{1}{n}$, then (1.3) we see that A_p-norm of polynomials $P_n(z)$ in G_R and G also is equivalent.

In [10] was obtained pointwise analog to the estimation (1.1) following type, where the norm $\|P_n\|_{C(\overline{G})}$ in right side of (1.1) was replaced with the norm $\|P_n\|_{A_2(G)}$:

Lemma A[10]. *Assume that L is quasiconformal and rectifiable. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$*

$$|P_n(z)| \leq \frac{c}{d(z, L)} \sqrt{n} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{1.4}$$

where $c = c(L) > 0$ constant independent from z and n.

On the other hand, using the mean value theorem, for an arbitrary Jordan region G , polynomials $P_n \in \wp_n$, $n \in \mathbb{N}$ and any $p > 0$ we can find

$$|P_n(z)| \leq \left(\frac{1}{\sqrt{\pi} d(z,L)} \right)^{\frac{2}{p}} \|P_n\|_{A_p(G)}, \quad z \in G. \tag{1.5}$$

Thus, combining a relations (1.4) and (1.5), we obtain an pointwise estimate on the growth of $|P_n(z)|$ in the whole complex plane. In particular, for $p = 2$ we have

Corollary B. *Assume that L is quasiconformal and rectifiable. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$*

$$|P_n(z)| \leq \frac{c_2}{d(z,L)} \|P_n\|_{A_2(G)} \begin{cases} 1, & z \in G, \\ \sqrt{n} |\Phi(z)|^{n+1}, & z \in \Omega. \end{cases} \tag{1.6}$$

In this work, we continue the study the similar problems (1.4), for polynomials $P_n^{(m)}(z)$, $m = 0, 1, 2, ...$, in regions with $K-$quasiconformal boundary.

Now, we give the main results:

Theorem 1.1. *Let $p > 1$ and assume that L is $K-$quasiconformal. Then, for any $P_n \in \wp_n$, $R > 1$, $n > m$ and $m = 0, 1, 2, ...$ we have*

$$\left| P_n^{(m)}(z) \right| \leq c_4 \frac{|\Phi(z)|^{n+1-m}}{[d(L, L_R)]^{m+\frac{2}{p}}} \|P_n\|_{A_p(G_R)}, \quad z \in \Omega, \tag{1.7}$$

where $c_4 = c_4(G, m, p) > 0$ constant independent from z and n.

Corollary 1.1. *For $R = 1 + \frac{c}{n}$, $c = const > 0$. we have*

$$\left| P_n^{(m)}(z) \right| \leq \frac{c_5}{\left[d(L, L_{1+\frac{c}{n}}) \right]^{m+\frac{2}{p}}} \|P_n\|_{A_p(G)} |\Phi(z)|^{n+1-m}, \quad z \in \Omega, \tag{1.8}$$

where $c_5 = c_5(G, m, p, c) > 0$ constant independent from z and n.

Remark 1.1. *If $z \in G_{1+\frac{c}{n}} \cap \Omega$ for some $c > 0$, then (1.8)for $p = 2$ and $m = 0$ is better than (1.4).*

Theorem 1.2. *Let $p > 1$ and assume that L is $K-$quasiconformal. Then, for any $P_n \in \wp_n$, $R > 1$ and $n > 1$ we have*

$$|P_n(z)| \leq c_6 \frac{n^{\mu - \mu^{-1} + \frac{2}{p} - 1}}{d(z,L)} \|P_n\|_{A_p(G)} \left| \Phi_{1+\frac{1}{n}}(z) \right|^{n+1}, \quad z \in \Omega, \tag{1.9}$$

where $\mu := \min \left\{ 2, K^4 \right\}$ and $c_6 = c_6(G, p) > 0$ constant independent from z and n.

Remark 1.2. *For $K \leq \sqrt[4]{\frac{1+\sqrt{17}}{4}}$ and for $z \in \Omega$ such that far away from L, (1.9) for $p = 2$ is better than (1.4).*

Theorem 1.3. *Let $p > 1$ and assume that L is $K-$quasiconformal. Then, for any $P_n \in \wp_n$, $n > 1$, we have*

$$|P_n(z)| \leq c_7 n^{\mu + \frac{2}{p} - 1} \|P_n\|_{A_p(G)} \left| \Phi_{1+\frac{1}{n}}(z) \right|^{n+1}, \quad z \in \Omega, \tag{1.10}$$

where $\mu := \min \left\{ 2, K^4 \right\}$ and $c_7 = c_7(G, p) > 0$ constant independent from z and n.

