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An Approach to Some Properties of Fuzzy Topological Spaces

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Abstract: In this manuscript, we use a specific definition of a fuzzy topological space. We define some basis structures of this topology and survey properties of them such as the concepts of F-closure, F-interior and F-limited points of a fuzzy subset of a topological space. Moreover, we redefine the concepts of the basis and subbasis of this topology and discuss about continuous related functions and prove some theorems about them.

Keywords: Fuzzy topological space, F-neighborhood, F-interior set and F-closure of a fuzzy set.

1. Introduction and Preliminaries

Fuzzy sets were introduced in 1965 by Zadeh [1]. Then fuzzy topological spaces were defined in 1968 by Chang [2] and later redefined in a different way by Lowen [3] and Hutton [4]. In 1985 Shostak [5] defined the gradation of openness and introduced the fundamental concept of a fuzzy topological structure. Then Ibedou [6] introduced the separation axioms which depend on the concept of valued fuzzy neighbourhoods and studied graded fuzzy topological spaces. In almost all of definitions such as [1-11], the fuzzy topology is defined on a crisp set, but we use the concept of a new fuzzy topology on fuzzy sets which is introduced by Taleshi [12]. It means that M is a crisp set and $X : M \to [0,1]$ is a fuzzy subset of M in Zadeh's sense [1]. Then (X,τ) is a fuzzy topological space if τ, as a collection of fuzzy subsets of M which are less than X, satisfies the following conditions:

i) $X,\phi \in \tau$.
ii) $\{A_i\}_{i\in I} \subseteq \tau \;\Rightarrow\; \bigcup_{i\in I} A_i \in \tau$.
iii) $A,B \in \tau \;\Rightarrow\; A\cap B \in \tau$.

Each element of τ is called an F-open subset of X and it's complement is called an F-closed subset.

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In this manuscript we fuzzify fundamental concepts of topology with the above fuzzy topology as follows:

In the first section, we define the F-neighborhood of a point in a fuzzy topological space (FTS) X and we show that a fuzzy subset which is an F-neighborhood of an element x is not necessarily an F-open subset of X.

This approach leads us to the F-neighborhood of a fuzzy subset A of X, which this theorem holds. Then we fuzzify some notions of topology such as F-interior and F-closure of a fuzzy subset of X.

In section 2, we introduce the concepts of an F-boundary and F-limited points of a fuzzy subset of X. Also, a related function of two FTS and continuity of it, is discussed.

In section 3, we define the basis and subbasis of an FTS and use them for continuous related functions. Then, we define a T_1 FTS and we finally prove a fundamental theorem.

2. F-Neighborhood of a Point in a Fuzzy Topology Space X

Definition 1. Let (X,τ) be a fuzzy topological space. Suppose $V \subseteq X$ and $x \in \text{supp } X$. If there exists an F-open subset U of X such that $x \in \text{supp } U$ and $U \subseteq V$, then V is called an F-neighborhood of x in X. We denote the set of all F-neighborhoods of x in X by $FN(x)$.

Remark 2. It is clear that every F-open subset A of X is an F-neighborhood of all elements of supp A but the following example shows that the converse is not true.

Example 3. Let $M = \mathbb{R}$ and X be a fuzzy subset of $\mathbb{R}$ which is defined by

$$X(x) = \begin{cases} 0.5 & x \in [1,2], \\ 0 & x \notin [1,2]. \end{cases}$$

For $\alpha \in (0, 0.5)$, define

$$U_\alpha(x) = \begin{cases} \alpha & x \in [1,2], \\ 0 & x \notin [1,2]. \end{cases}$$

Then $U_\alpha \subseteq X$.

