Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt 37, No. 3 (2016)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 37, No. 3 (2016)
ISSN: 1300-1949

http://dx.doi.org/10.17776/csj.03831

On The Fine Spectrum of Generalized Lower Triangular Double Band Matrıces Δ_{uv} Over The Sequence Space c_0

Nuh DURNA*, Mustafa YILDIRIM, Çağrı ÜNAL

Cumhuriyet University, Faculty of Science, Departman of Mathematic, Sivas, Turkey,

Received: 09.08.2016; Accepted: 30.09.2016

Abstract. The generalized difference operator Δ_{uv} has been defined by Fathi and Lashkaripour: $\Delta_{uv}x = \Delta_{uv}(x_n) = (v_n x_n + u_{n-2}x_{n-1})_{n=0}^{\infty}$ with $u_{-2} = u_{-1} = u_0 = 0$, (u_k), (v_k) are convergent sequences of nonzero real numbers satisfying certain conditions. The purpose of this paper is to completely determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $\Delta^{uv} = (\Delta_{uv})^t$ on the sequence space ℓ_1. Also, with this division, the fine spectrum of generalized lower triangular double band matrices Δ_{uv} which is given by Fathi and Lashkaripour in 2012, has been proved more shortly and different method on c_0 space.

Keywords: *Generalized difference operator, approximate point spectrum, defect spectrum, compression spectrum*.

Genelleştirilmiş Altüçgensel Çift Bant Matrisi Δ_{uv} nin c_0 Dizi Uzayı Üzerinde İnce Spektrumu

Özet. Genelleştirilmiş fark operatörü Δ_{uv} Fathi ve Lashkaripour tarafından tanımlanmıştır: (u_k), (v_k) bazı koşulları sağlayan sıfırdan farklı reel sayıların yakınsak dizileri olmak üzere $\Delta_{uv}x = \Delta_{uv}(x_n) = (v_n x_n + u_{n-1}x_{n-1})_{n=0}^{\infty}$ biçiminde tanımlanır. Burada $x_{-1} = u_{-1} = 0$ dır. Bu makalenin amacı ℓ_1 dizi uzayı üzerinde $\Delta^{uv} = (\Delta_{uv})^t$ operatörünün yaklaşık nokta sepktrumunu, eksik spektrumunu ve sıkıştırılmış spektrumunu tam olarak belirlemektir. Ayrıca bu ayrışım yardımıyla 2012 de Fathi ve Lashkaripour tarafından verilen Δ_{uv} genelleştirilmiş altüçgensel çift bant matrisinin ince spektrumu c_0 uzayı üzerinde daha farklı ve kısa bir yolla hesaplanmıştır.

Anahtar Kelimeler: *Genelleştirilmiş fark operatörü, yaklaşık nokta spektum, eksik spektrum, sıkıştırılmış spektrum.*

1. INTRODUCTION

Spectral theory is an important part of functional analysis. It has numerous applications in many parts of mathematics and physics including matrix theory, function theory, complex analysis, differential and integral equations, control theory and quantum physics. For example, in quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star.

In recent years, spectral theory has witnessed an explosive development. There are many types of spectra, both for one or several commuting operators, with important applications, for example the approximate point spectrum, Taylor spectrum, local spectrum, essential spectrum, etc.

* Corresponding author. *Email address:* ndurna@cumhuriyet.edu.tr

1.1 The Spectrum

By w, we shall denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. we shall write $\ell_\infty,\ c,\ c_0$ and bv for the space of all bounded, convergent, null and bounded variation sequences, respectively. Also by $\ell_1,\ \ell_p,\ bv_p$ we denote the spaces of all absolutely summable sequences, p-absolutely summable sequences and p-bounded variation sequences, respectively.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We summarize the knowledge in the existing literature concerned with the spectrum and the fine spectrum. The fine spectrum of the Cesáro operator on the sequence space ℓ_p for $1 < p < \infty$ has been studied by Gonzalez [14]. Also, Wenger [22] examined the fine spectrum of the integer power of the Cesáro operator over c, and Rhoades [19] generalized this result to the weighted mean methods. Reade [18] worked the spectrum of the Cesáro operator over the sequence space c_0. The spectrum of the Rhaly operators on the sequence spaces c_0 and c is studied by Yildirim [20] and the fine spectrum of the Rhaly operators on the sequence space c_0 is studied by Yıldırım [21].