Theorem 1.4. *Let $p>1$ and assume that L is $K-$quasiconformal. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have*

$$|P_n(z)| \leq c_8 \frac{n^{\nu-\nu^{-1}+\frac{2}{p}-1}}{d(z,L)} \|P_n\|_{A_p(G)} |\Phi(z)|^{n+1}, \quad z \in \overline{\Omega}_{1+\frac{1}{n}}, \tag{1.11}$$

where $\nu := \min\left\{2, K^2\right\}$ and $c_8 = c_8(G,p) > 0$ constant independent from z and n.

Remark 1.3. *For any $K \leq \frac{1}{2}\sqrt{1+\sqrt{17}}$ and the points $z \in \overline{\Omega}_{1+\frac{1}{n}}$, (1.11) for $p=2$ is better than (1.4).*

Theorem 1.5. *Let $p>1$ and assume that L is $K-$quasiconformal. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, we have*

$$|P_n(z)| \leq c_9 n^{\nu+\frac{2}{p}-1} \|P_n\|_{A_p(G)} |\Phi(z)|^{n+1}, \quad z \in \overline{\Omega}_{1+\frac{1}{n}}, \tag{1.12}$$

where $\nu := \min\left\{2, K^2\right\}$ and $c_9 = c_9(G,p) > 0$ constant independent from z and n.

We note that the Theorems 1.2 -1.5 are sharp in the some cases. This can be clearly seen from following theorem

Theorem 1.6. *For any $n = 1,2,3,...$, there exists a polynomial $P_n^*(z)$ and constant $c_{10} = c_{10}(G) > 0$ such that*

$$|P_n^*(z)| \geq c_{10} \frac{1}{d(L, L_{1+\frac{1}{n}})} \|P_n^*\|_{A_2(G)} |\Phi(z)|^{n+1},$$

for some $z \in \overline{\Omega}_{1+\frac{1}{n}}$.

2. Some auxiliary results

Let $G \subset \mathbb{C}$ be a finite region bounded by Jordan curve L and let $w = \varphi(z)$ be the univalent conformal mapping of G onto the B normalized by $\varphi(0) = 0, \varphi'(0) > 0$ and $\psi := \varphi^{-1}$.

The level curve (interior or exterior) can be defined for $t > 0$ as

$$L_t := \{z : |\varphi(z)| = t, \ if \ t < 1, \ |\Phi(z)| = t, \ if \ t > 1\}, \ L_1 \equiv L,$$

and let $G_t := intL_t$, $\Omega_t := extL_t$.

For $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" (order inequality), if $a \leq cb$ and "$a \asymp b$" are equivalent to $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively. Throughout this paper $\varepsilon, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (in general, different in different relations), which depend on G in general.

Let L be a $K-$quasiconformal curve. Then [5] there exists a quasiconformal reflection $y(.)$ across L such that $y(G) = \Omega, y(\Omega) = G$ and y(.) fixes the points of L. The quasiconformal reflection $y(.)$ is such that it satisfied the following condition [5], [7, p.26] :

$$\begin{aligned} &|y(\zeta)-z| \asymp |\zeta-z|, \ z\in L, \ \varepsilon<|\zeta|<\frac{1}{\varepsilon}, \\ &\left|y_{\overline{\zeta}}\right| \asymp |y_\zeta| \asymp 1, \ \varepsilon<|\zeta|<\frac{1}{\varepsilon}, \\ &\left|y_{\overline{\zeta}}\right| \asymp |y(\zeta)|^2, \ |\zeta|<\varepsilon, \ \left|y_{\overline{\zeta}}\right| \asymp |\zeta|^{-2}, |\zeta|>\frac{1}{\varepsilon}. \end{aligned} \tag{2.1}$$

and for the Jacobian $J_y = |y_z|^2 - |y_{\overline{z}}|^2$ of $y(.)$ the relation $J_y \asymp 1$ is hold.