Take $\tau = \{X, \phi\} \cup \{U_\alpha : 0 \le \alpha \le 0.3\}$. Then
1) $\phi = U_0 \in \tau$ and $X \in \tau$.
2) For each $\alpha, \beta \le 0.3$,

$$(U_\alpha \cap U_\beta)(x) = \begin{cases} \min\{\alpha,\beta\} & x \in [1,2], \\ 0 & x \notin [1,2]. \end{cases}$$

So $U_\alpha \cap U_\beta = U_{\min\{\alpha,\beta\}}$.
3) For each index set I of numbers less than 0.3, we have $\bigcup\limits_{\alpha\in I} U_\alpha = U_\gamma$ where $\gamma = \sup\{\alpha : \alpha \in I\} \leq 0.3$. Therefore (X,τ) is a FTS.

Let $A : M \to [0,1]$,

$$A(x) = \begin{cases} 0.4 & x \in [1,2], \\ 0 & x \notin [1,2], \end{cases}$$

then $A \in F_x(M)$ and $A \notin \tau$ but A is an F-neighborhood of all elements of supp A; because for all $x \in \text{supp } A = [1,2]$, we consider $U_{0.3} \in \tau$, then $x \in \text{supp } U_{0.3}$ and $U_{0.3} \subset A$, then $A \in FN(x)$.

We define an F-neighborhood of a fuzzy set, then we prove Theorem 5.

Definition 4. A fuzzy set V in an FTS (X,τ) is called an F-neighborhood of a fuzzy set A, if there exists an F-open set U such that $A \subseteq U \subseteq V$. We denote the set all fuzzy neighborhoods of A by $FN(A)$.

Theorem 5. A fuzzy set A is F-open iff for each fuzzy set B contained in A, $A \in FN(x)$.

Proof. It is straightforward. ■

Definition 6. Let A and B be fuzzy sets in an FTS (X,τ) and $B \subseteq A$. Then B is called an F-interior set of A iff $A \in FN(B)$. The union of all F-interior sets of A is denoted by A°.

Theorem 7. Let A be a fuzzy set in an FTS, (X,τ). Then, A° is F-open and is the largest F-open set contained in A. The fuzzy set A is F-open iff $A = A^\circ$. Hence

$$A^\circ = \sup\{U : U \subseteq A, U \in \tau\}.$$

Proof. By Definition 6, A° is itself an F-interior fuzzy set of A. Therefore, $A \in FN(A^\circ)$. Hence, there exists an open fuzzy set U such that $A^\circ \subseteq U \subseteq A$. Since U is an F-interior set of A, it follows that $U \subseteq A^\circ$. Therefore, $A^\circ = U$. Thus, A° is F-open and is the largest F-open set contained in A. If A is open, then $A \subseteq A^\circ$, A is an F-interior set of A. Hence $A = A^\circ$. The converse is obvious. ■

Theorem 8. Let (X,τ) be FTS and $x \in \text{supp } X$. Then we have:

1) $\forall y \in FN(x) \Rightarrow y \in \text{supp } V$,
2) $V_1 \in FN(x), V_1 \subseteq V_2 \Rightarrow V_2 \in FN(x)$,
3) $V_1, V_2 \in FN(x) \Rightarrow V_1 \cap V_2 \in FN(x)$,
4) $V \in FN(x) \Rightarrow \exists U \in FN(x)$ s.t. $\forall y \in \text{supp } U$, we have $V \in FN(y)$.

Proof. We prove item 3). The other items are clear.

If $V_i \in FN(x)$, $i = 1,2 \Rightarrow \exists U_i \in \tau$, $x \in \text{supp } U_i, U_i \subseteq V_i$. Therefore, $U_1 \cap U_2(x) \neq 0$ and $\forall y$, $U_1 \cap U_2(y) \subseteq (V_1 \cap V_2)(y) \Rightarrow V_1 \cap V_2 \in FN(x)$. ■

Definition 9. Let (X,τ) be an FTS and for all $x,y\in \text{supp}\,X$, $x\neq y$, there exists two F-neighborhoods $U_x\in FN(x), U_y\in FN(y)$ such that $U_x\cap U_y=0$. Then (X,τ) is called a Hausdorff fuzzy topological space.

