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Timelike Surfaces With Constant Angle in de -Sitter Space S_1^3

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Abstract. In this paper, we study a special class of timelike surface which is called constant timelike angle surfaces in de Sitter space S_1^3. In S_1^3, conditions being a constant angle timelike surface have been determined and invariants of these surface have been investigated. In here, we use the angle between unit normal vector field of surfaces and a fixed spacelike axis in ambient space.

Keywords: Constant angle surfaces , de Sitter space , Helix , Timelike surface

De-Sitter Uzayında Sabit Açılı Zamansal Yüzeyler

Özet. Bu çalışmada, yüzeyin birim normal vektör alanı ve R_1^4 *de sabit bir uzaysal eksen arasındaki açıyı kullanarak, de-Sitter uzayında sabit zamansal açılı yüzeyler olarak adlandırılan zamansal yüzeylerin özel bir sınıfı geliştirilmiştir*.

Anahtar Kelimeler: Sabit açılı yüzeyler, de-Sitter uzayı, Helis, zamansal yüzeyler

1. INTRODUCTION

In three dimensioan Euclidean space E^3, a constant angle surfaces are a surfaces whose tangents make constant angle with a fixed direction in ambient space. A surface whose tangent planes makes a constant angle with a fixed vector field is called constant angle surface in ambient space. M.I. Munteanu and A.I. Nistor studied constant angle surface and obtained all class of constant angle surface in E^3 [6]. Constant angle surface have been studied by A.J. Scale and G.R. Hernandez in n-dimension Euclidean space E^n [13,14]. The Constant angle surface were applied to liquid layers and liquid crystals by P. Germelli and A.J. Scala [12]. Constant angle surface have been studied recently in product spaces $S^2 \times R$ [15] , $H^2 \times R$ [16] or different ambient spaces Nil_3 [17]. In [1], Lopez and Munteanu studied constant hyperbolic angle surfaces whose unit normal timelike vector field makes a constant hyperbolic angle with a fixed timelike axis in Minkowski space R_1^4 . In the literature constant timelike and spacelike angle surface have not been investigated both in hyperbolic space H^3 and de sitter space S_1^3. A constant timelike and a spacelike angle surface in Hyperbolic space H^3 and constant angle spacelike surface in de sitter space S_1^3 are developed in our paper [19] , [20] and [21]. Constant timelike angle surface is a surface whose tangent planes makes a constant angle with a fixed vector field of S_1^3 . De Sitter space is a good model for a physical phenomenon. This kind of surfaces in de Sitter space S_1^3 involved with our daily life such as architecture and geometrical design.

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Probably, architectural structures and geometrical designs that use de Sitter curves enter into our life in the future. In this paper we introduce constant timelike angle timelike surfaces in de Sitter space S_1^3.

2. PRELIMINIARIES

Let R_1^4 be 4-dimensional vector space equipped with the scalar product $\langle , \rangle$ which is defined by

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 .$$

R_1^4 is 4-dimensional vector space equipped with the scalar product $\langle , \rangle$, than R_1^4 is called Lorentzian 4- space or 4-dimensional Minkowski space. The Lorentzian norm (length) of x is defined to be

$$\|x\| = \left|\langle x, x \rangle\right|^{\frac{1}{2}} .$$

If $\left(x_0^i, x_1^i, x_2^i, x_3^i\right)$ is the coordinate of x_i with respect to canonical basis $\{e_0, e_1, e_2, e_3\}$ of R_1^4, then the lorentzian cross product $x_1 \times x_2 \times x_3$ is defined by the symbolic determinant

$$x_1 \times x_2 \times x_3 = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix} .$$

On can easily see that

$$\langle x_1 \times x_2 \times x_3, x_4 \rangle = \det\left(x_1, x_2, x_3, x_4\right) .$$