For $R>1$, we denote $L^* := y(L_R)$, $G^* := intL^*$, $\Omega^* := extL^*$; $w=\Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty)=\infty$, $\Phi_R^{'}(\infty)>0$; $\Psi_R := \Phi_R^{-1}$; For $t>1$, let $L_t^* := \{z : |\Phi_R(z)| = t\}$, $G_t^* := intL_t^*$, $\Omega_t^* := extL_t^*$.

According to [6], for all $z \in L^*$ and $t \in L$ such that $|z-t| = d(z,L)$ we have

$$\begin{aligned} d(z,L) \asymp d(t,L_R) \asymp d(z,L_R^*). \\ |\Phi_R(z)| \le |\Phi_R(t)| \le 1 + c(R-1). \end{aligned} \tag{2.2}$$

Lemma 2.1. *[1] Let L be a $K-$quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z-z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j=1,2,3$. Then*

a) The statements $|z_1-z_2| \prec |z_1-z_3|$ and $|w_1-w_2| \prec |w_1-w_3|$ are equivalent. So are $|z_1-z_2| \asymp |z_1-z_3|$ and $|w_1-w_2| \asymp |w_1-w_3|$.

b) If $|z_1-z_2| \prec |z_1-z_3|$, then

$$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^{-2}} \prec \left|\frac{z_1-z_3}{z_1-z_2}\right| \prec \left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^2},$$

where $0<r_0<1$ a constant, depending on G.

In particular, for arbitrary $z_1 \in L$, $1<R<R_0$ and fixed $z_3 \in L_{R_0}$ we have

$$(R-1)^s \prec d(z_1, L_R) \prec (R-1)^{\frac{1}{K^2}}, \ s := \min\left\{2, K^2\right\}. \tag{2.3}$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ is defined as the following:

$$h(z) = h_0(z) \prod_{j=1}^m |z-z_j|^{\gamma_j}, \tag{2.4}$$

where $\gamma_j > -2$ for $j=\overline{1,m}$ and $h_0(z)$ is uniformly separated from zero in G:

$$h_0(z) \ge c_0 > 0, \ \forall z \in G.$$

Lemma 2.2. *[4] Let L be a $K-$quasiconformal curve; $h(z)$ is defined in (2.4). Then, for arbitrary $P_n(z) \in \wp_n$, any $R>1$ and $n=1,2,...,$ we have*

$$\|P_n\|_{A_p(h, G_{1+c(R-1)})} \le c_1 R^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \ p>0, \tag{2.5}$$

where c, c_1 are independent of n and R.

3. Proof of Theorems

3.1. Proof of Theorem 1.1.

Proof. Since L is a $K-$quasiconformal, we conclude that any L_R, $R>1$, is also quasiconformal. Therefore, we can construct a $c(K)$-quasiconformal reflection $y_R(z)$, $y_R(0)=\infty$, across L_R such that $y_R(G_R)=\Omega_R$, $y_R(\Omega_R)=G_R$ and $y_R(.)$ fixes the points of L_R that satisfies conditions (2.1) described for $y_R(z)$. By using this constructed $y_R(z)$, we can write the following integral representations for $P_n(z)$ [7, p.105]

$$P_n^{(m)}(z)=-\frac{(m+1)!}{\pi}\iint\limits_{G_R}\frac{P_n(\zeta)\,y_{R,\bar{\zeta}}}{(y_R(\zeta)-z)^{m+2}}d\sigma_\zeta\ ,\quad z\in G_R. \tag{3.1}$$

For $\varepsilon>0$, let us set $U_\varepsilon(z):=\{\zeta:|\zeta-z|<\varepsilon\}$ and without loss of generality we may take $U_\varepsilon:=U_\varepsilon(0)\subset G^*$. For arbitrary fixed point $z\in L$ we have

$$\begin{aligned}\left|P_n^{(m)}(z)\right| &\le \frac{(m+1)!}{\pi}\iint\limits_{U_\varepsilon}\frac{|P_n(\zeta)|\left|y_{R,\bar{\zeta}}\right|}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta \\ &+\frac{(m+1)!}{\pi}\iint\limits_{G_R\setminus U_\varepsilon}\frac{|P_n(\zeta)|\left|y_{R,\bar{\zeta}}\right|}{|y_R(\zeta)-z|^{m+2}}d\sigma_\zeta \\ &=:J_1+J_2.\end{aligned} \tag{3.2}$$