Example 10. Let (X,τ) be an FTS. Then $(X,F_X(M))$ is a Hausdorff FTS. Let $x,\,y\in \text{supp}\,X$, then $\chi_{\{x\},X},\ \chi_{\{y\},X}\in\tau$ and, for all z, we have $\min\{\chi_{\{x\},X}(z),\ \chi_{\{y\},X}(z)\}=0$. So $\chi_{\{x\},X}\cap\chi_{\{y\},X}=0$.

Note that for every fuzzy subset A of X,

$$\chi_{\{x\},A}(z)=\begin{cases}0 & z\neq x,\\ A(x) & z=x.\end{cases}$$

From now on we write $\chi_{\{x\}}$ instead of $\chi_{\{x\},X}$.

Definition 11. Let (X,τ) be an FTS and $A\in F_X(M)$. The intersection of all F-closed subsets containing A is called F-closure of A and denoted by $\overline{A}$.

If A is an F-closed subset, then $\overline{A}\subseteq A$, therefore $\overline{A}=A$. Conversely, if $\overline{A}=A$, then A is an F-closed subset; because every intersection of F-closed subsets is closed too.

We now demonstrate a distinction between ordinary topology and fuzzy topology.

Let $(X,\,\tau)$ be an FTS, $x\in\text{supp}\,X$ and $A\subseteq X$. Then, $x\in\text{supp}\,\overline{A}$ does not imply that for all F-open sets U in X, we have $\{U-\chi_{\{a\},X}\}\cap A\neq\phi$. It is sufficient that we give an example.

Example 12. Let $M=\{a,b,c,d,e\}$, $\quad X:M\to I$,

$$X(x)=\begin{cases}1 & x\neq e,\\ 0 & x=e.\end{cases}$$

and

$$\tau=\{\phi,\,x,\,G_\alpha,\,V_\alpha,\,W_\alpha\,|\,\alpha\in(0,1]\},$$

where

$$G_\alpha=\alpha\chi_{\{a\}},\quad V_\alpha=\alpha\chi_{\{b\}}\quad \textit{and}\quad W_\alpha=\alpha\chi_{\{a,b\}}.$$

Since we have $\forall\alpha,\beta\in[0,1]$

$$G_\alpha\cap V_\beta=\phi,\qquad G_\alpha\cap W_\beta=G_\alpha,\qquad V_\alpha\cap W_\beta=V_\alpha,$$

$$G_\alpha\cup V_\beta=W_{\max\{\alpha,\beta\}},\qquad G_\alpha\cup W_\beta=W_\beta,\qquad V_\alpha\cup W_\beta=W_\beta.$$

One can simply prove that τ is a fuzzy topology on X.

Let $A=\chi_{\{a,d\}}$. Then, for all α, $X-V_\alpha$, $X-W_\alpha$ and $X-G$ are all F-closed sets containing A. Therefore, the intersection of all of them equals to $X-V_1$. Therefore, $\overline{A}=X-V_1=\chi_{\{a,c,d\}}$. Now, we see $a\in\text{supp}\ U=G_1$ but $(U-\chi_{\{a\}})\cap A=\phi$, since $U-\chi_{\{a\}}=G_1-\chi_{\{a\}}=\chi_{\{a\}}-\chi_{\{a\}}=\phi$.

Definition 13. Let X be an FTS and $A \subseteq X$. A is said to be an F-dense subset of X, if $\overline{A} = X$.

Definition 14. Let X be an FTS and $A \subset X$. $x \in \text{supp } A$ is called an F-boundary point of A, if for every F-neighborhood V of x, we have $V \nsubseteq A$. The set of these points is called F-boundary of A and is denoted by A^m.

Definition 15. Let X be an FTS, $A \subset X$ and $x \in \text{supp } X$. If x belongs to the support of closure of $A - \chi_{\{x\},A}$, then x is called a F-limited point of A and the set of these points is denoted by A'.

