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Research Article / Araştırma Makalesi

ON VALUE GROUPS AND RESIDUE FIELDS OF VALUED FUNCTION FIELDS

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Abstract: In this paper studying on value groups and residue fields of valued rational function fields and valued function fields of conics is purposed. Let F be a function field over K ; v be a valuation on K ; w be an extension of v to F ; k_w , k_v and G_w , G_v be residue fields and value groups of w and v respectively. If F is rational function field over K then either k_w / k_v is an algebraic extension or k_w is a simple transcendental extension of any finite extension of k_v . If F is a function field of conic over K and $\mathrm{char}k_v \neq 2$ then either k_w / k_v is an algebraic extension or k_w is a regular function field of conics over any finite extension of k_v . In the both case either G_w / G_v is a torsion group or there exists a subgroup G_1 of G_w such that G_1 / G_v is a torsion group and G_w is the direct sum of G_1 and an infinite cyclic group.

Key words: Conics, extension of valuations, value group, valued function fields, residue field.

Değerlenmiş Fonksiyon Cisimlerinin Rezidü Cisimleri Ve Değer Grupları Hakkında

Özet: Bu çalışmada değerlenmiş rasyonel fonksiyon cisimlerinin ve değerlenmiş konik fonksiyon cisimlerinin değer gruplarının ve rezidü cisimlerinin incelenmesi amaçlanmıştır. F , K cismi üzerinde bir fonksiyon cismi; v , K cismi üzerinde bir değerlendirme; w , v nin F cismine bir genişlemesi; G_w , G_v ve k_w , k_v sırasıyla w ve v nin değer grupları ve rezidü cisimleri olsun. Eğer F , K cismi üzerinde bir rasyonel fonksiyon cismi ise k_w / k_v ya bir cebirsel genişlemedir ya da k_w , k_v nin bir sonlu genişlemesinin bir basit transandant genişlemesidir. Eğer F , K cismi üzerinde bir konik fonksiyon cismi ise k_w / k_v ya bir cebirsel genişlemedir ya da k_w , k_v nin bir sonlu genişlemesi üzerinde bir regüler konik fonksiyon cismidir. Her iki durumda da G_w / G_v ya bir torsion gruptur ya da G_1 / G_v bir torsion grup ve G_w , G_1 ile sonsuz devirli bir grubun direkt toplamı olacak şekilde G_w nın bir G_1 altgrubu vardır.

Anahtar kelimeler: Değer grubu, değerlendirmelerin genişlemeleri, değerlenmiş fonksiyon cisimleri, konikler, rezidü cismi.

Introduction

Let K be a field, v be a valuation on K . The old and important problem is finding all extensions of v to $K(x_1, x_2, \ldots, x_n)$ where $x_1, x_2, \ldots, x_n$ are indeterminates. This problem is solved completely for only $K(x)$. The other problem is describing residue fields and residue fields of $K(x_1, x_2, \ldots, x_n)$ for the valuation which is extension of v . In this paper there are some results on this problem.

Let $K(x)$ be a rational function field over K with valuation w which is extension of the valuation v . Let k_v and k_w be residue fields; G_v and G_w be value grups of v and w respectively. k_w / k_v is either an algebraic extension or k_w a simple transcendental extension of any algebraic extension of k_v. G_w/G_v is either torsion group or there exists a subgroup G_1 of G_w such that $G_v \subseteq G_1$, $[G_1 : G_v] < \infty$ and G_w is direct sum of G_1 and an infinite cyclic group.

Let F be a function field of a conic over a subfield K, v be a valuation on K with residue field k_v of characteristic $\neq 2$ and w be an extension of v to F having residue field k_w, G_v and G_w be value grups of v and w respectively. Either k_w is an algebraic extension of k_v or k_w is a regular function field of a conic over a finite extension of k_v. Either G_w/G_v is a torsion group or there exists a subgroup G_1 of G_w containing G_v with $[G_1 : G_v] < \infty$ together with an element γ of G_w such that G_w is the direct sum of G_1 and the cyclic group $Z\gamma$.