Quite recently, several authors have investigated spectral divisions of generalized differance matrices. For example, Akhmedov and El-Shabrawy, [1, 2] have studied the spectrum and fine spectrum of the generalized lower triangle double-band matrix Δ_v over the sequence spaces $c,\ c_0$ and ℓ_p, where $1 < p < \infty$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is investigated by Kayaduman and Furkan [21] and c_0 and c, is investigated by Altay and Başar [3]. Karaisa et al., [15] have studied the fine spectra of triangular triple-band matrices on sequence spaces c and ℓ_p $(0 < p < 1)$. Also, Karakaya et al. [16] examined the fine spectrum of the second order difference operator over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$ etc.

Let X and Y be the Banach spaceces, and $L : X \to Y$ also be a bounded linear operator. By $R(L)$, we denote the range of L, i.e.,

$$R(L) = \{ y \in Y : y = Lx,\ x \in X \}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach spaces and $L \in B(X)$ then the adjoint L^* of L is a bounded linear operator on the dual X^* of X defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$.

Let $L : D(L) \to X$ be a linear operator, defined on $D(L) \subset X$, where $D(L)$ denote the domain of L and X is a complex normed linear space. For $L \in B(X)$ we associate a complex number λ with the operator $(\lambda I - L)$ denoted by L_λ defined on the same domain $D(L)$, where I is the identity operator. The inverse $(\lambda I - L)^{-1}$, denoted by L_λ^{-1} is known as the resolvent operator of L_λ.

A regular value of L is a complex number λ of L such that L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X.

The resolvent set of L is the set of all such regular values a of L, denoted by $\rho(L, X)$. Its complement is given by $\mathbb{C} - \rho(L,X)$ in the complex plane $\mathbb{C}$ is called the spectrum of L, denoted by $\sigma(L, X)$. Thus the spectrum $\sigma(L, X)$ consist of those values of $\lambda \in \mathbb{C}$, for which L_λ is not invertible.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The point (discrete) spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigen value of L. We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $\lambda I - L$) is not dense in X; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two sub spectra form a disjoint subdivision

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1.1}$$

of the spectrum of L.

1.2 Goldberg's Classification of Spectrum

Let X be a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$,

(B) $\overline{R(T)} = X$, but $R(T) \neq X$,

(C) $\overline{R(T)} \neq X$.

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous,

(2) T^{-1} exists but discontinuous,

(3) T^{-1} does not exist

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exist but is discontinuous (see [13]).

If λ is a complex number such that $T = \lambda I - L \in A_1$ or $T = \lambda I - L \in B_1$ then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L. The further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L. That is, $\sigma(L, X)$ can be divided into the subsets $A_2\sigma(L, X) = \emptyset, A_3\sigma(L, X), B_2\sigma(L, X), B_3\sigma(L, X), C_1\sigma(L, X), C_2\sigma(L, X), C_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, C_2 (say), then we write $\lambda \in C_2\sigma(L, X)$.

Also the spectrum $\sigma(L, X)$ is partitioned into three sets which are not necessarily disjoint as follows:

1.3 Aproximate Point Spectrum, Defect Spectrum and Compression Spectrum.

The above-mentioned articles, concerned with the decomposition of spectrum which defined by Goldberg. However, in [4] Amirov et al. have investigated subdivision of the spectra for Cesáro, Rhaly and weighted mean operators on c, c_0 and ℓ_p and in [7] Başar et al. have investigated subdivisions of the spectra for genarilized difference operator over certain sequence spaces.

Let X be a Banach space, and $L \in B(X)$. We call a sequence (x_k) in X a Weyl sequence for L if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$.

