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# Positive Integral Operators With Analytic Kernels

*Can Murat DİKMEN*[1]

**Abstract**

In this paper we construct examples of positive definite integral kernels which are also analytic.

***Key words:*** *Integral operators, Cauchy integral formula, Positive definite kernels,*

**Abstract**

Bu çalışmada aynı zamanda analitik olan pozitif tanımlı integral çekirdek örneklerini oluşturacağız.

***Anahtar Kelimeler:*** *İntegral operatörler, Cauchy integral formülü, Pozitif tanımlı çekirdekler.*

## 1. INTRODUCTION

To construct examples of positive definite integral kernels which are also analytic, we need to recall the following definitions (see [2], [3], [4], [5]).

Throughout, let us denote the inner product on any complex Hilbert space *H* by $\langle .,. \rangle$. We let $\langle f, f \rangle^{1/2} = \|f\|$ and call it the norm of $f$.

**Definition 1.1.** (i) Let denote any interval (finite or infinite) on the real line. $L^2$ *(I)* is the space of Lebesgue measurable complex valued functions

$$f : I \to \mathbb{C}$$

1 Karaelmas Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Zonguldak
e-mail: canmuratdikmen@karaelmas.edu.tr

which are square integrable, in the sense that $\int_I |f(t)|^2 dt < \infty$, with pointwise operations and inner product

$$\langle f, g\rangle = \int_I f(t)\overline{g(t)}dt .$$

So the norm of $f$ is

$$\|f\|^2 = \int_I |f(t)|^2 dt < \infty .$$

(ii) Given two intervals $I, J$ $L^2(I \times J)$ = all measurable complex valued functions $k$ on $I \times J$ such that

$$\int_I \int_J |k(s,u)|^2 duds < \infty .$$

**Definition 1.2.** Let $H, H_1$ be Hilbert spaces. A linear operator $S : H_1 \to H$ is bounded if there exists some $M \in \mathbb{R}$ such that

$$\|Sf\| \le M\|f\| \text{ for all } f \in H_1 .$$

A linear operator $S : H_1 \to H$ is compact if given a bounded sequence $(f_n) \subseteq H_1$, there exists a subsequence $(f_{n_r}) \subseteq f_n$, $g \in H$ such that

$$Sf_{n_r} \to g .$$

We use $B(H_1, H)$ and $K(H_1, H)$ for the space of all bounded linear operators and for all compact operators from $H_1$ into $H$ respectively.

**Theorem 1.1.** If $S \in B(H_1, H)$, there exists a unique $S^* \in B(H, H_1)$, called adjoint of $S$, such that

$$\langle Sf, g\rangle_H = \langle f, S^* g\rangle_{H_1} .$$

If $H = H_1$ and $S = S^*$, then $S$ is called self-adjoint or symmetric.

**Definition 1.3.** Let $T$ be a self-adjoint linear operator on a Hilbert space $(H, \langle .,.\rangle)$. Then $T$ is called positive, written $T \ge 0$, if $\langle Tf, f\rangle \ge 0$ for all $f \in H$.

**Definition 1.4.** Let $I, J \subset \mathbb{R}$ be intervals and suppose $k \in L^2(I \times J)$, then the formula

$$Sf(s) = \int_J k(s,u) f(u) du$$

where $s \in I, f \in L^2(J)$, defines a compact linear operator $S$ mapping $L^2(J)$ into $L^2(I)$. The adjoint $S^* : L^2(I) \to L^2(J)$ is given by

$$S^* g(u) = \int_I g(t)\overline{k(t,u)}dt .$$

So if $g \in L^2(I)$

$$SS^*g(s) = \int_J S^*g(u)k(s,u)du$$
$$= \int_I \int_J g(t)\overline{k(t,u)}k(s,u)dtdu$$
$$= \int_I g(t)K(s,t)dt$$

where $K(s,t) = \int_J k(s,u)\overline{k(t,u)}du \quad s,t \in I.$

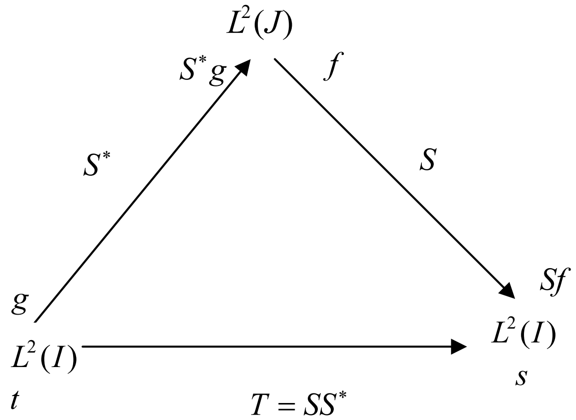


**Figure 1.1.**

It is well known that, because $k \in L^2(I \times J)$, interchanging the order of integral is legitimate and that $K \in L^2\left(I \times I\right)$.

**Theorem 1.2.** Here $T = SS^*$ is necessarily positive written $T \geq 0$ meaning that $\langle Tf, f \rangle \geq 0$ for all $f \in H$.

