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Automorphisms of Klein Surfaces of Algebraic Genus One

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Abstract

Klein surfaces of algebraic genus one are the Möbius band, the annulus and the Klein bottle. In this paper, the automorphisms of these surfaces are determined.

Keywords: *Klein surface, Möbius band, annulus, Klein bottle, automorphism.*

Özet

Cebirsel cinsi bir olan Klein yüzeyleri; Möbius şeridi, silindir ve Klein şişesidir. Bu çalışmada bu yüzeylerin otomorfizmaları belirlenmiştir.

Anahtar Kelimeler: *Klein yüzeyi, Möbius şeridi, silindir, Klein şişesi, otomorfizma.*

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 1$. An automorphism of X is a conformal or anti-conformal homeomorphism $f : X \to X$. X is called *symmetric* if it admits an anti-conformal involution $s : X \to X$ which we call a *symmetry* of X. The quotient surface $S = X / \langle s \rangle$ is a *Klein surface.* By a Klein surface we mean a surface with a dianalytic structure (see [1]). Here X is called the *complex double* of S. The *algebraic genus* of S is then defined to be the topological genus of X. It is known that the Klein surfaces of algebraic genus one are the Möbius band, the annulus and the Klein bottle. In this paper we study the automorphisms of these surfaces. We do not claim originality of the work. However, it contains something demonstrative of the method, not readily available in the literature, which may be helpful to those who are not experts but wish to understand the subject.

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2. Lattices and tori

A *lattice* is a group generated by two linearly independent translations in the complex plane **C**. Let Ω be a lattice. Then, the quotient space $\mathbf{C}/\Omega$ is a torus. Let $z \in \mathbf{C}$ and let $[z]_\Omega$ denote the Ω-orbit of z. Then, $[z]_\Omega$ is a single point on the torus $\mathbf{C}/\Omega$. If Ω is a lattice generated by the translations $z \to z + z_1$ and $z \to z + z_2$, and $a \in \mathbf{C} - \{0\}$, then $a\Omega$ is a lattice and is generated by $z \to z + az_1$ and $z \to z + az_2$.

We shall use the following theorems to determine the automorphisms of tori and automorphisms of their quotients by symmetries.

Theorem 2.1 ([3]) Let Ω be a lattice. Then, the conformal automorphisms of the torus $\mathbf{C}/\Omega$ are the transformations $[z]_\Omega \to [az+b]_\Omega$ such that $a, b \in \mathbf{C}$ and $a\Omega = \Omega$.

Theorem 2.2 ([1]) Let X be a Riemann surface and $s : X \to X$ a symmetry of X. Then, the automorphisms of the Klein surface $X/\langle s \rangle$ consist of the conformal automorphisms of X commuting with s.

Theorem 2.3 ([1]) Let X be a Riemann surface, and s_1, s_2 be symmetries of X. Then, the Klein surfaces $X/\langle s_1 \rangle$ and $X/\langle s_2 \rangle$ are isomorphic (dianalytically equivalent) if and only if s_1 and s_2 are conjugate in the group of automorphisms of X.

It is known that every complex torus is conformally equivalent to $\mathbf{C}/\Omega$, where Ω is a lattice generated by the translations $z \to z+1$ and $z \to z+\gamma$. Here γ lies in the usual fundamental region F for the modular group $\mathrm{PSL}(2, \mathbf{Z})$, that is, $\gamma \in F = \{z \in \mathbf{C} \mid -1/2 \le \mathrm{Re}(z) \le 1/2, \mathrm{Im}(z) > 0, |z| \ge 1\}$ (see [1]). The symmetric tori correspond to the points of F on the imaginary axis and on the boundary of F. The following table is given in [1], where in each case, Ω is the lattice generated by the translations $z \to z+1$ and $z \to z+\gamma$, and T is the torus $\mathbf{C}/\Omega$. The quotient surface in the last column is the quotient of T by the corresponding symmetry.

Table 1

Case	Symmetry of T	Quotient surface
1. $\mathrm{Re}(\gamma)=0, \mathrm{Im}(\gamma)>1$	$[z]_\Omega \to [\bar{z}]_\Omega$	Annulus
	$[z]_\Omega \to [-\bar{z}]_\Omega$	Annulus
	$[z]_\Omega \to [\bar{z}+1/2]_\Omega$	Klein Bottle
	$[z]_\Omega \to [-\bar{z}+\gamma/2]_\Omega$	Klein Bottle
2. $\gamma = i$	$[z]_\Omega \to [\bar{z}]_\Omega$	Annulus
	$[z]_\Omega \to [i\bar{z}]_\Omega$	Möbius Band
	$[z]_\Omega \to [\bar{z}+1/2]_\Omega$	Klein Bottle
3. $\lvert\gamma\rvert = 1$, $0<\mathrm{Re}(\gamma)<1/2$	$[z]_\Omega \to [\gamma\bar{z}]_\Omega$	Möbius Band
	$[z]_\Omega \to [-\gamma\bar{z}]_\Omega$	Möbius Band
4. $\gamma = (1+i\sqrt{3})/2$	$[z]_\Omega \to [\bar{z}]_\Omega$	Möbius Band
	$[z]_\Omega \to [-\bar{z}]_\Omega$	Möbius Band
5. $\mathrm{Re}(\gamma)=1/2$, $\mathrm{Im}(\gamma)>\sqrt{3}/2$	$[z]_\Omega \to [\bar{z}]_\Omega$	Möbius Band
	$[z]_\Omega \to [-\bar{z}]_\Omega$	Möbius Band