Theorem 16. Let X be an FTS, $A, B \subset X$, then we have:

1) If $A \subseteq B$ then $A' \subseteq B'$.
2) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.
3) $A' \cup B' \subseteq (A \cup B)'$.
4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. 1) Let C be a closed fuzzy subset of X containing $B - \chi_{\{x\},B}$. Then, $\forall y$, $(B - \chi_{\{x\},B})(y) \leq C(y)$. Since $\forall y \neq x$, $\chi_{\{x\},B}(y) = 0$, so $B(y) \leq C(y)$. We supposed $A \subseteq B$, hence
$A(y) \subseteq B(y) \leq C(y) \quad \forall y \neq x$.

On the other hand, $(A - \chi_{\{x\},A})(x) = A(x) - A(x) = 0 \leq C(x)$, therefore $(A - \chi_{\{x\},A}) \leq C$.

If $x \in A'$, then $x \in \text{supp } (\overline{A - \chi_{\{x\},A}})$, therefore $C(x) \neq 0$. Thus, x belongs to the support of intersection of all closed fuzzy subset C containing $(B - \chi_{\{x\},B})$. Hence, $x \in \text{supp } (B - \chi_{\{x\},B})$, implying that $x \in B'$.

2) Let C be an F-closed subset s.t. $B \subseteq C$. Then, $A \subseteq C$. Hence, $\overline{A} \subseteq C$, therefore $\overline{A}$ is contained in the intersection of all F-closed subsets containing B so $\overline{A} \subseteq \overline{B}$.

3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows from 1)that $A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$. Therefore, $\forall x \in \text{supp } X$, $\max\{A'(x), B'(x)\} \leq (A \cup B)'(x)$ so $A' \cup B' \subseteq (A \cup B)'$.

4) Since $\overline{A}$ and $\overline{B}$ are F-closed subsets of x containing A and B, respectively, then $\overline{A} \cup \overline{B}$ is an F-closed set containing $A \cup B$. Since $\overline{A \cup B}$ is the smallest F-closed set containing $A \cup B$, hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. The converse direction is obvious. Hence, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. ■

Theorem 17. Let (X, τ_X) be an FTS, $Y \subseteq X$ and (Y, τ_Y) be an FTS. Every F-open subset of Y is an F-open subset of X if and only if $Y \in \tau_X$.

Proof. Assume that $\forall U \in \tau_Y$, we have $U \in \tau_X$. Since $Y \in \tau_Y$, therefore $Y \in \tau_X$. Conversely, Let $Y \in \tau_X$. Then, $\forall U \in \tau_Y$ by means of the fuzzy topological subspace (FTSS) $U = V \cap Y$ such that $V \in \tau_X$. Therefore, $U \in \tau_X$. ■

Theorem 18. Let (X, τ) be an FTS and $A \subseteq X$. Then,

i) $\overline{(X-A)} \subseteq X - A^{\circ}$,
ii) $(X-A)^{\circ} \subseteq X - \overline{A}$.

Proof. i) Let $\{W_i, i \in I\}$ be the set of all F-open subsets S of A, then $A^{\circ} = \underset{i\in I}{\cup} W_i$. Therefore,

$$X - A^{\circ} = X - \underset{i\in I}{\cup} W_i = \underset{i\in I}{\cap}(X - W_i),$$

since $\forall i \in I$, $X - W_i$ is an F-closed subset of X containing $X - A_1$. Therefore, $\overline{(X-A)} \subseteq \underset{i\in I}{\cap}(X - W_i)$. Hence, $\overline{(X-A)} \subseteq X - A^{\circ}$. Therefore, $\overline{(X-A)} \subseteq \underset{i\in I}{\cap}(X - W_i)$. Hence, $\overline{(X-A)} \subseteq X - A^{\circ}$.

ii) If we set $X - A$ instead of A in i), we have $\overline{(X-(X-A))} \subseteq X - (X - A^{\circ})$. Hence, $\overline{A} \subseteq X - (X - A^{\circ}) \Rightarrow (X-A)^{\circ} \subseteq X - \overline{A}$. ■

3. Basis and Subbasis of a Topological Space

Definition 19. A family β of members of τ is called a basis of fuzzy topological space (X, τ), if each element of τ is a union of members of β.