In [2],[3]and [5] Izimuya at all introduced and investigated differantial geometry of curves and surfaces in Hyperbolic 3-space. In the rest of this section, we give background of context in [22]. Given a vector $v \in R_1^4$ and a real number c, the hyperplane with pseudo normal v is defined by

$$HP(v,c) = \left\{ x \in R_1^4, \langle x, v \rangle = c \right\} .$$

We say that $HP(v,c)$ is a spacelike hyperplane, timelike hyperplane or lightlike hyperplane if v is timelike, spacelike or lightlike respectively. We have following three types of pseudo-spheres in R_1^4:

Hyperbolic-3 space : $H^3(-1) = \left\{ x \in R_1^4, \langle x, x \rangle = -1, x_0 \geq 1 \right\}$,

de Sitter 3- space : $S_1^3 = \left\{ x \in R_1^4, \langle x, x \rangle = 1 \right\}$,

(open) lightcone : $LC^* = \left\{ x \in R_1^4 / \{0\}, \langle x, x \rangle = 0, x_0 = 1 \right\}$.

We also define the lightcone 3-sphere

$$S_+^3 = \left\{ x \in R_1^4, \langle x, x \rangle = 0, x_0 = 1 \right\} .$$

A hypersurface given by the intersection of S_1^3 with a spacelike (resp.timelike) hyperplane is called an elliptic hyperquadric (resp. hyperbolic hyperquadric). If $c \neq 0$ and $HP(v,c)$ is lightlike , then $HP(v,c) \cap S_1^3$ is a de Sitter horosphere.

Let $U \subset IR^2$ be open subset, and let $x : U \to S_1^3$ be an embedding. If the vector subspace $\tilde{U}$ which generated by $\{x_{u_1}, x_{u_2}\}$ is spacelike, then x is called spacelike surface, if U contain at least a timelike vector field then x is called timelike surface in S_1^3.

In point of view Kasedou [22], we construct the extrinsic differantial geometry of curves in S_1^3. Since S_1^3 is a Riemannian manifold, the regular curve $\gamma : I \to S_1^3$ is given by arclength parameter.

Theorem 1

i) if $\gamma : I \to S_1^3$ is a spacelike curve with unit speed, then Frenet-Serre type formulae is obtained

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = \kappa_d(s)n(s) - \gamma(s) \\ n'(s) = -\kappa_d(s)t(s) - \tau_d(s)e(s) \\ e'(s) = -\tau_d(s)n(s) \end{cases}$$

where $\kappa_d(s) = \| t'(s) + \gamma(s) \|$ and $\tau_d(s) = -\dfrac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa_d^2(s)}$.

ii) If $\gamma : I \to S_1^3$ is a timelike curve with unit speed, then Frenet-Serre type formulae is obtained

$$\begin{cases} \gamma'(s) = t(s) \\ t'(s) = \kappa_d(s)n(s) + \gamma(s) \\ n'(s) = \kappa_d(s)t(s) + \tau_d(s)e(s) \\ e'(s) = -\tau_d(s)n(s) \end{cases}$$

where $\kappa_d(s) = \| t'(s) - \gamma(s) \|$ and $\tau_d(s) = -\dfrac{\det(\gamma(s), \gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa_d^2(s)}$.

It is easily see that $\kappa_d(s) = 0$ if and only if there exists a lightlike vector c such that $\gamma(s) - c$ is a geodesic.

Now we give extrinsic differential geometry on surfaces in S_1^3 due to Kasedou [22].

Let $U \subset IR^2$ is an open subset, and $x : U \to S_1^3$ is a regular surface $M = x(u)$. Since M is a timelike surface, there is e($e(u) = \dfrac{x(u) \wedge x_1(u) \wedge x_2(u)}{\| x(u) \wedge x_1(u) \wedge x_2(u) \|}$ such that $\langle e, x \rangle = \langle e, x_{u_i} \rangle = 0, \langle e, e \rangle = 1$. Thus there is de Sitter Gauss image of x which is defined by mapping $E : U \subset IR^2 \to S_1^3, E(u) = e(u)$. The lightcone Gauss image of x is defined by map $L^{\pm} : U \to LC^*$

$$L^{\pm}(u) = x(u) \pm e(u).$$

The derivative $dx(u_0)$ can be identify by the mapping 1_{T_pM} on the tangent space T_pM.