To estimate the integral J_1, applying the Hölder inequality we get

$$\begin{aligned}J_1 &\le \left(\iint\limits_{U_\varepsilon}|P_n(\zeta)|^p\,d\sigma_\zeta\right)^{\frac{1}{p}}\cdot\left(\iint\limits_{U_\varepsilon}\frac{\left|y_{R,\bar{\zeta}}\right|^q}{|y_R(\zeta)-z|^{mq+2q}}d\sigma_\zeta\right)^{\frac{1}{q}} \\ &\prec \|P_n\|_{A_p(G)}\left(\iint\limits_{U_\varepsilon}\frac{\left|y_{R,\bar{\zeta}}\right|^q}{|y_R(\zeta)-z|^{mq+2q}}d\sigma_\zeta\right)^{\frac{1}{q}},\ \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

According to (2.1), $\left|y_{R,\bar{\zeta}}\right|\asymp|y_R(\zeta)|^2$, for all $\zeta\in U_\varepsilon$, because of $|\zeta-z|\ge\varepsilon$, $|y_R(\zeta)-z|\asymp|y_R(\zeta)|$ for $z\in L$ and $\zeta\in U_\varepsilon$. On the other hand, if $J_{y,R}:=|y_{R,\zeta}|^2-\left|y_{R,\bar{\zeta}}\right|^2$ is Jacobian of the reflection $y_R(\zeta)$, we can obtain

$$|J_{y,R}|\succ\left|y_{R,\bar{\zeta}}\right|^2$$

as in [2]. Then, we can find

$$J_1 \prec \|P_n\|_{A_p(G)} \left(\iint\limits_{y_R(U_\varepsilon)} \frac{\left|y_{R,\bar{\zeta}}\right|^q}{|J_{y,R}| \, |\zeta - z|^{mq+2q}} d\sigma_\zeta \right)^{\frac{1}{q}} \tag{3.3}$$

$$\prec \|P_n\|_{A_p(G)} \left(\iint\limits_{|\zeta - z| \geq c_1} \frac{d\sigma_\zeta}{|\zeta - z|^{mq+2q}} \right)^{\frac{1}{q}}$$

$$\prec \|P_n\|_{A_p(G)} .$$

For the J_2, we get

$$J_2 = \left(\iint\limits_{G_R \setminus U_\varepsilon} \frac{\left|y_{R,\bar{\zeta}}\right|^q d\sigma_\zeta}{|y_R(\zeta) - z|^{mq+2q}} \right)^{\frac{1}{q}} \cdot \left(\iint\limits_{G_R \setminus U_\varepsilon} |P_n(\zeta)|^p \, d\sigma_\zeta \right)^{\frac{1}{p}} =: J_{21} \cdot J_{22}. \tag{3.4}$$

For the integral J_{21}, we get

$$J_{21} \prec \left(\iint\limits_{y(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z|^{mq+2q}} \right)^{\frac{1}{q}} \tag{3.5}$$

$$\leq \left(\iint\limits_{|\zeta - z| \geq d(z, L_R)} \frac{d\sigma_\zeta}{|\zeta - z|^{mq+2q}} \right)^{\frac{1}{q}} \prec \frac{1}{d^{m+2-\frac{2}{q}}(L, L_R)}.$$

and for J_{22} :

$$J_{22} = \left(\iint\limits_{G_R \setminus U_\varepsilon} |P_n(\zeta)|^p \, d\sigma_\zeta \right)^{\frac{1}{p}} \leq \left(\iint\limits_{G_R} |P_n(\zeta)|^p \, d\sigma_\zeta \right)^{\frac{1}{p}} = \|P_n\|_{A_p(G_R)} .$$

Then,

$$J_2^2 = J_{21} \cdot J_{22} \prec \frac{\|P_n\|_{A_p(G_R)}}{d^{m+2-\frac{2}{q}}(L, L_R)}. \tag{3.6}$$