We call the set

$$\sigma_{ap}(L, X) := \{\lambda \in \mathbb{C} : there\ is\ a\ Weyl\ sequence\ for\ \lambda I - L\}, \quad (1.2)$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\sigma_\delta(L, X) := \{\lambda \in \sigma(L, X) : \lambda I - L \text{ is not surjective}\} \quad (1.3)$$

is called defect spectrum of L. There is another subspectrum,

$$\sigma_{co}(L, X) := \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\} \quad (1.4)$$

which is often called compression spectrum in the literature. Clearly, $\sigma_p(L) \subseteq \sigma_{ap}(L)$ and $\sigma_{co}(L) \subseteq \sigma_\delta(L)$. Moreover, comparing these subspectra with those in (1.1) we note that

$$\sigma_r(L) = \sigma_{co}(L) \setminus \sigma_p(L) \tag{1.5}$$

and

$$\sigma_c(L) = \sigma(L) \setminus \left[\sigma_p(L) \cup \sigma_{co}(L)\right] \tag{1.6}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1 *([5], Proposition 1.3). The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(L^*, X^*) = \sigma(L, X)$,

(b) $\sigma_c(L^*, X^*) \subseteq \sigma_{ap}(L, X)$,

(c) $\sigma_{ap}(L^*, X^*) = \sigma_\delta(L, X)$,

(d) $\sigma_\delta(L^*, X^*) = \sigma_{ap}(L, X)$,

(e) $\sigma_p(L^*, X^*) = \sigma_{co}(L, X)$,

(f) $\sigma_{co}(L^*, X^*) \supseteq \sigma_p(L, X)$,

(g) $\sigma(L, X) = \sigma_{ap}(L, X) \cup \sigma_p(L^*, X^*) = \sigma_p(L, X) \cup \sigma_{ap}(L^*, X^*)$.

By the definitions given above, in [8], Amirov et al. have written following table:

Table 1.

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
A	$R(\lambda I - L) = X$	$\lambda \in \rho(L)$	–	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$
B	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L)$	$\lambda \in \sigma_c(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$
C	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$

2. THE FINE SPECTRUM OF THE OPERATOR Δ_{uv} ON c_0

The generalized upper triangular double band matrices Δ^{uv} has been defined by Fathi and Lashkaripour [12]. Let (u_k) be a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $u = \lim_{k\to\infty} u_k \neq 0$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k < u + v$.

In [12] Fathi and Lashkaripour define the operator Δ^{uv} on sequence space ℓ_1 as follows:

$$\Delta^{uv} x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1} x_{n+1})_{n=0}^{\infty}.$$

It is easy to verify that the operator Δ^{uv} can be represented by a upper triangular double band matrix of the form

$$\Delta^{uv}=\begin{pmatrix} v_0 & u_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2.1 Subdivision of the spectrum of Δ^{uv} on ℓ_1.

If $T:\ell_1\to\ell_1$ is a bounded linear operator with matrix A, then the adjoint operator $T^*:\ell_1^*\to\ell_1^*$ is defined by the transpoze of the matrix A. The dual space of ℓ_1 is isomorphic to ℓ_∞, the space of all bounded sequences, with the norm $\|x\|=\sup_k|x_k|$.

The spectra and the fine spectra of the operator Δ^{uv} over the sequence space ℓ_1 has been studied by Fathi and Lashkaripour [12]. In this subsection we summarize the main results.