Therefore, we have

$$dL^{\pm}(u_0) = 1_{T_pM} \pm dE(u_0).$$

The linear transformations

$$S_p^{\pm} := -dL^{\pm}(u_0) : T_pM \to T_pM$$

and

$$A_p := -dE(u_0) : T_pM \to T_pM$$

is called the hyperbolic shape operator and de Sitter shape operator of M at $p = x(u_0)$.

Let $\overline{K_i^{\pm}}(p)$ and $K_i(p),(i=1,2)$ be the eigenvalues of $S_p^{\pm}$ and A_p. Since

$$S_p^{\pm} = -1_{T_pM} \pm A_p,$$

$S_p^{\pm}$ and A_p have same eigenvectors and relations

$$\overline{K_i^{\pm}}(p) = -1 \pm K_i(p).$$

$\overline{K_i^{\pm}}(p)$ and $K_i(p),(i=1,2)$ are called hyperbolic and de Sitter principal curvetures of M at $p = x(u_0)$.

Let $\gamma(s)$ be a unit speed curve on M, with $p = \gamma(u_1(s), u_2(s))$. We consider the hyperbolic curvature vector $k(s) = t'(s) - \gamma(s)$ and the de Sitter normal curvature

$$K_n^{\pm}(s_0) = \langle k(s_0), L^{\pm}(u_1(s_0), u_2(s_0)) \rangle = \langle t'(s_0), L^{\pm}(u_1(s_0), u_2(s_0)) \rangle + 1$$

of $p = \gamma(u_1(s_0), u_2(s_0))$. The de Sitter normal curvature depends only on the point p and the unit tangent vector of M at p. Hyperbolic normal curvature of $\gamma(s)$ is defined to be

$$\bar{K}_n^{\pm}(s) = K_n^{\pm}(s) - 1.$$

The Hyperbolic Gauss curvature and mean curvature of M at $p = x(u_0)$ is given by

$$\overline{K_h^{\pm}}(u_0) = \det S_p^{\pm} = \overline{K_1^{\pm}}(p)\overline{K_2^{\pm}}(p)$$

and

$$H_h^{\pm}(u_0) = \frac{1}{2} Trace S_p^{\pm} = \frac{\overline{K_1^{\pm}}(p) + \overline{K_2^{\pm}}(p)}{2}.$$

And also the extrinsic (de Sitter) Gauss curvature and mean curvature of M at $p = x(u_0)$ is given by

$$K_e(u_0) = \det Ap = K_1(p)K_2(p)$$

and

$$H_d(u_0) = \frac{1}{2} TraceAp = \frac{K_1(p)+K_2(p)}{2}.$$

Let $x: M \to R_1^4$ be an immersion of a surface M into R_1^4. We say that x is timelike (resp. spacelike, lightlike) if the induced metric on M via x is Lorentzian (resp. Riemannian, degenerated). If $\langle x, x\rangle = 1$, then x is an immersion of S_1^3.

Let $x: M \to S_1^3$ be a spacelike immersion, and let ξ be a unit normal vector field to M . If there exists spacelike direction W such that timelike angle $\theta(\xi, U)$ is constant on M , then M is called constant timelike angle surfaces with spacelike axis.

Let $x: M \to S_1^3$ be a spacelike immersion and let ξ be a unit normal vector field to M. If there exists spacelike direction W such that spacelike angle $\theta(\xi, U)$ is constant on M, then M is called constant spacelike angle surfaces with spacelike axis.

From now on, the constant angle surface is proposed in de Sitter space S_1^3.