Let $G \in \tau$ and $x \in \text{supp } G$. Since $G = \bigcup_{k\in\omega} B_k$, it holds that $\exists k \in \omega$ s.t. $x \in \text{supp } B_k$ and $B_k \subseteq G$.

Lemma 20. Suppose that $\beta = \{B_i\}_{i\in I} \subseteq F_x(M)$ has the following properties

i) $\bigcup_{i\in I} B_i = X$,
ii) $B_1, B_2 \in \beta, B_1 \cap B(x) > 0 \Rightarrow \exists B_3 \in \beta,\ B_3 \subseteq B_1 \cap B_2$ and $B_3(x) = B_1 \cap B_2(x)$.

Then, $\tau = \{\bigcup_{j\in J} B_j, J \subseteq I\}$ is a fuzzy topology on X and β is a basis of τ.

Theorem 21. Let $S = \{S_k \,|\, k \in \omega\}$ be a collection of some fuzzy subsets of X such that $X = \bigcup_{k\in\lambda} S_k$. If $\beta = \{\beta_i | i \in I\}$ is a collection of a finite intersection of elements of S and $\tau = \{\bigcup_{j\in J} B_j \,|\, B_j \in \beta, J \subseteq I\}$, then β is a basis for fuzzy topology τ on X.

Proof. Since $S_k \in \beta$, so $X = \bigcup_{i\in I} B_i$. Also $\forall i, j \in I$, $B_i \cap B_j$ is a finite intersection of elements of S. So $B_i \cap B_j \in \beta$. Therefore, β satisfies the two conditions of Lemma 20 and β is a basis for τ. ■

Definition 22. Let $f : M_1 \to M_2$ be a function and X $\in F(M_1)$, $Y \in F(M_2)$ such that $f(X) \subseteq Y$. Then, f is called a related function of X and Y, and we write $f = Rf(X,Y)$

Lemma 23. If $f = Rf(X,Y)$, f is one-to-one and onto, then

i) $\forall x,\quad f[X](f(x)) = X(x)$,
ii) $\forall x,\ X(x) \leq Y(f(x))$.

Proof. i) Since f is one-to-one and onto, then for all y, there exists a unique x such that $y = f(x)$ so $f^{-1}(y) = \{x\}$. By definition of the image of f, we have

$$f[X](y) = \begin{cases} \sup\{\, X(z) \; : z \in f^{-1}(y)\} & f^{-1}(y) \neq \phi, \\ \\ 0 & \text{otherwise} . \end{cases}$$

Therefore, for all x, we have $f[X](f(x)) = \sup\{\, X(z) \; : z \in \{x\}\} = X(x)$.

ii) Since f is a related function of X and Y, we have $f[X] \subseteq Y$. Then, $\forall x,\; X(x) \leq Y(f(x))$. Then, by i), we have $\forall y,\; f[X](y) \leq Y(y)$. ∎

Definition 24. Let (X, τ_X), (Y, τ_Y) be two FTS and $f = Rf(X,Y)$. Then,

i) f is called open if $f(A) \in \tau_Y, \forall A \in \tau_X$.
ii) f is called continuous if $f^{-1}[G] \cap X \in \tau_X,\; \forall G \in \tau_Y$.
iii) f is called a homeomorphism if it is one-to-one, onto, continuous and open.

Definition 25. $S \subseteq F_X(M)$ is said to be a subbase of (X, τ) if each element of τ is an arbitrary union of finite intersections of elements of S.