**Proof:** $\langle Tg, g \rangle_{L^2(I)} = \langle SS^*g, g \rangle_{L^2(I)} = \langle S^*g, S^*g \rangle_{L^2(J)} = \left\| S^*g \right\|_{L^2(J)}^2 \geq 0$.

Similarly $S^*S$ is positive operator on $L^2(J)$.

This gives us a method of constructing examples of positive integral operators on $L^2(I)$. Whenever $k \in L^2(I \times J),\ T = SS^*$ will be a positive integral operator on $L^2(I)$ with kernel

$$K(s,t) = \int_J k(s,u)\overline{k(t,u)}du.$$

**Definition 1.5.** Here $k$ is called kernel of $S$ and $K$ is called the kernel of $T$.

**Remark 1.3.** If $k(s,u) = l(s,u)h(u)$ where $|h(u)| = 1$ then

$$\int_J k(s,u)\overline{k(t,u)}du = \int_J l(s,u)\overline{l(t,u)}du.$$

**Remark 1.4.** A result analogous to theorem is true if the Lebesque measure on $J$ is nultiplied by a positive constant $m$ (usually $(1/2\pi)$). In this case we have

$$Sf(s) = \int_J k(s,u)f(u)(mdu)$$

where $s \in I, f \in L^2(J)$ and

$$S^* g(u) = \int_I \overline{k(s,u)}g(t)dt$$

where $u \in J, t \in I$ and $g \in L^2(I)$

$$Tf(s) = SS^* f(s) = \int_I K(s,t)g(t)dt$$

where $K(s,t) = \int_J k(s,u)\overline{k(t,u)}(mdu)$ .

Now, we will use this theorem to give examples of positive definite kernels $K$ using kernels $k$ which arise in a natural way in mathematical analysis. Specifically we consider $k$ 's which arise from Cauchy's integral formula (C.I.F.).

As a sequel we hope to give some more examples using same techniques considering the Fourier transformation and the Laplace transform (see [1]). In all cases $K$ will be an analytic kernel of $s$ and $t$.

## 2. Examples suggested by C.I.F.

In this section we will give some examples of positive integral operators suggested by Cauchy's integral formula which were obtained during my M.Sc. study (see [1] ).

We recall the parameterized Cauchy's integral formula. We parameterize the integral by taking $z = \varphi(u)$.

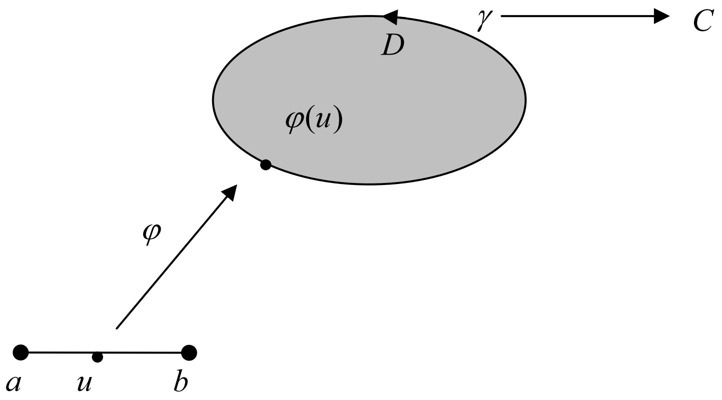


**Figure 2.1.**

Here γ is a positively oriented rectifiable Jordan curve and $D$ is its inner domain. Let $f$ be an analytic neighborhood of $D$ and $s \in D$

$$f(s) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-s} dz$$
$$= \frac{1}{2\pi i} \int_{a}^{b} \frac{f(\varphi(u))\varphi'(u)}{\varphi(u)-s} du.$$

**Example 2.1.** Suppose γ is the unit circle, $I = [a,b] \subseteq (-1,1)$. Here we shall take $J = [-\pi, \pi]$.

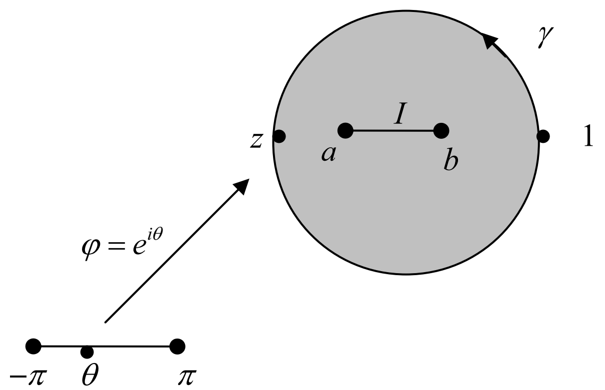


**Figure 2.2.**

We write the Cauchy's integral formula (C.I.F) to get our integral kernel

$$f(s) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(z)}{z-s}dz \quad (s \in I).$$

If we substitute $z = e^{i\theta}$ then $dz = ie^{i\theta}d\theta$ and

$$f(s) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(e^{i\theta})ie^{i\theta}}{e^{i\theta}-s}d\theta.$$