This facts are proved by J. Ohm, S.K. Khanduja and U. Garg in 1988,1991,1993,1994.

Preliminaries:

Let F/K be a finitely generated field extension. F/K is said to be a function field of a conic over K if the transcendence degree of F/K is one and if $F = K(x,y)$ where x and y satisfy an irreducible polynomial relation total degree 2 over K.

F/K is said to be a regular function field of a conic over K if

i) F/K is seperable extension i.e. either x is seperably algebraic over $K(y)$ or y is seperably algebraic over $K(x)$

ii) K is algebraic closed in F.

Throughout the paper if K is a field and v is a valuation on K then G_v and k_v will denote the value group and residue field of v respectively. For any η in the valuation ring of v, η^* will denote its $v-$residue i.e. the image of η in k_v.

If k_v' algebraic closure of k_v in k_w, we shall denote by $I = [G_w : G_v]$, $R = [k_v' : k_v]$ and by D the henselian defect of the finite extension $(F,w)/(K(\xi), v^{\xi})$ where v^{ξ} is the restriction of w to $K(\xi)$.

Results:

Theorem 1: Let v be a valuation of K with value group G_v and the residue field k_v. Let w be an extension of v to $K(x)$ with value group G_w and residue field k_w such that G_w/G_v is not a torsion group. Then there exists $\beta \in \overline{K}$ with minimal polynomial say $P(x)$ of degree n over K and $\theta \in G_w$, θ not torsion mod G_v such that if $f(x) = \sum_{i=0}^{r} f_i(x)P(x)^i$ is the canonical representation of $f(x) \in K[x]$ with respect o $P(x)$, one has

$$w(f(x)) = \min_{0 \le i \le r}(\overline{v}(f_i(\beta)) + i\theta).$$

Then G_w is the direct sum of $G_1 = v(K(\beta) \setminus \{0\})$ and an infinite cyclic group.

Proof: Let $\overline{K}$ be an algebraic closure of K and $\overline{w}$ be an extension of w $\overline{K}(x)$. Since G_w/G_v is not a torsion group, $G_{\overline{w}}/G_v$ satisfies the same property. So the subset M of $\overline{K}$ defined by

$$M = \{\alpha \in \overline{K} \mid \overline{w}(x-\alpha) \text{ is not torsion mod } G_v\}$$

is non-empty. Choose an element β of M so that $[K(\beta):K] \le [K(\alpha):K]$ for all α in M.

We denote by $P(x)$ the minimal polynomial of β over K of degree n (say). Its roots $\beta = \beta_1 ..., \beta_n$ are arranged such that $\overline{w}(x-\beta_i)$ is not torsion mod G_v for $1 \le i \le m$ and $\overline{w}(x-\beta_i)$ is torsion mod G_v for $m+1 \le i \le n$. We define μ and θ by

$$\mu = \overline{w}(x-\beta), \qquad \theta = w(P(x)).$$

Observe that for any element α of M, $\overline{w}(x-\alpha)$ must be μ; for $\overline{w}(x-\beta)$ cannot be equal to $\overline{w}(x-\beta_i)$ which is torsion mod G_v and hence by the strong triangle law

$$\overline{w}(x-\alpha) = \min(\overline{w}(x-\beta),\ \overline{w}(\beta-\alpha)) = \overline{w}(x-\beta) \tag{1}$$

A similar argument yield that if $\delta \in \overline{K} \setminus M$, then

$$\overline{w}(x-\delta) = \overline{w}(\beta-\delta) < \mu . \tag{2}$$

Using (1) and (2), it is immediately verified that

$$\theta = \overline{\mathrm{w}}(\mathrm{P(x)}) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \overline{\mathrm{w}}(\mathrm{x}-\beta_{\mathrm{i}}) = \mathrm{m}\mu + \sum_{\mathrm{i=m+1}}^{\mathrm{n}} \overline{\mathrm{v}}(\beta-\beta_{\mathrm{i}}) , \tag{3}$$

which shows that θ is not torsion mod G_v.