3. TIMELIKE SURFACE WITH CONSTANT TIMELIKE ANGLE

Let call $\chi(M)$ is tangent vector fields space over M. Let write $\overline{\overline{D}}, \overline{D}$ and D are Levi-Civita connections of R_1^4, S_1^3 and M respectively. For any $X, Y \in \chi(M)$, we have

$$D_X Y = (\overline{\overline{D}}_X Y)^T \; D_X Y = (\overline{\overline{D}}_X Y)^T, \; \tilde{V}(X,Y) = (\overline{\overline{D}}_X Y)^{\perp}$$

here $\tilde{V}$ is second fundamental form of M over R_1^4 and

$$\overline{\overline{D}}_X Y = \overline{D}_X Y - \langle X, Y\rangle x, \overline{\overline{D}}_X Y = D_X Y + \tilde{V}(X,Y) \tag{3.1}$$

where the superscript T and $\perp$ are the tangent and normal component of $\overline{\overline{D}}_X Y$. Equations in (3.1) are called the Gauss formula of M on S_1^3. If ξ is a normal vector field of M over S_1^3, then $A_\xi(X)$ and $B_x(X)$ Weingarten Endomorphism are defined by the tangent components of $-\overline{\overline{D}}_X \xi$ and -$-\overline{\overline{D}}_X x$. So the Weingarten equations of the vector field ξ and x will be as follows

$$\begin{cases} A_\xi(X) = -\overline{\overline{D}}_X \xi - \langle \overline{\overline{D}}_X x, \xi \rangle x \\ B_x(X) = -\overline{\overline{D}}_X x + \langle \overline{\overline{D}}_X x, \xi \rangle \xi \end{cases} \tag{3.2}$$

It is clear that $A_\xi(X)$ and $B_x(X)$ operators for each $p \in M$ are both linear and self adjoint operators. That is

$$\left\langle A_{\xi}(X),Y\right\rangle=\left\langle X,A_{\xi}(Y)\right\rangle \text{ and } \left\langle B_{x}(X),Y\right\rangle=\left\langle X,B_{x}(Y)\right\rangle.$$

Let called eigenvalues $K_i(P)$ of $\left(A_{\xi}\right)_p$ over S_1^3 and eigenvalues $\tilde{K}_i(P)$ of $\left(B_x\right)_p$ over R_1^4 are principal curvetures and also, for any $X,Y\in\chi(M)$ we have

$$\left\langle A_{\xi}(X),Y\right\rangle=\left\langle \widetilde{V}(X,Y),\xi\right\rangle,\ \left\langle B_{x}(X),Y\right\rangle=\left\langle \widetilde{V}(X,Y),x\right\rangle.$$

Since $\tilde{V}(X,Y)$ is second fundamental form of M over IR_1^4, so we can write $\tilde{V}(X,Y)$ as follows

$$\widetilde{V}(X,Y)=\left\langle A_{\xi}(X),Y\right\rangle\xi+\left\langle B_{x}(X),Y\right\rangle x.$$

Let call $\{v_1,v_2\}$ is a base of TpM tangent plane and let us denote

$$a_{ij}=\left\langle \widetilde{V}(v_i,v_j),\xi\right\rangle=\left\langle A_{\xi}(v_i),v_j\right\rangle \tag{3.3}$$

$$b_{ij}=\left\langle \widetilde{V}(v_i,v_j),x\right\rangle=\left\langle B_{x}(v_i),v_j\right\rangle \tag{3.4}$$

So $\overline{\overline{D}}_X Y=D_X Y+\tilde{V}(X,Y)$. On the other hand for $\{v_1,v_2\}$ base , we get

$$\overline{\overline{D}}_{v_i}v_j=D_{v_i}v_j-\left\langle A_{\xi}(v_i),v_j\right\rangle\xi-\left\langle v_i,v_j\right\rangle x \tag{3.5}$$

If $\{v_1,v_2\}$ is orthonormal base, then we have from (3.1) and (3.2)

$$\overline{\overline{D}}_{v_i}v_j=D_{v_i}v_j-a_{ij}\xi \tag{3.6}$$

and also we get

$$\overline{\overline{D}}_{v_i}\xi=-a_{i1}v_1-a_{i2}v_2 \tag{3.7}$$

$$\overline{\overline{D}}_{v_i}x=b_{i1}v_1-b_{i2}v_2 \tag{3.8}$$

3.1 Constant Timelike Angle Surfaces With Spacelike Axis

Definition 1 Let $U\subset IR^2$ be open set and let $x:U\to S_1^3$ be an embedding where $M=x(U)$. Let $x:M\to S_1^3$ and ξ is spacelike unit normal vector field on M, if there exist a constant spacelike vector W which has a constant timelike angle with ξ, then M is called **constant timelike angle surface with spacelike axis.**