Theorem 1 *([12], Theorem 2.2).* $\sigma_p\left(\left(\Delta^{uv}\right)^*,\ell_1^*\right)=\varnothing.$

Theorem 2 *([12], Corollary 2.4).* $\sigma_r\left(\Delta^{uv},\ell_1\right)=\varnothing.$

In [12] the authors define the operator Δ_{uv} on sequence space c_0 follows:

$$\Delta_{uv}x=\Delta_{uv}\left(x_n\right)=\left(u_{n-2}x_{n-1}+v_nx_n\right)_{n=0}^{\infty} \text{ with } u_{-2}=u_{-1}=u_0=0.$$

It is easy to verify that the operator Δ_{uv} can be represented by a lower triangular double band matrix of the form

$$\Delta_{uv}=\begin{pmatrix} v_0 & 0 & 0 & 0 & \cdots \\ u_1 & v_1 & 0 & 0 & \cdots \\ 0 & u_2 & v_2 & 0 & \cdots \\ 0 & 0 & u_3 & v_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that, if v_k and u_k is a constant sequence, say $v_k=r\neq 0$ and $u_k=s\neq 0$ for all $k\in\mathbb{N}$, then the operator Δ_{uv} is reduced to the operator $B(r,s)$ and the results for the subdivisions of the spectra for generalized difference operator Δ_{uv} over c_0, c, ℓ_p and bv_p have been studied in [7].

Theorem 3 *([12], Theorem 2.5).* $\sigma_p(\Delta_{uv}, c_0) = \varnothing$.

If $T : c_0 \to c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpoze of the matrix A. It is well known that the dual space c_0^* of c is isomorphic to ℓ_1.

Theorem 4 *([12], Theorem 2.6).* $\sigma_r(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$.

Theorem 5 *([12], Theorem 2.7).* $\sigma_p(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$.

Theorem 6 *([12], Theorem 2.8).* $\sigma(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| \le u\}$.

Theorem 7 *([12], Theorem 2.9).* $\sigma_c(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$.

Theorem 8 *([12], Theorem 2.10).* $A_3\sigma(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$.

Corollary 1. $B_3\sigma(\Delta^{uv}, \ell_1) = C_3\sigma(\Delta^{uv}, \ell_1) = \varnothing$.

Proof. Since $\sigma_p(\Delta^{uv}, \ell_1) = A_3\sigma(\Delta^{uv}, \ell_1) \cup B_3\sigma(\Delta^{uv}, \ell_1) \cup C_3\sigma(\Delta^{uv}, \ell_1)$ from Table 1, by using Theorem 5 and Theorem 8, we get the required result.

Corollary 2. $C_1\sigma(\Delta^{uv}, \ell_1) = C_2\sigma(\Delta^{uv}, \ell_1) = \varnothing$.

Proof. Since $\sigma_r(\Delta^{uv}, \ell_1) = C_1\sigma(\Delta^{uv}, \ell_1) \cup C_2\sigma(\Delta^{uv}, \ell_1)$ from Table 1, by using Theorem 4, we get the required result.

Theorem 9 ***(a)*** $\sigma_{ap}(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| \le u\}$,

(b) $\sigma_\delta(\Delta^{uv}, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$.

(b) $\sigma_{co}(\Delta^{uv}, \ell_1) = \varnothing$.

Proof **(a)** Since $\sigma_{ap}(\Delta^{uv}, \ell_1) = \sigma(\Delta^{uv}, \ell_1) - C_1\sigma(\Delta^{uv}, \ell_1)$ from Table 1, by using Theorem 6 and Corollary 2, we get the required result.

(b) Since $\sigma_\delta(\Delta^{uv}, \ell_1) = \sigma(\Delta^{uv}, \ell_1) - A_3\sigma(\Delta^{uv}, \ell_1)$ from Table 1, by using Theorem 6 and Theorem 8, we get the required result.

(c) Since $\sigma_{co}(\Delta^{uv}, \ell_1) = C_1\sigma(\Delta^{uv}, \ell_1) \cup C_2\sigma(\Delta^{uv}, \ell_1) \cup C_3\sigma(\Delta^{uv}, \ell_1)$ from Table 1, by using Corollary 1 and Corollary 2, we get the required result.

Theorem 10. $\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| \le u\}$.

Proof The proof is clear from Proposition 1 (a) and Theorem 6.