Definition 26. Let (X, τ) be an FTS, $x \in \text{supp } X$ and $\beta_x = \{B_{ix} \,|\, i \in I, B_{ix} \in \tau_X, B_{ix}(x) > 0\}$. If, for every open fuzzy subset A and for every $x \in \text{supp } A$, there exists $B_{kx} \in \beta_x$ such that $B_{kx} \subseteq A$ and $B_{kx}(x) = A(x)$, then β_x is called a local basis of τ at x.

Example 27. Let $M = \mathbb{R}^n$ and X be a constant fuzzy subset of M equal to 1. Then, $\beta = \{B(p,q,r),$ $p \in \mathbb{R}^n, q \in \mathbb{R}^+, r \in [0,1]\}$ is a basis for Euclidean topology on X. $B(p,q,r)$ is a fuzzy subset that equals to zero outside the sphere $B(p,q)$ and equals to r inside $B(p,q)$. It is clear that $\beta_x = \{B(p,q,r),\, p \in \mathbb{R}^n, q \in \mathbb{R}^+, r \in [0,1], x \in B(p,q)\}$ is a local basis for the Euclidean topology.

Example 28. Let $M = \mathbb{R}^2$ and X be a constant fuzzy subset equal to 1. Let $T([a,b] \times [c,d], r)$ be a fuzzy subset that is zero outside $[a,b] \times [c,d]$ and r inside this rectangle. Then, $\beta = \{T([a,b] \times [c,d], r),\, a,b,c,d \in R, r \in \mathbb{R}^+\}$ is a basis for the Euclidean topology on X.

Lemma 29. Let X, Y be two FTS, $f = Rf(X,Y)$ and β be a basis for τ_Y. f is continuous if and only if $f^{-1}[B] \cap X$ is an open fuzzy subset of Y for all $B \in \beta$.

Proof. Let f be continuous. Since $\beta \subseteq \tau_Y$, it holds that $f^{-1}[B] \cap X \in \tau_X,\; \forall B \in \beta$.

Conversely, let $f^{-1}[B] \cap X \in \tau_X,\; \forall B \in \beta$. Since β is a basis for τ_Y, then for each open fuzzy subset G of Y, we have $G = \bigcup_{k\in\omega} B_k,\; B_k \in \beta,\; \forall k \in \omega$. Therefore, $f^{-1}[G] = f^{-1}[\bigcup_{k\in\omega} B_k] = \bigcup_{k\in\omega} f^{-1}[B_k]$. Since $f^{-1}[B_k] \cap X \in \tau_X,\; \forall k \in \omega$. Hence, $f^{-1}[G] \cap X \in \tau_X$. Thus, f is continuous. ∎

Proposition 30. Let $f = Rf(x,y)$, then for each $x \in suppX$ and each fuzzy neighborhood V of $f(x)$ in Y, $f^{-1}[V]$ is fuzzy neighborhood of x.

Proof. Let f be continuous related function of X and $x, y \in \operatorname{supp} X$. Let $V \in FN(f(x))$. Then, there exists a $U \in \tau_Y$, s.t. $f(x) \in \operatorname{supp} U$, $U \subseteq V$. Therefore, $f^{-1}[U] \subseteq f^{-1}[V]$, $f^{-1}[U] \in \tau_X$ and $f^{-1}[U](x) = U(f(x)) > 0$. Hence, $x \in \operatorname{supp} f^{-1}[U]$. Therefore, $f^{-1}[V] \in FN(x)$. ■

Theorem 31. Let $S = \{S_i, i \in I\}$ be a subbase of a fuzzy topological space Y and $f = Rf(X,Y)$. Then f is continuous if and only if $\forall i \in I, f[S_i] \cap X \in \tau_X$.

Proof. ($\Leftarrow$) Let β be the basis of τ_X. Then, $f^{-1}[B] \cap X \in \tau_X$, $\forall B \in \beta$. Since each element of β is a finite intersection of elements of the subbase S, it holds that $S \subseteq \beta$. Therefore, $f^{-1}[S_i] \cap X \in \tau_X$ $\forall i \in I$.