To complete the proof of Theorem 1.1, according the maximum modulus principle, for any $z \in \Omega$ we have

$$\left| \frac{P_n^{(m)}(z)}{\Phi^{n+1-m}(z)} \right| \leq \max_{z \in \overline{\Omega}} \left| \frac{P_n^{(m)}(z)}{\Phi^{n+1-m}(z)} \right| = \max_{z \in L} \left| P_n^{(m)}(z) \right| \prec \frac{\|P_n\|_{A_p(G_R)}}{d^{m+2-\frac{2}{q}}(L, L_R)},$$

or

$$\left| P_n^{(m)}(z) \right| \prec \frac{\|P_n\|_{A_p(G_R)}}{d^{m+2-\frac{2}{q}}(L, L_R)} \cdot |\Phi(z)|^{n+1-m} . \tag{3.7}$$

Since $\frac{2}{q} = 2 - \frac{2}{p}$, then combining (3.2), (3.3), (3.4), (3.6) and (3.7), we complete the proof. □

Taking $R = 1 + \frac{1}{n}$, according to Lemma 2.2, we obtain the proof of Corollary 1.1.

3.2. Proof of Theorem 1.2

Proof . For the arbitrary fixed $R > 1$, let us set $L^* := y(L_R)$. According to (2.2), the number ε_1(consequently $\rho_1 := 1 + \varepsilon_1(R-1)$) can be chosen such that $\overline{G}^*_{\rho_1} \subseteq G$. Let $R_1 := 1 + \frac{\rho_1 - 1}{2}$. For $z \in \Omega$ and $w = \Phi_R(z)$ let us get:

$$h_R(w) := \frac{P_n(\Psi_R(w))}{w^{n+1}}.$$

Cauchy integral representation for unbounded region gives

$$h_R(w) = -\frac{1}{2\pi i} \int\limits_{|t|=R_1} h_R(t) \frac{dt}{t-w} .$$

For all $|t| = R_1 > 1$, $|t|^{n+1} = R_1^{n+1} > 1$, then

$$\begin{aligned} A_n &:= |P_n(\Psi_R(w))| \\ &\leq |w|^{n+1} \frac{1}{2\pi} \int\limits_{|t|=R_1} |P_n(\Psi_R(t))| \frac{|dt|}{|t-w|}. \end{aligned} \tag{3.8}$$

Applying the Hölder inequality, we get

$$\begin{aligned} A_n &\prec |w|^{n+1} \left(\int\limits_{|t|=R_1} |P_n(\Psi_R(t)) \cdot \Psi_R'(t)|^p |dt| \right)^{\frac{1}{p}} \\ &\times \left(\int\limits_{|t|=R_1} \frac{1}{|\Psi_R'(t)|^q |t-w|^q} |dt| \right)^{\frac{1}{q}} \\ &=: |w|^{n+1} \left(A_n^1\right)^{\frac{1}{p}} \cdot \left(B_n^1\right)^{\frac{1}{q}} . \end{aligned} \tag{3.9}$$

Let us set

$$f_n(t) := P_n(\Psi_R(t)) \cdot \Psi_R'(t) .$$

Now, we separate the circle $|t| = R_1$ to n equal part δ_n with $mes\delta_n = \frac{2\pi R_1}{n}$ and by applying the mean value theorem to the integral A_n^1 we get

$$A_n^1 = \sum_{k=1}^{n} \int\limits_{\delta_k} |f_n(t)|^p |dt| = \sum_{k=1}^{n} \left| f_n\left(t_k'\right)\right|^p mes\delta_k, \quad t_k' \in \delta_k.$$

On the other hand, by applying mean value estimation,

$$\left| f_n\left(t_k'\right)\right|^p \leq \frac{1}{\pi\left(\left|t_k'\right| - 1\right)^2} \iint\limits_{\left|\xi - t_k'\right| < \left|t_k'\right| - 1} |f_n(\xi)|^p \, d\sigma_\xi,$$

we obtain

$$A_n^1 \prec \sum_{k=1}^{n} \frac{mes\delta_k}{\pi\left(\left|t_k'\right|-1\right)^2} \iint\limits_{\left|\xi-t_k'\right|<\left|t_k'\right|-1} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi, \qquad t_k' \in \delta_k.$$

Taking into account that discs with origin at the points t_k' at most two may be crossing, we have

$$A_n^1 \prec \frac{mes\delta_1}{\left(\left|t_1'\right|-1\right)^2} \iint\limits_{1<|\xi|<\rho_1} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi \prec n \iint\limits_{1<|\xi|<\rho_1} \left|f_n\left(\xi\right)\right|^p d\sigma_\xi.$$