Theorem 11 ***(a)*** $\sigma_{ap}(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$,

(b) $\sigma_\delta(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| \le u\}$,

Proof **(a)** Since $\sigma_{ap}\left(\Delta_{uv}, c_0\right) = \sigma_{\delta}\left(\Delta_{uv}^{*}, c_0^{*}\right) = \sigma_{\delta}\left(\Delta^{uv}, \ell_1\right)$ from Proposition 1 (d), by using Theorem 9 (b), we get the required result.

(b) Since $\sigma_{\delta}\left(\Delta_{uv}, c_0\right) = \sigma_{ap}\left(\Delta_{uv}^{*}, c_0^{*}\right) = \sigma_{\delta}\left(\Delta^{uv}, \ell_1\right)$ from Proposition 1 (c), by using Theorem 9 (a), we get the required result.

Theorem 12. $C_1\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$.

Proof. Since $\sigma_{ap}\left(\Delta_{uv}, c_0\right) = \sigma\left(\Delta_{uv}, c_0\right) - C_1\sigma\left(\Delta_{uv}, c_0\right)$ from Table 1, by using Theorem 10 and Theorem 11 (b), we get the required result.

Theorem 13. $B_2\sigma(\Delta_{uv}, c_0) \cup C_2\sigma(\Delta_{uv}, C_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$.

Proof. Since $B_2\sigma\left(\Delta_{uv}, c_0\right) \cup C_2\sigma\left(\Delta_{uv}, c_0\right) = \sigma_{\delta}\left(\Delta_{uv}, c_0\right) - \left\{\sigma_p\left(\Delta_{uv}, c_0\right) \cup C_1\sigma\left(\Delta_{uv}, c_0\right)\right\}$ from Table 1, by using Theorem 3, Theorem 11 (b) and Theorem 12, we get the required result.

Lemma 1 *([13], Theorem II 3.7). A linear operator T has a dense range if and only if the adjoint operator T^* is one to one.*

Theorem 14. $C_2\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$.

Proof. From Lemma 1 $\overline{R\left(\left(\Delta_{uv}\right)_{\lambda}\right)} = c_0$ if and only if $\left(\Delta_{uv}\right)_{\lambda}^{*} = \left(\Delta^{uv}\right)_{\lambda}$ is one to one. From Theorem 5, if $|\lambda - v| \geq u$ then $\left(\Delta^{uv}\right)_{\lambda}$ is one to one. Thus if $||\lambda - v| = u$ then $\lambda \in C$ from Lemma 1. Hence we get $C_2\sigma(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| = u\}$ from Theorem 13.

Corollary 3. $\sigma_c\left(\Delta_{uv}, c_0\right) = B_2\sigma\left(\Delta_{uv}, c_0\right) = \varnothing$.

Proof. Since $\sigma_c\left(\Delta_{uv}, c_0\right) = B_2\sigma\left(\Delta_{uv}, c_0\right)$ from Table 1, by using Theorem 13 and Theorem 14, we get the required result.

Remark 1. *In here, the proof of Theorem 12-Theorem 14 and Corollary 3 which have been forementioned [11], give in different ways and the shorter than proof of the Theorems in [11].*

3. CONCLUSION

Many researchers have determined the spectrum and the fine spectrum of a matrix operator in some sequence spaces. Although the fine spectrum with respect to the Goldberg's classification of the generalized difference operator Δ_{uv} over the sequence space c_0 were studied by Fathi and Lashkaripour [11], in the present paper, the concepts of the approximate point spectrum, defect spectrum and compression spectrum are introduced, and given the subdivisions of the spectrum of the the generalized difference operator Δ_{uv} over the sequence space c_0, as the new subdivisions of spectrum. It is immediate that our new results cover a wider class of linear operators which are represented by infinite lower triangular double-band matrices on the sequence space c_0. For this reason, our study is more general and more comprehensive than the previous work. We note that our new results in this paper improve and generalize the results which have been stated in [6,7,8].