We next show that if $h(x)$ is a non-zero polynomial over K, none of whose roots lies in M, then

$$\overline{w}((h(x)/h(\beta))-1) > 0 \tag{4}$$

For this write $h(x) = a\prod(x-\delta_i)$ as a product of linear factors over $\overline{K}$. By hypothesis $\delta_i \notin M$, so by (2)

$$\overline{w}(x-\delta_i) = \overline{w}(\beta-\delta_i) < \mu .$$

Consequently

$$\overline{w}\left(\frac{x-\delta_i}{\beta-\delta_i} - 1\right) = \overline{w}\left(\frac{x-\beta}{\beta-\delta_i}\right) = \mu - \overline{w}(\beta-\delta_i) > 0,$$

which shows that the $\overline{w}$-residues of $(x-\delta_i)/(\beta-\delta_i)$ and 1 are the same on taking product over i, one concludes that (4) holds.

An immediate consequence of the assertion proved above is that if all irreducible factors of a non-zero polynomial $h(x)$ of $K[x]$ are of degree less than n, then $\overline{w}((h(x)/h(\beta))-1) > 0$, in particular

$$w(h(x)) = \overline{v}(h(\beta)). \tag{5}$$

We now prove (a) and (b). Let $\mathrm{f(x)} = \sum_{\mathrm{i=0}}^{\mathrm{r}} \mathrm{f_i(x)P(x)^i}$ be the canonical representation of a non-zero polynomial $f(x)$ with respect to $P(x)$. Since deg $f(x) < n$, by (5) $w(f_i(x)) = \overline{v}(f_i(\beta))$ holds for each i. So the triangle law gives

$$w(f(x)) \ge \min_{0\le i\le r}(w(f_i(x)p(x)^i)) = \min_i(\overline{v}(f_i(\beta) + i\theta) \tag{6}$$

It is to be shown that equality holds in (6). Suppose that strict inequality holds, then the minimum in (6) is attained for at least two subscripts i an j, which implies that a non-zero integral multiple of θ is free mod G_0 proved in (3). □

Let K be a field, v be a valuation on K , w be an extension of v to $K(x)$ and W be valuation ring of w. We shall define that $S=\{\xi \in W | \xi^* \ trans/k_v\}$ and $\min S=\{\eta \in S | \deg \eta \leq \deg \xi \ for\ all\ \xi \in S\}$.

Theorem 2: k_w is not algebraic over k_v and let $\xi \in$ min S. Then $k_w = k_v'(\xi^*)$, where k_v' is the algebraic closure of k_v in k_w.

Proof: The inclusion $\supseteq$ is immediate, so it remains to show $\subseteq$. By [5, Lemma 3.1] we may assume $\xi = f/g$, where $\deg f = n > \deg g$. For any $\xi \in V$, we may write, by [5, Lemma3.2],

$$\xi = (a_m \xi^m + a_{m-1}\xi^{m-1} + \ldots + a_0)/(b_m \xi^m + b_{m-1}\xi^{m-1} + \ldots + b_0),$$

where a_i, b_i are elements of $K[x]$ which are either of $\deg < n$ or are 0. Let d be an element of least value from among $\{a_i, b_i | i = 0, \ldots, m\}$, and let $\alpha_i = a_i/d, \beta_i = b_i/d$.

Then

$$\xi = (a_m \xi^m + a_{m-1}\xi^{m-1} + \ldots + a_0)/(\beta_m \xi^m + \beta_{m-1}\xi^{m-1} + \ldots + \beta_0),$$

where now the coefficients α_i, β_i are elements of $K(x)$ which are either of $\deg < n$ or are 0. Moreover, $v(\alpha_i), v(\beta_i), i = 0, \ldots, m$, are all ≥ 0. We may therefore consider the equality

$$(\beta_m^* \xi^{*m} + \ldots + \beta_0^*)\xi^* = \alpha_m^* \xi^{*m} + \ldots + \alpha_0^*.$$