This suggests the linear operator $S : L^2([-\pi,\pi]) \to L^2(I)$ defined by

$$Sf(s) = \int_{-\pi}^{\pi} f(\theta)\frac{1}{e^{i\theta}-s}\frac{d\theta}{2\pi} \quad \left(k(s,\theta) = \frac{1}{e^{i\theta}-s}\right).$$

Hence

$$S^*g(\theta) = \int_I g(t)\frac{1}{e^{-i\theta}-t}dt \quad \left(\overline{k(t,\theta)} = \frac{1}{e^{-i\theta}-t}\right).$$

Here

$$k(s,\theta) = \frac{1}{e^{i\theta}-s} \in L^2(I \times J).$$

For this we need to show that they are square integrable:

$$\int_{-\pi}^{\pi}\int_I \frac{1}{\left|e^{i\theta}-s\right|^2}dsd\theta < \infty \tag{2.1}$$

Then, equation (2.1) is true since $k(s,\theta)$ is continuous on $I \times J$. So is $k(t,\theta)$.
So $SS^*$ has kernel

$$\begin{aligned} K(s,t) &= \int_{-\pi}^{\pi} k(s,t)\overline{k(t,\theta)}\, d\theta \\ &= \int_{-\pi}^{\pi}\frac{1}{\left(\mathrm{e}^{i\theta}-s\right)\left(\mathrm{e}^{-i\theta}-t\right)}\, d\theta. \end{aligned} \tag{2.2}$$

In general, if $h$ is a function on $\partial\Delta$ then

$$\int_{-\pi}^{\pi} h(e^{i\theta})ie^{i\theta}d\theta = \int_{\partial\Delta} h(z)\frac{dz}{2\pi}$$

so that

$$\int_{-\pi}^{\pi} h(e^{i\theta})d\theta = \int_{\partial\Delta} h(z)\frac{1}{iz}\frac{dz}{2\pi}. \tag{2.3}$$

Now if we use (2.3) in (2.2), then we get

$$K(s,t) = \frac{1}{2\pi}\int_{\partial\Delta}\frac{1}{(z-s)\left(\frac{1}{z}-t\right)}\frac{dz}{iz}$$
$$= \frac{1}{2\pi i}\int_{\partial\Delta}\frac{dz}{(z-s)(1-zt)}.$$

The poles of integrand are at $z = s$ and $z = 1/t$. Since $s,t \in I$, we know that $|s| < 1$, $|1/t| > 1$. Then we have only one pole at $z = s$.
Therefore,

$$K(s,t) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\frac{1}{1-zt}}{z-s}dz = \operatorname{Res}(f(z),s) = \frac{1}{1-st}.$$

Since $K$ is the kernel of $SS^*$, $K$ is positive definite on $L^2(I)$ where $I \subseteq (-1,1)$.
Now we will find another positive definite kernel for vertical strip.

**Example 2.2.** Let $\beta \in \mathbb{R}$ and let $D$ be the open half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > -\beta\}$. Let $\gamma$ be the boundary line of $D$ and suppose $I = [a,b] \subseteq D$, (i.e. $a > -\beta$), so that $s,t > -\beta$ where $s,t \in I$.

We shall now construct a positive integral operator on $L^2(I)$ whose kernel is derived from the Cauchy integral formula for functions analytic in a neighborhood of $D$.

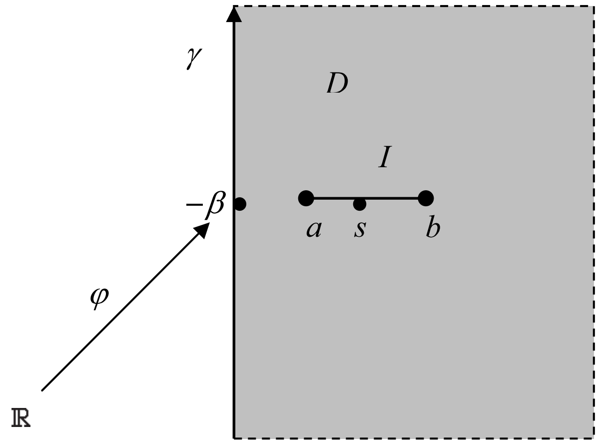


**Figure 2.3.**

We can parameterize γ by $\gamma = \varphi(u) = -\beta + iu,\ \varphi'(u) = i \quad -\infty < u < \infty$.
Then we have by C.I.F.

$$f(s) = -\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(z)}{z-s}dz = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(-\beta+iu).i}{-\beta-s+iu}du\,.$$

This suggests us the operator $S : L^2(\mathbb{R}) \to L^2(I)$ such that

$$Sf(s) = \int_{\mathbb{R}}\frac{f(u)}{\beta+s-iu}\frac{du}{2\pi}$$

so we have

$$k(s,u) = \frac{1}{\beta+s-iu}\,.$$

Here we have that $k(s,u) \in L^2(I \times \mathbb{R})$ because

$$\int_{\mathbb{R}}\int_I \frac{1}{|\beta+s-iu|^2}dsdu = \int_{\mathbb{R}}\int_I \frac{1}{(\beta+s)^2+u^2}dsdu.$$