REFERENCES

1. Akhmedov, A.M, El-Shanrawy, S.R: The spectrum of the generalized lower triangle double-band matrix Δ_a over the sequence space c. *Al-Azhar Univ. Eng. J., JAUES (speacial issue)*, 5(9), 54-60 (2010).
2. Akhmedov, A.M, El-Shanrawy, S.R: On the fine spectrum of theoperator Δ_v over the sequence space c and ℓ_p, $(1 < p < \infty)$, *Appl. Math. Inf. Sci.*, 5(3), 635-654 (2011).
3. Altay, B, Başar, F: On the fine spectrum of the difference operator on c_0 and c. *Inform. Sci.* 168, 217-224 (2004).
4. Amirov, R. Kh, Durna, N, Yildirim, M: Subdivision of the spectra for Cesaro, Rhaly and weighted mean operators on c_0, c and ℓ_p. *IJST*, A3, 175-183 (2011).
5. Appell, J, Pascale, E.D, Vignoli, A: Nonlinear Spectral Theory. *Walter de Gruyter · Berlin · New York*, (2004).
6. Başar, F, Durna, N, Yildirim, M: Subdivisions of the spectra for genarilized difference operator Δ_v on the sequence space ℓ_1. *CP1309, ICMS, International Conference on Mathematical Science*, 254-260 (2010).
7. Başar, F, Durna, N, Yildirim, M: Subdivisions of the spectra for genarilized difference operator over certain sequence spaces. *Thai J. Math.*, 9 (2), 285--295 (2011).
8. Başar, F, Durna, N, Yildirim, M: Subdivision of the spectra for difference operator over certain sequence spaces. *Malaysian J. Math. Sci*., 6 (S), 151--165 (2012).
9. El-Shabrawy, S.R: On the fine spectrum of the generalized difference operator Δ_{ab} over the sequence space ℓ_p, $(1 < p < \infty)$. *Appl. Math. Inf. Sci.*, 6(1), 111-118 (2012).
10. El-Shabrawy, S.R: Spectra and fine spectra of certain lower triangular double band matrices as operators on c_0. *J. Ineq. Appl*., 241(1), 1-9 (2014).
11. Fathi, J, Lashkaripour, R: On the fine spectra of the generalized difference operator Δ_{uv} over the sequence space c_0. *J. Mahani Math. Res. Cent.*, 1(1), 1-12 (2012).
12. Fathi, J, Lashkaripour, R: On the fine spectrum of the generalized upper double band matrices Δ^{uv} over the sequence space ℓ_1. *Math. Vesnik*, 65(1), 64-73 (2013).
13. Goldberg, S: Unbounded Linear Operators. *McGraw Hill, New York*, (1966).
14. Gonzalez, M: The fine spectrum of the Cesaro operator in ℓ_p, $(1 < p < \infty)$. *Arch. Math.*, 44, 355-358 (1985).

15. Karaisa, A, Asar, Y, Tollu, D.T: Fine spectra of triangular triple-band matrices on sequence spaces c and ℓ_p, $(0 < p < 1)$. *Math. Commun*, 20, 1-19 (2015).
16. Karakaya V, Manafov, M, Şimşek N: On the fine spectrum of the second order difference operator over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$. *arXiv:1105.4839v1, http://arxiv.org/abs/1105.4839v1*.
17. Kayaduman, K, Furkan, H: On the fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv. *Int. Math. Forum*, 24(1), 1153-1160 (2006).
18. Reade, J.B: On the spectrum of the Cesaro operator. *Bull. Lond. Math. Soc.*, 17, 263-267 (1985).
19. Rhoades BE: The fine spectra for weighted mean operators. *Pacific J. Math*., 104, 263-267 (1983).
20. Yildirim, M: On the spectrum of the Rhaly operators on c_0 and c. *Indian J. Pure Appl. Math*., 29, 1301-1309 (1998).
21. Yildirim, M: The Fine Spectra of the Rhaly Operators on c_0. *Turkish J. Math*., 26(3), 273-282 (2002).
22. Wenger, R.B: The fine spectra of Hölder summability operators. *Indian J. Pure Appl. Math*., 6, 695-712 (1975).