Since $\xi \in \min S$ and the α_i, β_i are either 0 or of $\deg < \deg \xi$, it follows that the β_i^* are all algebraic over k_0. But ξ^* is tr. over k_0 and we know some α_i^* or β_i^* is 1, some β_i^* must be $\neq 0$. Therefore $\beta_m^* \xi^{*m} + \ldots + \beta_0^* \neq 0$, and hence

$$\xi^* = (\alpha_m^* \xi^{*m} + \ldots + \alpha_0^*)/(\beta_m^* \xi^{*m} + \ldots + \beta_0^*) \in k_0' \ (\xi^*).$$

Theorem 3: Let v be a valuation of a field K and w be an extension of v to an overfield $F = K(x, y)$ of transcendence degree one over K where $y^2 = P(x)$ is in $K[x]$. If $G_v \subseteq G_w$ are the value groups of v and w then either G_w / G_v is a torsion group or there exists a subgroup G_1 of G_w containing G_v with $[G_1 : G_v] < \infty$ and an element γ of G_w such that G_w is the direct sum of G_1 and the cyclic group $Z\gamma$ generated by γ.

Proof: Assume that G_w / G_v is not a torsion group. Let H denote the value group of the valuation w restricted to the subfield $K(x)$ of F. Then $[G_w : H] < [F : K(x)] \leq 2$, and H / G_v is not a torsion group. It is known that there exists an (explicitly constructible) subgroup H_1 of H containing G_v with $[H_1 : G_v] < \infty$ and an element of θ of H such that H is the direct sum of H_1 and $Z\theta$. [2.Corollary 1.2] So we assume that $[G_w : H] = 2$.

Two cases are distinguished:

If $(\lambda + \vartheta)/2 = \theta_1$ belongs to G_w for some λ in H, then

$$H = H_1 \oplus Z\theta \subset H_1 \oplus Z\theta_1 \subseteq G_w$$

and hence $G_w = H_1 \oplus Z\theta_1$ in this case. Suppose that $(h_1+\theta)/2 \notin G_w$ for any h_1 in H_1. It will be shown that $G_w = (G_w \cap \frac{1}{2}H_1) \oplus Z\theta$ in this case. Let g be an element of G_w. Since $2g \in H$, we can write

$$g = \frac{h_1}{2} + \frac{n\theta}{2}$$

for some h_1 in H_1 and some integer n. The claim is that n must be even. If n were odd, then on writing g as

$$g = \frac{h_1 + \theta}{2} + \frac{n-1}{2}\theta,$$

we derive that $\frac{h_1+\theta}{2} \in G_w$, contrar to the assumption.

Lemma 4: Let K be a field of char $\neq 2$ and let F be a function field of a conic over K_v. Then there exist explicitly constructible elements $c-d \in K$ such that the K irreducible polynomial $X^2 - y^2 - d$ is a defining polynominal for F/K_v. [3]

Theorem 5: Let F be a function field of a conic over a field K Let v be a valuation of K and w be an extension of v to F. Assume that $chark_v \neq 2$. Then the residue field k_w of w is either an algebraic extension of k_v or k_w is a regular function field of a conic over a finite extension of k_v.

Proof : We may assume that k_w / k_v is not an algebraic extension. In view of Lemma 4, we may write $F = K(x, y)$ where (x, y) satisfies an irreducible polynomial $X^2 - cY^2 - d$ over K. Observe that y is transcendental over K and that $[F : K(y)] \leq 2$. We denote by v_1, the valuation w restricted to $K(y)$ and by k_1, G_1 the residue field and the value group of v_1. Then $[k_w : k_1] \leq 2$ and k_w / k_1 is not an algebraic extension.

When $k_w = k_1$, the desired result follows from the Theorem 2 applied to the simple transcendental extension $K(y)/K$ and the observation that a simple transcendental extension $L(t)$ of a field L is the regular function field of a conic over L which can be visualized by writing $L(t)$ as $L(t, 1/t)$ where $(t, 1/t)$ satisfies $XY - 1 = 0$.