Since the nearest point of $I$ to the line γ is $a$, we have that $(\beta+s)^2 \geq (\beta+a)^2$ for all $s \in I$. Then,

$$\int_{\mathbb{R}}\int_{I} \frac{1}{(\beta+s)^2+u^2}\,dsdu \leq \int_{\mathbb{R}}\int_{I} \frac{1}{(\beta+a)^2+u^2}\,dsdu$$

$$= \int_{\mathbb{R}} \frac{(b-a)}{(\beta+a)^2+u^2}\,du < \infty.$$

Hence

$$K_1(s,t) = \int_{\mathbb{R}} \frac{1}{(\beta+s-iu)(\beta+t+iu)}\frac{du}{2\pi}$$

$$= \int_{\mathbb{R}} \frac{1}{(u+i(\beta+s))(u-i(\beta+t))}\frac{du}{2\pi}.$$

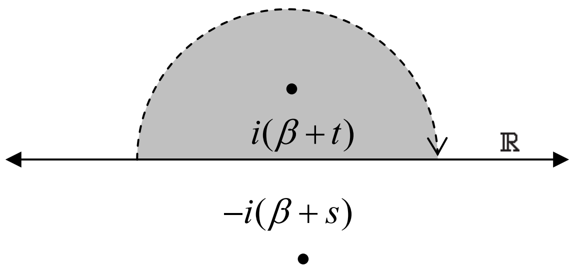


**Figure 2.4.**

The pole in the upper half plane is at $i(\beta+t)$. Say

$$\frac{1}{(u+i(\beta+s))(u-i(\beta+t))} = h(u)$$

then

$$\begin{aligned} K_1(s,t) &= i\operatorname{Res}(h(u), i(\beta+t)) \\ &= i\frac{1}{i(\beta+t)+i(\beta+s)} \\ &= \frac{1}{2\beta+s+t}. \end{aligned}$$

Since $K_1(s,t)$ is the kernel of $SS^*$, (2.4) is positive definite on $L^2(I)$.

Suppose now $\gamma = \beta + iu,\ u \in \mathbb{R}$ and $D$ is all points to the left of γ, that is $D = \{z \in £ : \operatorname{Re} z < \beta\}$ and that $I = [a,b] \subseteq D$ (i.e. $b < \beta$).

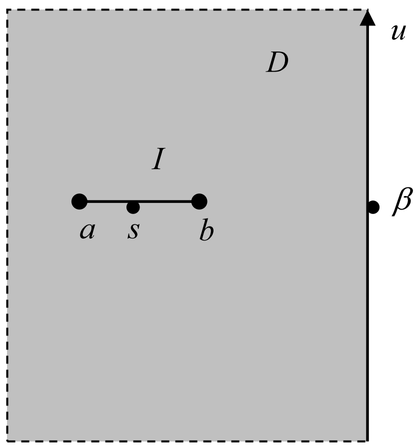


**Figure 2.5.**

In this case C.I.F. reads

$$f(s) = \frac{1}{2\pi i}\int \frac{f(z)}{z-s}dz = \frac{1}{2\pi i}\int \frac{if(\beta+iu)}{\beta+iu-s}du$$

which suggests the linear operator $S : L^2(\mathbb{R}) \to L^2(I)$ such that

$$Sf(s) = \int_{\mathbb{R}} \frac{f(u)}{\beta+iu-s}\frac{du}{2\pi}.$$

Then we have

$$k_2(s,u) = \frac{1}{\beta + iu - s} \quad s \in I, u \in \mathbb{R}.$$

Here $k_2(s,u) \in L^2$ (I x $\mathbb{R}$), because

$$\int_{\mathbb{R}} \int_{I} \frac{1}{|\beta + iu - s|^2} dsdu = \int_{\mathbb{R}} \int_{I} \frac{1}{(\beta + a)^2 + u^2} dsdu$$

$$\leq \int_{\mathbb{R}} \int_{I} \frac{1}{(\beta + a)^2 + u^2} dsdu$$

$$= \int_{\mathbb{R}} \frac{(b-a)}{(\beta - b)^2 + u^2} du < \infty.$$

So

$$K_2(s,t) = \int_{\mathbb{R}} \frac{1}{(\beta + iu - s)(\beta - iu - t)} \frac{du}{2\pi}$$

$$= \int_{\mathbb{R}} \frac{1}{(u - i(\beta - s))(u + i(\beta - t))} \frac{du}{2\pi}.$$

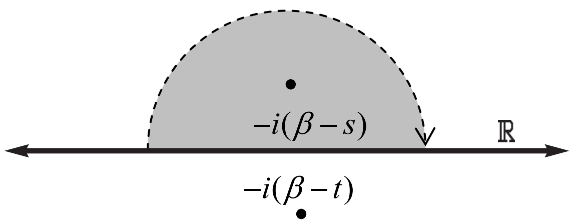


**Figure 2.6.**

The pole in the upper half plane is $i(\beta - s)$. Say

$$\frac{1}{(u - i(\beta - s))(u + i(\beta - t))} = h(u)$$

then $K_2(s,t) = i\operatorname{Res}(h(u), i(\beta - s))$. Hence

$$K_2(s,t) = \frac{1}{2\beta - s - t}. \tag{2.5}$$

Since $K_2$ is kernel of $SS^*$, (2.5) is positive definite on $L_2(I)$.