Since our surface is timelike, the orthogonal base of tangent space TpM has a timelike tangent vector. Let M be constant timelike angle surface with spacelike axis, and let ξ and W be unit normal and fixed axis of M. If θ is an timelike angle between spacelike vectors ξ and W then

$$\langle \xi, W \rangle = -\cosh\theta .$$

If $\theta = 0$, then $\xi = W$. From now on the rest of the paper, without loss of generality we assume that θ. If W^T is the projection of W on the tangent plane of M, then we decompose W as

$$W = W^T + W^N$$

So that we write

$$W = W^T + \lambda_1 \xi + \lambda_2 x .$$

If we take inner product of both sides of this equality first with ξ, then with x

$$\lambda_1 = -\cosh\theta, \lambda_2 = \langle W, x \rangle .$$

On the other hand since W and x are spacelike vector fields, then we can use define of spacelike and timelike angle between these vectors.

Theorem 2 If φ is spacelike angle between spacelike vectors W and x, then we can write from [11]

$$W = \sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\, e_1 - (\cosh\theta)\xi + (\cos\varphi)x$$

and de Sitter projection W_d of W as follows

$$W_d = \sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\, e_1 - (\cosh\theta)\xi \tag{3.9}$$

Remark 1 Let φ be timelike angle between spacelike vectors W and x, then we can write for [11]

$$\langle W, x \rangle = -\cosh\varphi$$

or

$$\lambda_2 = -\cosh\phi .$$

Therefore W can be written as follows

$$W = W^T - (\cosh\theta)\xi - (\cosh\varphi)x .$$

On the other hand, since

$$\left\|W^T\right\|^2 = -\sinh^2\theta - \cosh^2\varphi$$

then there is not any timelike angle between W and x.

Let $e_1 = \dfrac{W^T}{\|W^T\|}$, and let assume e_2 be a unit vector field on M orthogonal to e_1. Then we have an ortonormal basis $\{e_1, e_2, \xi, x\}$ in R_1^4 for each point of M. Since W_d is constant vector field on S_1^3, we have

$$\overline{\overline{D}}_{e_2} W_d = \overline{D}_{e_2} W_d = 0$$

hence we have

$$\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,\overline{\overline{D}}_{e_2} e_1 - (\cosh\theta)\overline{\overline{D}}_{e_2}\xi = 0 \tag{3.10}$$

By (3.10) ξ, we obtain

$$-\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,a_{21} = 0 .$$

Since $-\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|} \neq 0$, we conclude $a_{21} = a_{12} = 0$. Using (3.7) in (3.10), we find

$$\overline{\overline{D}}_{e_2} e_1 = \frac{-\cosh\theta}{\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}}\, a_{22} e_2 \tag{3.11}$$

Similarly, since W_d is a constant vector field on S_1^3, then we have

$$\overline{D}_{e_1} W_d = 0 \text{ and } \overline{\overline{D}}_{e_1} W_d = \sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,x \tag{3.12}$$

By (3.9), we see that

$$\overline{\overline{D}}_{e_1} W_d = \sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,\overline{\overline{D}}_{e_1} e_1 - (\cosh\theta)\overline{\overline{D}}_{e_1}\xi \tag{3.13}$$

By (3.12) and (3.13), we conclude that

$$\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,\overline{\overline{D}}_{e_1} e_1 - (\cosh\theta)\overline{\overline{D}}_{e_1}\xi = \sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,x \tag{3.14}$$

By (3.14), then we get

$$\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\left\langle \overline{\overline{D}}_{e_1} e_1, \xi \right\rangle = 0$$

or

$$-\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}\,a_{11} = 0$$

Since $-\sqrt{\left|\sin^2\varphi-\cosh^2\theta\right|}\neq 0$, we conclude $a_{11}=0$. Using (3.7) in (3.12) , we obtain

$$\overline{\overline{D}}_{e_1}e_1 = x \tag{3.15}$$

Hence, we have proved the following theorem.