According to (2.2), for A_n^1, we get

$$A_n^1 \prec n \iint\limits_{G_{\rho_1}^* \setminus G^*} \left|P_n\left(z\right)\right|^p d\sigma_z \prec n.\left\|P_n\right\|_{A_2(G)}^p. \tag{3.10}$$

To estimate the integral B_n^1, taking into account the estimation for the Ψ_R' (see, for instance, [7, Th.2.8]) and Lemma 2.1 written for $\Phi_R(z)$, from (2.3) we get

$$\begin{aligned}
B_n^1 &\prec \int\limits_{|t|=R_1} \frac{\left(|t|-1\right)^q}{d^q(\Psi_R\left(t\right), L^*)} \frac{|dt|}{|t-w|^q} \\
&= \int\limits_{|t|=R_1} \frac{\left(|t|-1\right)^q}{d^q(\Psi_R\left(t\right), L^*)} \frac{|dt|}{|t-w|^{q-\frac{q}{\mu}} \left|t-w\right|^{\frac{q}{\mu}}} \\
&\prec \frac{1}{d^q(z, L_{R_1}^*)} \int\limits_{|t|=R_1} \frac{\left(|t|-1\right)^q}{\left(|t|-1\right)^{q\mu}} \frac{|dt|}{|t-w|^{q-\frac{q}{\mu}}} \\
&= \frac{1}{d^q(z, L_{R_1}^*)} \int\limits_{|t|=R_1} \frac{1}{\left(|t|-1\right)^{q(\mu-1)}} \frac{|dt|}{|t-w|^{q(1-\frac{1}{\mu})}} \\
&\prec \frac{1}{d^q(z, L_{R_1}^*)} \cdot n^{q(\mu-\mu^{-1})-1} \\
&\prec \frac{1}{d^q(z, L)} \cdot n^{q(\mu-\mu^{-1})-1},
\end{aligned} \tag{3.11}$$

where $\mu := \min\left\{2, K^4\right\}$. Relations (3.8), (3.9), (3.10), and (3.11) yield

$$\begin{aligned}
\left|P_n(z)\right| &\prec |w|^{n+1} n^{\frac{1}{p}}.\left\|P_n\right\|_{A_p(G)} \frac{1}{d(z, L)} \cdot n^{(\mu-\mu^{-1})-\frac{1}{q}} \\
&= \frac{n^{(\mu-\mu^{-1})+\frac{2}{p}-1}}{d(z, L)} \left|\Phi_R(z)\right|^{n+1} \left\|P_n\right\|_{A_p(G)}, \ z \in \Omega.
\end{aligned}$$

The proof is complete. □

3.3. Proof of Theorem 1.3

Proof. Proof of the Theorem 1.3 will be similar to proof of Theorem 1.2. The term in (3.11) will be treated as the following:

$$
\begin{aligned}
B_n^1 &\prec \int\limits_{|t|=R_1} \frac{(|t|-1)^q}{d^q(\Psi_R(t), L^*)} \frac{|dt|}{|t-w|^q} \\
&\prec \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{q\mu-q}} \frac{|dt|}{|t-w|^q} \\
&\prec \int\limits_{|t|=R_1} \frac{|dt|}{(|t|-1)^{q\mu}} \\
&\prec n^{q\mu-1}.
\end{aligned} \tag{3.12}
$$

And, consequently,

$$
\begin{aligned}
|P_n(z)| &\prec |w|^{n+1} n^{\frac{1}{p}}. \|P_n\|_{A_p(G)} n^{\mu-\frac{1}{q}} \\
&= n^{\mu+\frac{2}{p}-1} \|P_n\|_{A_p(G)} |\Phi_R(z)|^{n+1}, \ z \in \Omega.
\end{aligned}
$$