For the last part of our example we use the fact that the sum of two positive operators is positive. So if $\beta > 0$ and $I = [a,b] \subseteq (-\beta, \beta)$, we obtain a positive operator on $L_2(I)$ with kernel $K(s,t)$ which is analytic in $D \times D$.

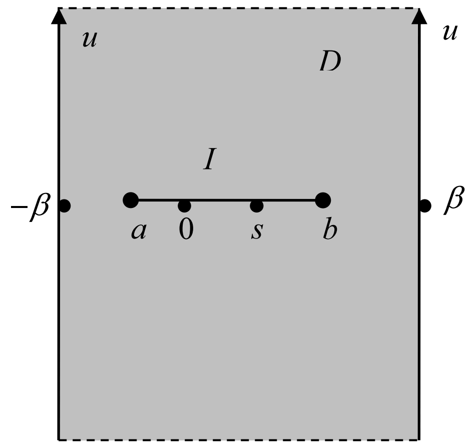


**Figure 2.7.**

Hence we have

$$\begin{aligned} K(s,t) &= K_1(s,t) + K_2(s,t) \\ &= \frac{1}{2\beta + s + t} + \frac{1}{2\beta - s - t} \\ &= \frac{4\beta}{4\beta^2 - (s+t)^2}. \end{aligned} \tag{2.6}$$

Again since $K$ is kernel of $S_1S_1^* + S_2S_2^*$, (2.6) is positive definite on $L_2(I)$.

We now give another example which is similar to the last one. This time $D$ will be the horizontal strip.

**Example 2.3.** Let $\beta > 0$ and let $D_1$ be the open half-plane $\{z \in £ : \operatorname{Im} z < \beta\}$. Let γ be the boundary line of $D_1$ and suppose $I = [a,b] \subseteq D_1$, $s,t \in I$.

We shall now construct a positive integral operator on $L_2(I)$ whose kernel is derived from the Cauchy integral formula for functions analytic in a neighborhood of $D_1$.

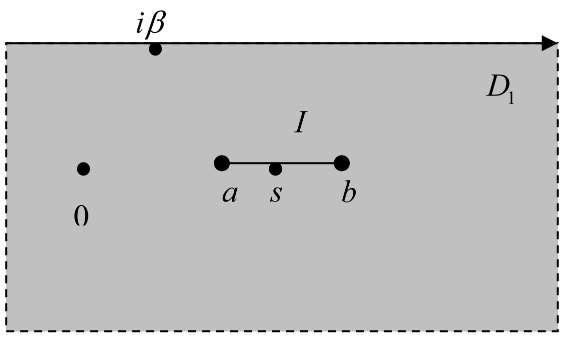


**Figure 2.8.**

This time C.I.F. reads

$$f(s) = -\frac{1}{2\pi i}\int_{\mathbb{R}+i\beta}\frac{f(z)}{z-s}\,dz.$$

If we put $z = i\beta + u$ then $dz = du$, then we get

$$f(s) = -\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(i\beta+u)}{i\beta+u-s)}\,du.$$

This suggests the linear operator $S : L^2(\mathbb{R}) \to L^2(I)$ defined by

$$Sf(s) = \int_{\mathbb{R}}\frac{f(u)}{i\beta+u-s}\frac{du}{2\pi}.$$

Hence

$$S^* g(u) = \int_I \frac{g(t)}{-i\beta+u-t}\,dt.$$

Then we have

$$k_1(s,t) = \frac{1}{i\beta + u - s} \text{ and } \overline{k_1(u,t)} = \frac{1}{-i\beta + u - t}.$$

Here $k_1(s,u) \in L^2(\text{I x } \mathbb{R})$, because

$$\int_{\mathbb{R}} \int_I \frac{1}{|i\beta + u - s|^2} dsdu = \int_{\mathbb{R}} \int_I \frac{1}{(u-s)^2 + \beta^2} dsdu$$

Let $u = s + \beta \tan\theta$ and $du = \beta \sec^2 \theta d\theta$. Then,

$$\int_{\mathbb{R}} \int_I \frac{1}{(u-s)^2 + \beta^2} dsdu = \int_a^b \int_{-\pi/2}^{\pi/2} \frac{\beta \sec^2 \theta}{\beta^2 \tan^2 \theta + \beta^2} d\theta ds$$

$$= \int_a^b \int_{-\pi/2}^{\pi/2} \frac{1}{\beta} d\theta ds$$

$$= \int_a^b \frac{\pi}{\beta} ds = \frac{(b-a)\pi}{\beta} < \infty.$$

Then we have

$$K_1(s,t) = \int_{\mathbb{R}} \frac{1}{(i\beta + u - s)(-i\beta + u - t)} \frac{du}{2\pi}$$