Theorem 3 If D is Levi-Civita connection for a constant timelike angle surface M of S_1^3, then

$$D_{e_1}e_1 = 0 \qquad D_{e_2}e_1 = \frac{-\cosh\theta}{\sqrt{\left|\sin^2\varphi-\cosh^2\theta\right|}}a_{22}e_2$$

$$D_{e_1}e_2 = 0 \qquad D_{e_2}e_2 = \frac{-\cosh\theta}{\sqrt{\left|\sin^2\varphi-\cosh^2\theta\right|}}a_{22}e_1.$$

Corollary 1 Let M be a timelike surface with a constant timelike angle in S_1^3. Then, there exist local coordinates u,v such that the metric on M writes as $\langle,\rangle := -du^2+\beta^2dv^2$, where $\beta=\beta(u,v)$ is a smooth function on M, i.e. the coefficients of the first fundamental form are $E=-1, F=0, G=\beta^2$.

Let we find the $x=x(u,v)$ parametrization of the surface M with respect to the metric $\langle,\rangle := -du^2+\beta^2dv^2$ on M. By Theorem-1, one can obtain the following corollary.

Corollary 2 There exist an equation system for a timelike surface with a constant timelike angle in S_1^3 which is

$$\begin{cases} x_{uu} = x \\ x_{uv} = \dfrac{\beta_u}{\beta}x_v \\ x_{vv} = \beta\beta_u x_u + \dfrac{\beta_v}{\beta}x_v - \beta^2 a_{22}\xi - \beta^2 x \end{cases} \tag{3.16}$$

Corollary 3 Let ξ be unit normal vector of the a timelike surface with a constant timelike angle M. Then the equation below hold

$$\begin{cases} \xi_u = \overline{\overline{D}}_{x_u}\xi = 0 \\ \xi_v = \overline{\overline{D}}_{x_v}\xi = -a_{22}x_v \end{cases} \tag{3.17}$$

Since $\xi_{uv}=\xi_{vu}$, we have $\overline{\overline{D}}_{x_u}(-a_{22}x_v)=0$. Using $a_{22}=0$, $\overline{\overline{D}}_{x_u}x_v=\overline{\overline{D}}_{x_v}x_u$ and Theorem 1, we obtain

$$(a_{22})_u - \frac{\cosh\theta}{\sqrt{\left|\sin^2\varphi - \cosh^2\theta\right|}}(a_{22})^2 = 0 \tag{3.18}$$

or

$$(a_{22})_u + \frac{\beta_u}{\beta}(a_{22}) = 0 \tag{3.19}$$

Hence, we have

$$(\beta a_{22})_u = 0 \tag{3.20}$$

By (3.20),we see that there exist a smooth function $\psi = \psi(v)$ depending on v such that

$$\beta a_{22} = \psi(v) \tag{3.21}$$

Prposition 1 Let $x = x(u,v)$ be parametrization of a timelike surface with a constant timelike angle in S_1^3. If $a_{22} = 0$ on M, then the x describes an flat plane of de Sitter space S_1^3.

From now on, we are assume that $a_{22} \neq 0$. By solving equation (3.16), we obtain a function $\alpha = \alpha(v)$ such that

$$a_{22} = \frac{-\sqrt{\sin^2\varphi - \cosh^2\theta}}{u\cosh\theta + \alpha(v)}, \alpha(v) = \sqrt{\sin^2\varphi - \cosh^2\theta}\,\bar{\alpha}(v).$$

Therefore by (3.21) ,we obtain

$$\beta(u,v) = \frac{-\psi(v)}{\sqrt{\sin^2\varphi - \cosh^2\theta}}(u\cosh\theta + \alpha(v)).$$

If we choose $\psi(v) = -v\sqrt{\sin^2\varphi - \cosh^2\theta}$ and $\alpha(v) = \ln v$, then we have the following theorem.