□

3.4. Proof of Theorem 1.4

Proof. Let $R > 1$ be arbitrary fixed and let $R_1 := 1 + \frac{R-1}{2}$. For $z \in \overline{\Omega}_R$ and $w = \Phi(z)$ let us get:

$$h(w) := \frac{P_n(\Psi(w))}{w^{n+1}}.$$

Cauchy integral representation for unbounded region gives

$$h(w) = -\frac{1}{2\pi i} \int\limits_{|t|=R_1} h(t) \frac{dt}{t-w}.$$

Following the method used in proof of Theorem 1.2, similar terms are treated as below:

$$
\begin{aligned}
\widetilde{A_n} &:= |P_n(\Psi(w))| \\
&\leq |w|^{n+1} \frac{1}{2\pi} \int\limits_{|t|=R_1} |P_n(\Psi(t))| \frac{|dt|}{|t-w|}.
\end{aligned} \tag{3.13}
$$

$$\prec |w|^{n+1}\left(\int\limits_{|t|=R_1}|P_n(\Psi(t))\cdot\Psi'(t)|^p\,|dt|\right)^{\frac{1}{p}}$$
$$\times\left(\int\limits_{|t|=R_1}\frac{1}{|\Psi'(t)|^q\,|t-w|^q}\,|dt|\right)^{\frac{1}{q}}$$
$$=:|w|^{n+1}\left(\widetilde{A}_n^1\right)^{\frac{1}{p}}\cdot\left(\widetilde{B}_n^1\right)^{\frac{1}{q}}.$$

Let us set

$$\widetilde{f_n}(t):=P_n(\Psi(t))\cdot\Psi'(t).$$

Now, we separate the circle $|t|=R_1$ to n equal part δ_n with $mes\delta_n=\frac{2\pi R_1}{n}$ and by applying the mean value theorem to the integral $\widetilde{A}_n^1$ we get

$$\widetilde{A}_n^1=\sum_{k=1}^{n}\int\limits_{\delta_k}\left|\widetilde{f_n}(t)\right|^p|dt|=\sum_{k=1}^{n}\left|\widetilde{f_n}\left(t_k'\right)\right|^p mes\delta_k,\quad t_k'\in\delta_k.$$

On the other hand, applying mean value estimation, we obtain

$$\widetilde{A}_n^1\prec\sum_{k=1}^{n}\frac{mes\delta_k}{\pi\left(\left|t_k'\right|-1\right)^2}\iint\limits_{\left|\xi-t_k'\right|<\left|t_k'\right|-1}\left|\widetilde{f_n}(\xi)\right|^p d\sigma_\xi,\qquad t_k'\in\delta_k.$$

$$\widetilde{A}_n^1\prec\frac{mes\delta_1}{\left(\left|t_1'\right|-1\right)^2}\iint\limits_{1<|\xi|<R}\left|\widetilde{f_n}(\xi)\right|^p d\sigma_\xi\prec n\iint\limits_{1<|\xi|<R}\left|\widetilde{f_n}(\xi)\right|^p d\sigma_\xi.$$

According to (2.2), for $\widetilde{A}_n^1$, we get

$$\widetilde{A}_n^1\prec n\iint\limits_{G_R\setminus G}|P_n(z)|^p\,d\sigma_z\prec n.\,\|P_n\|_{A_p(G_R)}^p. \tag{3.14}$$

To estimate the integral $\widetilde{B}_n^1$, taking into account that the estimation for the Ψ' (see, for instance, [7, Th.2.8]) and Lemma 2.1, we get

$$\begin{aligned}\widetilde{B}_n^1 &\prec \int\limits_{|t|=R_1} \frac{(|t|-1)^q}{d^q(\Psi(t),L)} \frac{|dt|}{|t-w|^q} \\ &= \int\limits_{|t|=R_1} \frac{(|t|-1)^q}{d^q(\Psi(t),L)} \frac{|dt|}{|t-w|^{q-\frac{q}{\nu}}\,|t-w|^{\frac{q}{\nu}}} \\ &\prec \frac{1}{d^q(z,L_{R_1})} \int\limits_{|t|=R_1} \frac{(|t|-1)^q}{(|t|-1)^{q\nu}} \frac{|dt|}{|t-w|^{q-\frac{q}{\nu}}} \\ &= \frac{1}{d^q(z,L_{R_1})} \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{q(\nu-1)}} \frac{|dt|}{|t-w|^{q(1-\frac{1}{\nu})}} \\ &\prec \frac{1}{d^q(z,L_{R_1})} \cdot n^{q(\nu-\nu^{-1})-1},\end{aligned} \tag{3.15}$$

where $\nu := \min\left\{2, K^2\right\}$.