($\Rightarrow$) If $G \in \tau_Y$, then $G = \bigcup\limits_{i \in I}(S_{i1} \cap \ldots \cap S_{ik})$. Therefore, $f^{-1}[G] = \bigcup\limits_{i \in I}(f^{-1}[S_{i1}] \cap \ldots \cap f^{-1}[S_{ik}])$. Since $f^{-1}[S_{ij}] \in \tau_Y$, so $f^{-1}[G] \in \tau_X$. Hence, f is continuous. ■

Definition 32. Let (X, τ) be an FTS and for all $x, y \in \operatorname{supp} X, x \neq y$, there exist two F-open sets H, G such that $x \in \operatorname{supp} G$, $y \in \operatorname{supp} H$, $H(x) = 0$, $G(y) = 0$. Then, (X, τ) is called an T_1 space.

Theorem 33. (X, τ) is an T_1 FTS if and only if $\chi_{\{x\}}$ is an F-closed set for each $x \in \operatorname{supp} X$.

Proof. ($\Rightarrow$) We show that $X - \chi_{\{x\}}$ is an F-open set. Let $B \subseteq X - \chi_{\{x\}}$. Then, $B \cap \chi_{\{x\}} = \emptyset$. Since X is a T_1 space, for every $y \in \operatorname{supp} B$, there exist two F-open sets H_y, G_y such that $\chi_{\{x\}} \subseteq H_y$, $\chi_{\{y\}} \subseteq G_y$, $H_y(y) = 0$, $G_y(x) = 0$. Hence,

$$B = \bigcup_{y \in \operatorname{supp} B} \chi_{\{y\}} \subseteq \bigcup_{y \in \operatorname{supp} B} G_y.$$

If we set $G = \bigcup\limits_{y \in \operatorname{supp} B} G_y$, then $G(x) = \sup\{G_y(x) | y \in \operatorname{supp} B\} = 0$. Thus, $B \subseteq G \subseteq X - \chi$. Therefore, $X - \chi_{\{x\}} \subseteq FN(B)$. Then, by Theorem 5, $X - \chi_{\{x\}}$ is an F-open set.

($\Leftarrow$) Let $x, y \in \operatorname{supp} X, x \neq y$, then $F = X - \chi_{\{x\}}$ and $G = X - \chi_{\{y\}}$ are F-open sets and $x \in \operatorname{supp} G, y \in \operatorname{supp} H, H(x) = 0, G(y) = 0$. Therefore, X is an T_1 space. ■

Definition 34. Let (X, τ) be an FTS. If there exists a countable basis for τ, then (X, τ) is called a second countable FTS.

Lemma 35. Let (X, τ_x) and (Y, τ_y) be two FTS. Then, $(X \times Y, \tau_{X \times Y})$ defined by

$$\tau_{X \times Y} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$$

is an FTS.

Proof. Straightforward. ■

Theorem 36. Let (X, τ_x) and (Y, τ_y) be two FTS. Then, X and Y are second-countable spaces if and only if $X \times Y$ is a second-countable space.

Proof. $(\Rightarrow)$ Let W be an open fuzzy subset of $X\times Y$. There exists $U\in\tau_X$ and $V\in\tau_Y$ s.t. $W=U\times V$. If $\beta_x,\ \beta_Y$ are two countable bases for $\tau_X,\ \tau_Y$ respectively, then $\beta=\{B_1\times B_2\ :\ B_1\in\beta_x,\ B_2\in\beta_Y\}$ is a countable basis for $\tau_{X\times Y}$.

$(\Leftarrow)$ Let β be a countable basis for $\tau_{X\times Y}$. We set

$$\beta_x=\{P_1^{-1}(W):\ W\in\beta\},\ \ \beta_Y=\{P_2^{-1}(W):W\in\beta\},$$

where P_1,P_2 are projection functions. One can easily prove that $\beta_X,\ \beta_Y$ are two countable bases for $\tau_X,\ \tau_Y$, respectively. Hence X and Y are second-countable spaces. ■

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