Theorem 4 If M is satisfy (3.22), then there exist local coordinates u,v on M with having the parametrization

$$x_i(u,v) = \frac{-d_{1i}(v)}{2\cosh\theta(u\cosh\theta + \ln v)^2} + d_{2i}(v),\ i = 1,2,3,4 \tag{3.23}$$

Proof From (3.22), the proof is clear.

REFERENCES

[1]. R. Lopez, M.I. Munteanu, Constant angle surfaces in Minkowski space, Bulletin of the Belgian Math. So. Simon Stevin, Vo.18 (2011) 2, (271-286).

[2]. S.Izumıya, K.Sajı, M.Takahashı, Horospherical flat surfaces in Hyperbolic 3-space, J.Math.Soc.Japan, Vol.87 (2010), (789-849).

[3]. S.Izumıya, D.Peı, M.D.C.R. Fuster, The horospherical geometry of surfaces ın hyperbolic 4-spaces, Israel Journal of Mathematıcs, Vol.154 (2006), (361-379).

[4]. C.Thas, A gauss map on hypersurfaces of submanıfolds ın Euclıdean spaces, J.Korean Math.Soc., Vol.16, No.1, (1979).

[5]. S.Izumıya, D.Peı, T.Sano, Sıngularities of hyperbolic gauss map, London Math.Soc. Vol.3 (2003), (485-512).

[6]. M.I.Munteanu, A.I.Nistor, A new approach on constant angle surfaces in E^3 , Turk T.Math. Vol.33 (2009), (169-178).

[7]. C.Takızawa, K.Tsukada , Horocyclic surfaces in hyperbolic 3-space, Kyushu J.Math. Vol.63 (2009), (269-284).

[8]. S.Izumıya, M.D.C.R. Fuster, The horospherical Gauss-Bonnet type theorem in hyperbolic space, J.Math.Soc.Japan, Vol.58 (2006), (965-984).

[9]. B. O'Neill, Semi-Riemannıan Geometry with applications to relativity, Academic Press, New York, 1983.

[10].W.Fenchel , Elementary Geometry in Hyperbolic Space, Walter de Gruyter , New York , 1989.

[11].J.G.Ratcliffe, Foundations of Hyperbolic Manifolds, Springer 1948.

[12].P.Cermelli, A.J. Di Scala, Constant angle surfaces in liquid crystals, Phylos. Magazine,Vol.87 (2007),(1871-1888).

[13].A.J. Di Scala, G. Ruiz-Hernandez, Helix submanifolds of Euclidean space, Monatsh. Math. DOI 10.1007 / s00605-008-0031-9.

[14].G.Ruiz-Hernandez, Helix, shadow boundary and minimal submanifolds, Illinois J. Math. Vol.52 (2008),(1385-1397).

[15].F. Dillen, J. Fastenakels, J. Van der Veken, L. Vrancken, Constant angle surfaces in $S^2 \times R$, Monaths. Math. , Vol.152 (2007), (89-96).

[16].F. Dillen and M. I. Munteanu, Constant angle surfaces in $H^2 \times R$, Bull. Braz. Math. soc. , Vol.40 (2009), (85-97).

[17].J. Fastenakels, M. I: Munteanu , J. Van der Veken, Constant angle surfaces in the Heisenberg group, Acta Math. Sinica (English Series) Vol.27 (2011), (747-756).

[18].R.Lopez, Differantial Geometry of Curves and Surfaces in Lorentz-Minkowski space, arXiv: 0810.3351 (2008).

[19].Mert.T, Karliğa.B, Constant Angle Spacelike Surface in Hyperbolic Space H^3, J. Adv. Res. Appl. Math., 7(2015), no 2, (89-102).

[20].Mert.T, Karliğa.B, Constant Angle Spacelike Surface in de Sitter Space S_1^3, Boletim da Sociedade Paranaense de Mathematica (Accepted).

[21].Mert.T, Karliğa.B, On The Timelike Surface with Constant Angle in Hyperbolic Space H^3, CBU Journal of Science (Accepted).

[22].M. Kasedou , Spacelike submanifolds in de sitter space, demonstratio Mathematica , Vol.XLIII (2010) , 2.