Assume now that $[k_w : k_1] = 2$. Let Δ', Δ denote the algebraic closures of k_v in k_1 and k_w respectively. By the Theorem 2; k_1 is a simple transcendental extension of Δ' and Δ' is a finite extension of k_v. If $\Delta' \subseteq \Delta$, then

$$k_1 = \Delta'(t) \subseteq \Delta(t) \subseteq k_w .$$

In view of te assumption that $[k_w : k_1] = 2$, it is now clear that in the present case

$$[\Delta : \Delta'] = 2 \text{ and } k_w = \Delta(t).$$

The theorem remains to be proved when $\Delta' = \Delta$ and $[k_w : k_1] = 2$. Since

$$[F : K(y)] = [k_w : k_1] = 2, \tag{7}$$

it follows from the fundamental inequality (cf. [1, Chapter 6, §8.3, Theorem 1(b)]) relating the degree of extension with the ramification indices and residual degrees that the value group of w is G_1; in particular $w(x) \in G_1$. By [3,Lemma 2.2], there exists a non-zero polynomial $R(y) \in K[y]$ of degree less than $E = E(v_1 / v)$ such that $w(x) = v_1(R(y))$. Set

$$T = x / R(y) \text{ and } \eta = (cy^2 + d) / R(y)^2$$

Since $x^2 - cy^2 - d = 0$, the v-residue T^* of T satisfies the polynomial $X^* - \eta^*$ over k_1. In view of (7) and the fundamental inequality referred to above, w is the only extension to $F = (y, T)$ of the valuation v' defined on $K(y)$. Recall that char $k_1 \neq 2$; it now follows from [3,Lemma 2.4] applied to the extension $F / K(y)$ that $T^* = \sqrt{\eta^*}$ is not in k_1. Since k_1 contains Δ' which equals the algebraic closure of Δ' in k_w, we conclude that T^* and hence η^* is transcendental over Δ'. Therefore $k_w = k_1 = (\sqrt{\eta^*})$ is proved to be a function field and hence a regular function field of a conic over $\Delta' = \Delta$, as soon as we show that there exists a generator u of the simple transcendental extension k_1 / Δ' such that η^* is a polynomial in u of degree ≤ 2 with coefficients from Δ'. By [3,Lemma 2.3], η^* is itself a generator, say u, of the simple transcendental extension k_1 / Δ'. if $\deg(cy^2 + d) \leq E$; in fact in this situation $k_w = \Delta'(\sqrt{\eta^*})$ is a simple transcendental extension of Δ'. The remaining case is when $E = 1$, i.e., when there exist $a, b \in K$ such that $((y - a)/b)^* = u$ (say) is transcendental over k_v. In this case the polynomial $R(y)$ being of degree less than $E = 1$, must be a constant say R. Therefore on writing $\eta = (cy^2 + d) / R^2$ as a polynomial in $(y - a)/b$, we conclude that η^* is a polynomial of degree ≤ 2 in u over k_v. Then the theorem is completely proved.

Theorem 6: Let v be a valuation of a field K with residue field k_v of char $\neq 2$ and let w be an extension of v to an overfield $F = K(x, \sqrt{P(x)})$, $P(x)$ being a non-constant poylnomial in an indeterminate x over K. Assume that the residue field k_w of w is not algebraic over. Let's denote by D the henselian defect of the finite extension $(F, w)/(K(\xi), v^{\xi})$ where v^{ξ} is the restriction of w to $K(\xi)$, ξ is a element of the valuation ring of w such that ξ^* *trans* $/ k_v$.

Then one can determine (by an explicit algorithm) an element u transcendental over k_v and a polynomial $A(u)$ over the algebraic closure Δ of k_v in k_w with $\deg A(u) \leq \delta + (\deg P(x)) / IRD$ such that $k_w = \Delta(u, \sqrt{A(u)})$ where $\delta = 0$ or 2; indeed δ can be chosen to be 0 when $I = 1$.