$$= \int_{\mathbb{R}} \frac{1}{(u-(s-i\beta))(u-(i\beta+t))} \frac{du}{2\pi}.$$

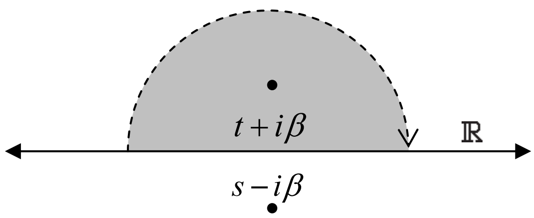


**Figure 2.9.**

The pole in the upper half plane is $\left(t+i\beta\right)$. Then,

$$\begin{aligned} K_1(s,t) &= i\operatorname{Res}(h(u),t+i\beta) \\ &= \frac{i}{t+i\beta-s+i\beta} \\ &= \frac{i}{t-s+2i\beta} \\ &= \frac{i\left(t-s-2i\beta\right)}{\left(t-s\right)^2+4\beta^2}. \end{aligned}$$

Hence

$$K_1(s,t) = \frac{2\beta+i(t-s)}{(t-s)^2+4\beta^2} = \overline{K_1(t,s)}\,.$$

Here $K_1(s,t)$ is symmetric and positive definite.

For the second part of our example we again let $\beta > 0$ and let $D_2$ be the open half-plane $\left\{z \in £ : \operatorname{Im} z > -\beta\right\}$. Let γ be the boundary line of $D_2$ and suppose $I=[a,b] \subseteq D_2$, $s,t \in I$. We shall now construct a positive integral operator on $L_2(I)$ whose kernel is derived from the Cauchy formula for functions analytic in a neighborhood of $D_2$.

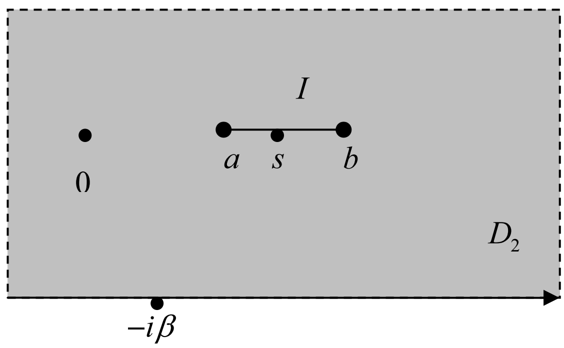


**Figure 2.10.**

We put $z=\varphi(u)=-i\beta+u$ and $dz=du$ in C.I.F.

$$f(s) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(-i\beta + u)}{-i\beta + u - s} du.$$

This suggests the linear operator $S : L^2(\mathbb{R}) \to L^2(I)$ such that

$$Sf(s) = \int_{\mathbb{R}} \frac{f(u)}{-(i\beta + s) + u} \frac{du}{2\pi}.$$

Here

$$k_2(s,u) = \frac{1}{-(i\beta + s) + u} \in L^2(I \times \mathbb{R}), \text{ because}$$

$$\int_{\mathbb{R}} \int_I \frac{1}{\left|-(i\beta + s) + u\right|^2} dsdu = \int_{\mathbb{R}} \int_I \frac{1}{(u - s)^2 + \beta^2} dsdu < \infty \qquad \text{by (2,7)}$$

Then,

$$K_2(s,t) = \int_{\mathbb{R}} \frac{1}{\left(-(i\beta + s) + u\right)\left(-(-i\beta + t) + u\right)} \frac{du}{2\pi}.$$

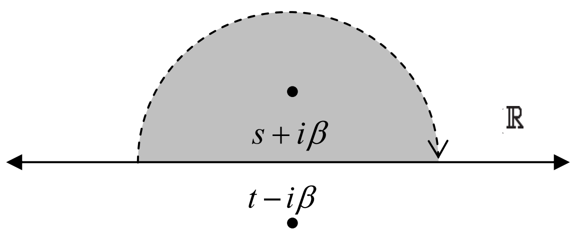


**Figure 2.11.**

The pole in the upper half plane is $s + i\beta$. Then,

$$\begin{aligned} K_2(s,t) &= i \operatorname{Res}(h(u), s + i\beta) \\ &= \frac{i}{(i\beta - t) + i\beta + s} \\ &= \frac{1}{2\beta + i(t - s)} \\ &= \frac{2\beta - i(t - s)}{(t - s)^2 + 4\beta^2} = \overline{K_2(t,s)}. \end{aligned}$$

Here $K_2(s,t)$ is symmetric and positive definite on $L_2(I)$.

For the last part of our example, we again use the fact that the sum of two positive operators is positive. So if $\beta > 0$ and $I = [a,b] \subseteq \mathbb{R}$, we obtain a positive operator on $L_2(I)$ with kernel $K(s,t)$.