Remark 2.1

(i) Any two points in the upper half complex plane lying in the same orbit under the action of the modular group correspond to conformally equivalent tori (see [3]). For this reason, the cases $\mathrm{Im}(\gamma)>1$, $\mathrm{Re}(\gamma)=-1/2$ and $\lvert\gamma\rvert=1$, $-1/2<\mathrm{Re}(\gamma)<0$ have not been considered in Table 1.

(ii) As shown in [1], the torus corresponding to $\gamma = i$ (case 2) admits three conjugacy classes of symmetries. Since the quotients of a torus by conjugate symmetries are isomorphic Klein surfaces by Theorem 2.3, in case 2, we consider only one symmetry from each conjugacy class. The same discussion applies to the case 4.

3. Automorphisms of the Möbius band

In this section, we shall determine the automorphisms of the Möbius bands given in Table 1. Let us choose, for example, the Möbius band M corresponding to the case 2 of Table 1. In this case, the translations $z \to z+1$ and $z \to z+i$ generate a lattice Ω such that $\mathbf{C}/\Omega$ is a torus T. T admits a symmetry $s : T \to T$ defined by $s([z]_\Omega) = [i\bar{z}]_\Omega$ such that $M = T/\langle s \rangle$. The Möbius band M can also be considered as $\mathbf{C}/\Lambda$, where Λ is the group generated by the translations $z \to z+1$, $z \to z+i$ and the reflection $z \to i\bar{z}$, that is, $\Lambda = \langle \Omega, s \rangle$. So, Λ is a group containing Ω with index 2.

Now let us calculate the automorphisms of M. By Theorem 2.2, the automorphisms of M consist of the conformal automorphisms of T commuting with s. Let $F_\Omega = \{z \mid 0 \le \mathrm{Re}(z) \le 1, 0 \le \mathrm{Im}(z) \le 1\}$, which is a fundamental region for Ω. It follows from [1] and Theorem 2.1 that T admits the following conformal automorphisms:

$$f_1([z]_\Omega) = [z+b]_\Omega \ (b \in F_\Omega),$$

$$f_2([z]_\Omega) = [-z+b]_\Omega \ (b \in F_\Omega),$$

$$f_3([z]_\Omega) = [iz+b]_\Omega \ (b \in F_\Omega),$$

$$f_4([z]_\Omega) = [-iz+b]_\Omega \ (b \in F_\Omega).$$

Let us now determine those conformal automorphisms of T commuting with s. The automorphism f_1 of T commutes with s if and only if for every $z \in \mathbf{C}$,

$$\begin{aligned}(f_1 \circ s)([z]_\Omega) = (s \circ f_1)([z]_\Omega) &\Leftrightarrow [i\bar{z}+b]_\Omega = [i\bar{z}+i\bar{b}]_\Omega \\ &\Leftrightarrow [b]_\Omega = [i\bar{b}]_\Omega \\ &\Leftrightarrow \mathrm{Re}(b) = \mathrm{Im}(b) \\ &\Leftrightarrow b = (1+i)x, \ (0 \le x < 1).\end{aligned}$$

Therefore, by using f_1, we obtain the following automorphisms for the Möbius band M:

$$[z]_\Lambda \to [z+(1+i)x]_\Lambda \quad (0 \le x < 1).$$

Similarly, from f_2 we obtain the following automorphisms for M:

$$[z]_\Lambda \to [-z+(1+i)x]_\Lambda \quad (0 \le x < 1).$$

Now, consider the automorphism $f_3([z]_\Omega) = [iz + b]_\Omega$ of T. It commutes with $s([z]_\Omega) = [i\bar{z}]_\Omega$ if and only if for every $z \in \mathbf{C}$,

$$(f_3 \circ s)([z]_\Omega) = (s \circ f_3)([z]_\Omega) \Leftrightarrow [i(i\bar{z}) + b]_\Omega = [i(\overline{iz + b})]_\Omega$$

$$\Leftrightarrow [-\bar{z} + b]_\Omega = [\bar{z} + i\bar{b}]_\Omega$$

$$\Leftrightarrow [-2\bar{z}]_\Omega = [i\bar{b} - b]_\Omega$$

$$\Leftrightarrow [-2\bar{z}]_\Omega = [(\mathrm{Re}(b) - \mathrm{Im}(b))(i-1)]_\Omega$$

$$\Leftrightarrow [z]_\Omega = [(1/2)(1+i)(\mathrm{Re}(b) - \mathrm{Im}(b))]_\Omega .$$