Proof : We write $F = K(x, y)$, where $y^2 = P(x) \in K[x] \setminus \mathrm{K}$. We denote by v' the valuation w restricted to $K(x)$ and by $k_{v'}$, $G_{v'}$ the residue field and the value group of v'. Then $[k_w : k_{v'}] \leq [F : K(x)] \leq 2$, and $k_{v'} / k_v$ is a non-algebraic extension as k_w / k_v is given to be so. By the Theorem 2, $k_{v'}$ is a simple tr. extension of a finite extension Δ' of k_0. Throughout the proof, t will stand for the particular generator of $k_{v'} / \Delta'$ described in the opening lines of the proof of [4 Lemma 3.2.]. If $k_w = k_{v'}$, the theorem needs no proof. From now on, it is assumed that $[k_w : k_{v'}] = 2$ and that $\Delta' = \Delta$, for $\Delta' \subseteq \Delta$ yields $k_w = \Delta(t)$.

Since

$$[F : K(x)] = [k_w : k_{v'}] \qquad (8)$$

it follows from the fundamental inequality [1 The 1 b] that the value group of w is $G_{v'}$; in particular $w(y) \in G_{v'}$. By [4 Lemma 3.1] we can choose a non-zero polynomial $h(x) \in F[x]$ of degree $< E' = E'(v'/v_0)$ such that $w(y) = v'(h(x))$; in the case $G_w = G_v$, we choose $h(x)$ of degree 0. Set

$$z = y / h(x),\ \eta = P(x)/h(x)^2$$

Then $z^2 = \eta$ and $v'(\eta) = 0$. In view of (1) and the fundamental inequality [3,§ 8.3,Theo. 2(b)], w is the only extension to $F = K(x,z)$ of the valuation v' defined on $K(x)$. It follows from [4,Lemma 3.4] applied to the extension $F/K(x)$ that $z^* = \sqrt{\eta^*}$ is not in $k_{v'}$. Keeping in view the assumptions $[k_w : k_{v'}] = 2$ and $\Delta = \Delta'$, it is now clear that

$$k_w = k_{v'}(\sqrt{\eta^*}) = \Delta(t, \sqrt{\eta^*}).$$

Recall that $\eta = P(x)/h(x)^2$, where $\deg h(x)^2 \le 2E' - 2$; in fact $\deg h(x)^2 = 0$ if $G_w = G_v$. By [4,Lemma 3.2]. $\eta^* = B(t)/C(t)$ with $B(t)$, $C(t)$ in $\Delta[t]$ satisfying $\deg B(t) \le (\deg P)/E'$ and $\deg C(t) \le 1$. Further by [4,Remark 3.3], the polynomial $C(t)$ may be chosen to be of degree 0 when $G = G_0$.

Let us assume the inepuality $E' \ge IRD$ to be proved below.

If $\deg C(t) = 1$, on taking $u = C(t)$ and writing the polynomial $B(t)$ as $B_1(u)$, we see that

$$k_w = \Delta(u, \sqrt{B_1(u)/u}) = \Delta(u, \sqrt{uB(u)})$$

as desired, for $\deg B_1(u) = \deg B(t) \le (\deg P)/IRD$.

In case $\deg C(t) = 0$, say $C(t) = C \in \Delta$, then the theorem is proved on taking $u = t$ and $A(u) = B(t)/C$.

It only remains to verify the inequality $E' \ge IRD$ with the assumptions $\Delta = \Delta'$ and $[F : K(x)] = [k_w : k_{v'}]$. The latter implies that $G_w = G_{v'}$ and that the henselian defect of the extension $(F,w)/(K(x),v')$ is 1. Fix any element ξ of $K(x)$ with $v'(\xi) = 0$ and ξ^* tr. over k_v. Since the henselian defect is multuplicative, it follows that

$$D = def^h(F/K(x))def^h(K(x)/K(\xi)) = def^h(K(x)/K(\xi)).$$

Thus D equals the number $D' = def^h(K(x)/K(\xi))$ and $E' \ge [G_{v'} : G_v][\Delta' : k_v]D'$, as $G_{v'} = G_w$ and $\Delta = \Delta'$.

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