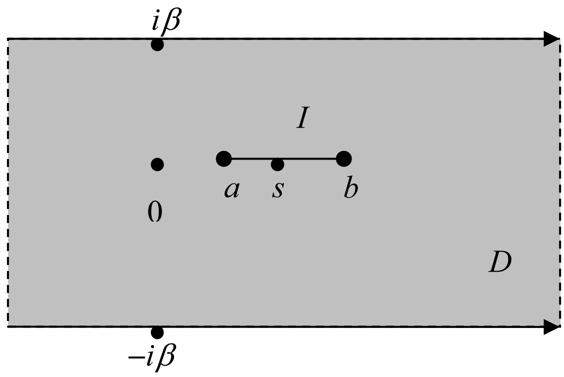


**Figure 2.12.**

$$K(s,t) = \frac{2\beta + i(t-s)}{(t-s)^2 + 4\beta^2} + \frac{2\beta - i(t-s)}{(t-s)^2 + 4\beta^2}$$
$$= \frac{4\beta}{4\beta^2 + (s-t)^2}. \qquad (2.8)$$

Then (2.8) is positive definite on $L_2(I)$.

We will now consider more general half-planes.

**Example 3.4.** Let $0 < \theta < \pi/2$. We define the two half planes by

$$D_1 = \{z \in \mathbb{C} : -\theta < \arg z < -\theta + \pi\}$$
$$D_2 = \{z \in \mathbb{C} : \theta - \pi < \arg z < \theta\}$$

$I = [a,b],\ a > 0$ so $I \subseteq D_1 \cap D_2$. Let $\gamma_1 = \partial D_1,\ \gamma_2 = \partial D_2$ and put $\omega = e^{i\theta}$. We can parameterize $\gamma_1$ by $\varphi(u) = \overline{\omega}\, u,\ u \in \mathbb{R}$.

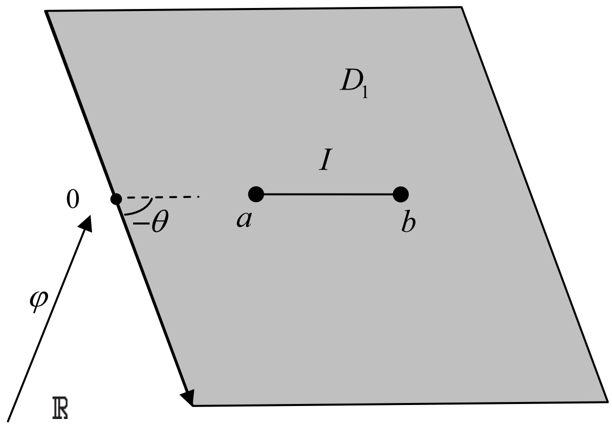


**Figure 2.13.**

So C.I.F. for $D_1$ can be written as

$$f(s)\,\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\overline{\omega} f(\overline{\omega}\, u)}{\overline{\omega}\, u - s}\, du$$

where we do not consider $\omega$ and $1/2\pi$ since from Remark 1.3. and Remark 1.4. This suggests the linear operator $S_1 : L^2(\mathbb{R}) \to L^2(I)$ defined by

$$Sf(s) = \int_{\mathbb{R}} \frac{f(u)}{\overline{\omega} u - s} \frac{du}{2\pi}.$$

Then,

$$k_1(s,u) = \frac{1}{\overline{\omega}\, u - s} \in L^2\,(\mathrm{I} \text{ x } \mathbb{R})$$

$$\int_{\mathbb{R}}\int_{I}\frac{1}{\left|\overline{\omega}u-s\right|^{2}}dsdu=\int_{\mathbb{R}}\int_{I}\frac{1}{|u\cos\theta-s-iu\sin\theta|}dsdu$$
$$=\int_{\mathbb{R}}\int_{I}\frac{1}{u^{2}+s^{2}-2us\cos\theta}dsdu$$
$$\leq\int_{\mathbb{R}}\int_{I}\frac{1}{\left(1-\cos\theta\right)(u^{2}+s^{2})}dsdu$$
$$\leq\frac{(b-a)}{(1-\cos\theta)}\int_{\mathbb{R}}\frac{1}{u^{2}+a^{2}}<\infty.$$

So that we have

$$K_{1}(s,t)=\int_{\mathbb{R}}\frac{1}{(\overline{\omega}u-s)(\omega u-t)}\frac{du}{2\pi}$$
$$=\int_{\mathbb{R}}\frac{1}{(u-\omega s)(u-\overline{\omega}t)}\frac{du}{2\pi}.$$

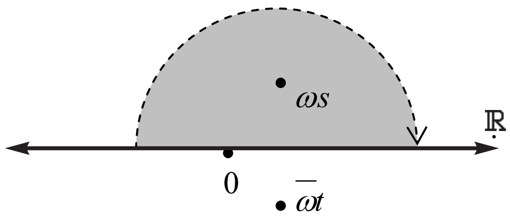


**Figure 2.14**

The pole in the upper half plane is $\omega s$ . Then,

$$\begin{aligned} K_1(s,t) &= i\operatorname{Res}(h(u),\omega s) = \frac{i}{\omega s - \overline{\omega t}} \\ &= \frac{i}{(s-t)\cos\theta + i(s+t)\sin\theta} \\ &= \frac{1}{(s+t)\sin\theta - i(s-t)\cos\theta} \\ &= \frac{(s+t)\sin\theta + i(s-t)\cos\theta}{(s+t)^2\sin^2\theta + (s-t)^2\cos^2\theta} = \overline{K_1(t,s)}. \end{aligned}$$

Then we know that $K_1(s,t)$ is symmetric and positive definite on $L_2(I)$.