So, the equation $(f_3 \circ s)([z]_\Omega) = (s \circ f_3)([z]_\Omega)$ is not satisfied for all z, and hence f_3 and s do not commute. In the same way, we can show that f_4 and s do not commute and therefore, from f_3 and f_4 we cannot obtain any automorphisms for the Möbius band M. As a result, all the automorphisms of M are as follows:

$$[z]_\Lambda \to [\pm z + (1+i)x]_\Lambda , \ (0 \le x < 1) .$$

By doing similar calculations to those above, we can determine the automorphisms of the Möbius bands corresponding to the other cases, and we give the results in Table 2.

Table 2

Case	Symmetry of T	Automorphisms of $T/\langle s \rangle$
$\gamma = i$	$s([z]_\Omega) = [i\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + (1+i)x]_\Lambda$, $(0 \le x < 1)$
$\lvert\gamma\rvert = 1$, $0 < \mathrm{Re}(\gamma) < 1/2$	$s([z]_\Omega) = [\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + (1+\gamma)x]_\Lambda$, $(0 \le x < 1)$
	$s([z]_\Omega) = [-\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + (1-\gamma)x]_\Lambda$, $(0 \le x < 1)$
$\gamma = (1 + i\sqrt{3})/2$	$s([z]_\Omega) = [\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + x]_\Lambda$, $(0 \le x < 1)$
	$s([z]_\Omega) = [-\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + ix]_\Lambda$, $(0 \le x < \sqrt{3})$
$\lvert\gamma\rvert > 1$, $\mathrm{Re}(\gamma) = 1/2$	$s([z]_\Omega) = [\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + x]_\Lambda$, $(0 \le x < 1)$
	$s([z]_\Omega) = [-\bar{z}]_\Omega$	$[z]_\Lambda \to [\pm z + ix]_\Lambda$, $(0 \le x < 2\mathrm{Im}(\gamma))$

4. Automorphisms of the Annulus

It follows from section two that we obtain annuli in the cases where $\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$ and $\gamma = i$. Let $\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$ and Ω be the lattice generated by $z \to z+1$ and $z \to z+\gamma$. Let T be the torus $\mathbf{C}/\Omega$. From [1] and Theorem 2.1, it follows that T admits the following conformal automorphisms:

$$[z]_\Omega \to [\pm z + b]_\Omega \ (b \in F_\Omega),$$

where $F_\Omega = \{z \in \mathbf{C} \mid 0 \le \mathrm{Re}(z) \le 1, 0 \le \mathrm{Im}(z) \le |\gamma|\}$ is a fundamental region for Ω. As shown in [1], T admits two non-conjugate symmetries $s_1([z]_\Omega) = [\bar{z}]_\Omega$ and $s_2([z]_\Omega) = [-\bar{z}]_\Omega$ such that the quotient spaces $T/\langle s_1 \rangle$ and $T/\langle s_2 \rangle$ are annuli. Let $A_1 = T/\langle s_1 \rangle$. By Theorem 2.2, the automorphisms of A_1 are the conformal automorphisms of T commuting with s_1. Let $b \in F_\Omega$. Then the automorphism $[z]_\Omega \to [z+b]_\Omega$ of T commutes with s_1 if and only if for every $z \in \mathbf{C}$,

$$[\overline{z+b}]_\Omega = [\bar{z}+b]_\Omega \Leftrightarrow [\bar{b}]_\Omega = [b]_\Omega$$

$$\Leftrightarrow 0 \le b < 1 \text{ or } b = x + \gamma/2, (0 \le x < 1).$$

Thus, the automorphisms of A_1 obtained from $[z]_\Omega \to [z+b]_\Omega$ are as follows:

$$[z]_\Lambda \to [z+x]_\Lambda \quad (0 \le x < 1),$$

$$[z]_\Lambda \to [z+x+\gamma/2]_\Lambda \quad (0 \le x < 1),$$

where $\Lambda = \langle \Omega, s_1 \rangle$.