Let us denote $\zeta = \Psi(\tau) \in L$, $\zeta_1 = \Psi(\tau_1) \in L_{R_1}$ such that: $d(z,L) = |z-\zeta|$, $d(z,L_{R_1}) = |z-\zeta_1|$, and denote this image from $\tau = \Phi(\zeta)$, $\tau_1 = \Phi(\zeta_1)$. Also we denote points $|\tau^*| = 1$, $|w-\tau^*| = |w|-1$, $|\tau_1^*| = R_1$, $|w-\tau_1^*| = |w|-R_1$. According to $R_1 := 1+\frac{R-1}{2}$ we have $|w-\tau_1| \asymp |w-\tau_1^*| \succ |w-\tau^*| \asymp |w-\tau|$. Then, by Lemma 2.1 we get $d(z,L_{R_1}) \succ d(z,L)$. Therefore, we obtain

$$\widetilde{B}_n^1 \prec \frac{1}{d^q(z,L)} \cdot n^{q(\nu-\nu^{-1})-1}. \tag{3.16}$$

Relations (3.13), (3.14), (3.16) and Lemma 2.2 yield

$$\begin{aligned}|P_n(z)| &\prec |w|^{n+1}\, n^{\frac{1}{p}}.\, \|P_n\|_{A_p(G_R)} \frac{1}{d(z,L)} \cdot n^{(\nu-\nu^{-1})-\frac{1}{q}} \\ &= \frac{n^{(\nu-\nu^{-1})+\frac{2}{p}-1}}{d(z,L)} |\Phi(z)|^{n+1} \|P_n\|_{A_p(G)},\ z \in \overline{\Omega}_{1+\frac{1}{n}}.\end{aligned}$$

The proof is complete. □

3.5. Proof of Theorem 1.5

Proof. Analogous to proof of the Theorem1.3, the proof of the Theorem 1.5 is identical to proof of the proof Theorem 1.4. In this case, the following the method used in the proof of Theorem 1.4 and (3.15) will be treated as the following:

$$\begin{aligned}
B_n^1 &\prec \int\limits_{|t|=R_1} \frac{(|t|-1)^q}{d^q(\Psi(t), L)} \frac{|dt|}{|t-w|^q} \\
&\prec \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{q\nu-q}} \frac{|dt|}{|t-w|^q} \\
&= \int\limits_{|t|=R_1} \frac{|dt|}{(|t|-1)^{q\nu}} \\
&\prec n^{q\nu-1}.
\end{aligned}$$

And, consequently,

$$\begin{aligned}
|P_n(z)| &\prec |w|^{n+1} n^{\frac{1}{p}}. \|P_n\|_{A_p(G_R)} n^{\nu-\frac{1}{q}} \\
&\prec n^{\nu+\frac{2}{p}-1} \|P_n\|_{A_p(G)} |\Phi(z)|^{n+1}, \ z \in \overline{\Omega}_{1+\frac{1}{n}}.
\end{aligned}$$

$\square$

3.6. Proof of Theorem 1.6

Proof. Let $P_n^*(z) := P_n(z) =_{j=0}^{n} (j+1)z^j$, $G = B$ and $p = 2$. In this case,

$$\|P_n\|_{C(\overline{G})} = \frac{(n+1)(n+2)}{2}; \quad \|P_n\|_{A_2(G)} = \sqrt{\frac{\pi(n+1)(n+2)}{2}}.$$

Then, for all $z \in L_{1+\frac{1}{n}}$ such that $|P_n(z)| = \|P_n\|_{C(\overline{G_R})}$, we have

$$\begin{aligned}
|P_n(z)| &\geq \|P_n\|_{C(\overline{G})} \geq \frac{1}{\sqrt{2\pi}} n \|P_n\|_{A_2(G)} \\
&= \frac{1}{\sqrt{2\pi}} \frac{d(L, L_{1+\frac{1}{n}})}{d(L, L_{1+\frac{1}{n}})} n \|P_n\|_{A_2(G)} \frac{|\Phi(z)|^{n+1}}{|\Phi(z)|^{n+1}} \\
&\geq c_{10} \frac{1}{d(L, L_{1+\frac{1}{n}})} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}.
\end{aligned}$$

$\square$

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