Now we will construct our kernel for $D_2$. We can parameterize $\gamma_2$ by $\varphi(u) = \omega u,\ u \in \mathbb{R}$ .

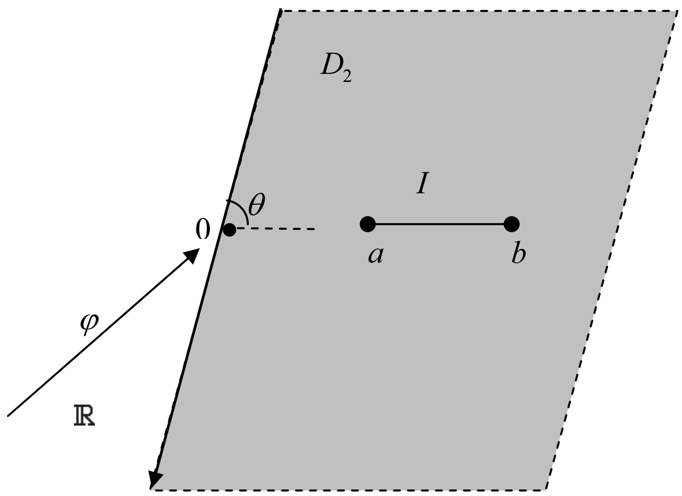


**Figure 2.15.**

So C.I.F. for $D_2$ can be written as

$$f(s) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\omega f(u)}{\omega u - s}\,du.$$

This suggests us the operator $S$ from $L_2(\mathbb{R})$ to $L_2(I)$ such that

$$Sf(s) = \int_{\mathbb{R}} \frac{f(u)}{\omega u - s}\frac{du}{2\pi}.$$

Hence

$$S^* g(u) = \int_I \frac{g(t)}{\overline{\omega} u - t}\,dt.$$

Similarly

$$k_2(s,u) = \frac{1}{\omega u - s} \in L^2(I \text{ x } \mathbb{R}).$$

Then we have

$$K_2(s,t) = \int_{\mathbb{R}} \frac{1}{(\omega u - s)(\overline{\omega} u - t)}\frac{du}{2\pi}$$

$$= \int_{\mathbb{R}} \frac{1}{(u - \overline{\omega} s)(u - \omega t)}\frac{du}{2\pi}.$$

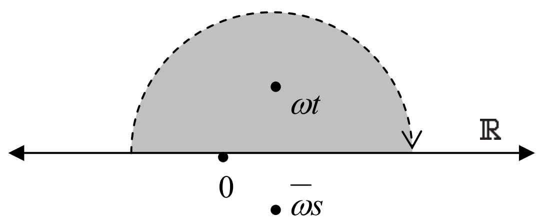


**Figure 2.16.**

The pole in the upper half plane is $\omega t$ . Then,

$$K_2(s,t) = i\,\mathrm{Res}(h(u),\omega t) = \frac{i}{\omega t - \overline{\omega s}}$$

$$= \frac{i}{(t-s)\cos\theta + i(t+s)\sin\theta}$$

$$= \frac{1}{(t+s)\sin\theta - i(t-s)\cos\theta}$$

$$= \frac{(t+s)\sin\theta + i(t-s)\cos\theta}{(s+t)^2\sin^2\theta + (s-t)^2\cos^2\theta} = \overline{K_2(t,s)}.$$

Then, we know that $K_2(s,t)$ is symmetric and positive definite on $L_2(I)$.

Now for the last part of the example we use the fact that the sum of two positive operators is positive.

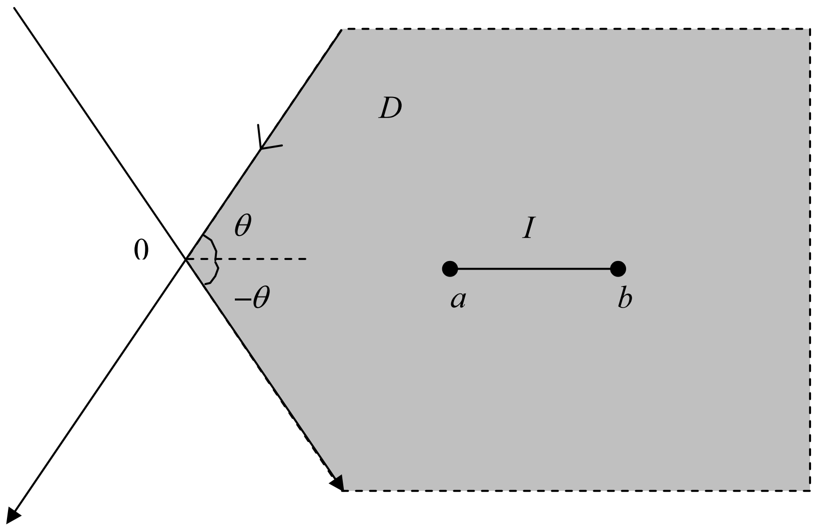


**Figure 2.17.**

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