In the same way, from the automorphism $[z]_\Omega \to [-z+b]_\Omega$ of T we obtain the following automorphisms for A_1:

$$[z]_\Lambda \to [-z+x]_\Lambda \quad (0 \le x < 1),$$

$$[z]_\Lambda \to [-z+x+\gamma/2]_\Lambda \quad (0 \le x < 1).$$

Now let $A_2 = T/\langle s_2 \rangle$ and $\Lambda = \langle \Omega, s_2 \rangle$. As before, we can observe that the automorphisms $[z]_\Omega \rightarrow [z+b]_\Omega$ and $[z]_\Omega \rightarrow [-z+b]_\Omega$ of T commute with s_2 if and only if $b = ix$ $(0 \le x < |\gamma|)$ or $b = 1/2 + ix$ $(0 \le x < |\gamma|)$. Therefore, we find that the authomorphisms of A_2 are as follows:

$$[z]_\Lambda \rightarrow [\pm z + ix]_\Lambda \quad (0 \le x < |\gamma|),$$

$$[z]_\Lambda \rightarrow [\pm z + 1/2 + ix]_\Lambda \quad (0 \le x < |\gamma|).$$

The automorphisms of the annulus corresponding to the case $\gamma = i$ can be found in the same way, and we give the results of this section in Table 3.

Table 3

Case	Symmetry of T	Automorphisms of $T/\langle s \rangle$
$\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$	$s([z]_\Omega) = [\bar{z}]_\Omega$	$[z]_\Lambda \rightarrow [\pm z + x]_\Lambda$, $(0 \le x < 1)$ $[z]_\Lambda \rightarrow [\pm z + x + \gamma/2]_\Lambda$, $(0 \le x < 1)$
	$s([z]_\Omega) = [-\bar{z}]_\Omega$	$[z]_\Lambda \rightarrow [\pm z + ix]_\Lambda$, $(0 \le x < |\gamma|)$ $[z]_\Lambda \rightarrow [\pm z + 1/2 + ix]_\Lambda$, $(0 \le x < |\gamma|)$
$\gamma = i$	$s([z]_\Omega) = [\bar{z}]_\Omega$	$[z]_\Lambda \rightarrow [\pm z + x]_\Lambda$, $(0 \le x < 1)$ $[z]_\Lambda \rightarrow [\pm z + x + i/2]_\Lambda$, $(0 \le x < 1)$

5. Automorphisms of the Klein bottle

We know that Klein bottles can be obtained as the quotients of tori by symmetries in the cases where $\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$ and $\gamma = i$. Let $\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$ and Ω be the lattice generated by $z \rightarrow z+1$ and $z \rightarrow z+\gamma$. Let T be the torus $\mathbf{C}/\Omega$. As shown in [1], T admits two non-conjugate symmetries $s_1([z]_\Omega) = [\bar{z} + 1/2]_\Omega$ and $s_2([z]_\Omega) = [-\bar{z} + \gamma/2]_\Omega$ such that the quotient spaces $T/\langle s_1 \rangle$ and $T/\langle s_2 \rangle$ are Klein bottles. The automorphisms of these Klein Bottles can be found as before. For example, The Klein bottle $T/\langle s_1 \rangle$ admits the following automorphisms:

$$[z]_\Lambda \rightarrow [\pm z + x]_\Lambda \quad (0 \le x < 1/2, \Lambda = \langle \Omega, s_1 \rangle),$$

$$[z]_\Lambda \to [\pm z + x + \gamma/2]_\Lambda \quad (0 \le x < 1/2,\ \Lambda = \langle \Omega, s_1 \rangle)\ .$$

These automorphisms have also been determined by Bârz_ [2]. The automorphisms of the other Klein bottles are given in Table 4.

Table 4

Case	Symmetry of T	Automorphisms of $T/\langle s \rangle$
$\mathrm{Re}(\gamma) = 0, \mathrm{Im}(\gamma) > 1$	$s([z]_\Omega) = [\bar{z} + 1/2]_\Omega$	$[z]_\Lambda \to [\pm z + x]_\Lambda,\ (0 \le x < 1/2)$ $[z]_\Lambda \to [\pm z + x + \gamma/2]_\Lambda\ ,\ (0 \le x < 1/2)$
	$s([z]_\Omega) = [-\bar{z} + \gamma/2]_\Omega$	$[z]_\Lambda \to [\pm z + ix]_\Lambda,\ (0 \le x < \lvert\gamma\rvert)$ $[z]_\Lambda \to [\pm z + 1/2 + ix]_\Lambda\ ,\ (0 \le x < \lvert\gamma\rvert)$
$\gamma = i$	$s([z]_\Omega) = [\bar{z} + 1/2]_\Omega$	$[z]_\Lambda \to [\pm z + x]_\Lambda\ ,\ (0 \le x < 1/2)$ $[z]_\Lambda \to [\pm z + x + i/2]_\Lambda\ ,\ (0 \le x < 1/2)$

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