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Assessment of an Analytical Approach in Solving Two Strongly Boundary Value Problems

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Abstract- In this research, a powerful analytical method called Reconstruction of Variational Method (RVIM) is introduced to handle two boundary value problems. One is a parameterized sixth order boundary value problem and the other is a nonlinear boundary value problem arising in the study of thin film flow of a third grade fluid down an inclined plane. With similarity method, the governing equations can be reduced to a system of nonlinear ordinary differential equations. The effectiveness of the method, which is independent of the small parameter, is investigated by comparing the results obtained with the numerical ones (4th order Rung-kutta method) and the exact ones. For the second problem the velocity profile is plotted and the effect of varying the material constant on the velocity profile is studied.

Keywords- *Rotating disk; Condensation film; Analytical approach; Reconstruction of Variational Method, RVIM*

I. INTRODUCTION

Nonlinear problems and phenomena play an important role in applied mathematics, physics, engineering and other branches of science. Except for a limited number of these problems, most of them do not have precise analytical solutions; therefore, these nonlinear equations should be solved using approximation methods.

The perturbation method is one well-known method to solve nonlinear equations. But since, using the common perturbation method is based on the existence of a small parameter, developing the method for different application is difficult. Therefore, many different new techniques have been recently introduced to eliminate the small parameter, such as the Adomian decomposition method [1-3], the Homotopy analysis method [4-6], the Homotopy perturbation method [7-10].

In this letter we employ a new and effective analytical method named Reconstruction of Variational Iteration method (RVIM) to solve two strongly boundary value problems. By applying Laplace Transform, RVIM overcomes the difficulty of the perturbation techniques and other variational methods in case of using small parameters and Lagrange multipliers, respectively. Reducing the size of calculations and omitting the difficulty arising in calculation of nonlinear intricately terms are other advantages of this method. Besides, it provides us with a simple way to ensure the convergence of solution series, so that we can always get accurate enough approximations even in first orders of the result iteration.

II. RVIM METHOD AND CONVOLUTION THEOREM

In this section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform will be introduced [11 -12]. Suppose x is the independent variable; applying Laplace transform to $u(x,t)$ with respect to x as variable, we have

$$\mathcal{L}\{u(x,t);s\} = \int_0^\infty e^{-st} u(x,t)dt \quad (1)$$

$$U(s) = \mathcal{L}\{u(x);s\} \quad (2)$$

We often come across functions which are not the transform of known functions. But, by means of the convolution theorem, we can take the inverse Laplace transform. The convolution of $u(x)$ and $v(x)$ is written as $u(x)*v(x)$. It is defined as the integral of the functions after one is reversed and shifted. If $U(s)$ and $V(s)$ are the Laplace transforms of $u(x)$ and $v(x)$, respectively. Then $U(s)*V(s)$ is the Laplace Transform of $\int_0^x u(x-\varepsilon)\cdot v(\varepsilon)d\varepsilon$ so by taking inverse Laplace Transform as below,

$$\mathcal{L}^{-1}\{U(s)*V(s)\} = \int_0^x u(x-\varepsilon)\cdot v(\varepsilon)d\varepsilon \quad (3)$$

To illustrate the concept of the RVIM, we consider the following general differential equation

$$L(u(x)) + N(u(x)) = f(x) \quad (4)$$

Where L and N are linear and nonlinear operators respectively and $f(x)$ is the forcing term. To facilitate our discussion of RVIM, with introducing the new function $h(u(x)) = f(x) - N(u(x))$ and considering the new equation, Eq. (4) can be rewritten as,

$$L(u(x)) = h(u, x) \tag{5}$$

Now, for implementation the RVIM technique based on new idea of Laplace transform, we apply Laplace Transform on both sides of the Eq. (5). Introducing all artificial initial conditions as zero for the main problem, the left side of the equation after transformation will be

$$\mathcal{L}\{L(u(x))\} = U(s)P(s) \tag{6}$$

Where $P(s)$ is polynomial with the highest order derivative of the selected linear operator

$$\mathcal{L}\{L(u(x))\} = U(s)P(s) = \mathcal{L}\{h(u, x)\} \tag{7}$$

$$U(s) = \frac{\mathcal{L}\{h(u, x)\}}{P(s)} \tag{8}$$

Suppose that $D(s) = \frac{1}{P(s)}$ and $\mathcal{L}\{h(u, x)\} = H(s)$. Using the convolution theorem we have;

$$U(s) = D(s) \cdot H(s) = \mathcal{L}\{d(x) * h(u, x)\} \tag{9}$$

Taking the inverse Laplace transform on both sides of Eq. (9)

$$u(x) = \int_0^x d(x - \varepsilon) \cdot h(u, \varepsilon) d\varepsilon \tag{10}$$

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x) = u_0(x) + \int_0^x d(x - \varepsilon) \cdot h(u_n, \varepsilon) d\varepsilon \tag{11}$$

III. A PARAMETERIZED SIXTH ORDER BOUNDARY VALUE

A. Governing Equation

Consider the following problem

$$u^{(6)}(x) = (1+c)u^{(4)}(x) - cu''(x) + cx, \quad 0 < x < 1, \tag{12}$$

With boundary conditions:

$$u(0) = 1,\ u'(0) = 1,\ u''(0) = 0,$$
$$u(1) = \frac{7}{6} + \sinh(1),\ \ u'(1) = \frac{1}{2} + \cosh(1),\ \ u''(1) = 1 + \sinh(1). \tag{13}$$

The exact solution of this problem is:

$$u(x) = 1 + \frac{1}{6}x^3 + \sinh(x). \tag{14}$$

We see the exact solution of this problem does not depend on the parameter c but the problem itself does. This can be viewed by rewriting Eq.(12) as

$$\{u^{(6)}(x) - u^{(4)}(x)\} - c\{u^{(4)}(x) + u''(x) + x\} = 0, \tag{15}$$

Which shows that, no matter what the value of c is, a solution of fourth order problem is also a solution of the sixth-order problem.

B. Implementation of RVIM

Considering the initial approximations for u as follow

$$u_0(x) = ax^5 + bx^4 + dx^3 + x + 1, \tag{16}$$

Rewriting equations (12), based on selective linear operator we have

$$u^{(6)}(x) = (1+c)u^{(4)}(x) - cu''(x) + cx, \tag{17}$$

Now Laplace transform is implemented with respect to independent variable x on both sides of Eqs. (17). Using the new artificial boundary conditions (which all of them are zero) we have

$$s^6U(s) = L\left\{(1+c)u^{(4)}(x) - cu''(x) + cx\right\}, \tag{18}$$

By using the Laplace inverse Transform and convolution theorem, it is concluded that

$$u(x) = \int_0^x \frac{(x-\varepsilon)^5}{120}\left((1+c)u^{(4)}(\varepsilon) - cu''(\varepsilon) + c\varepsilon\right)d\varepsilon \tag{19}$$

Hence, we arrive at the following iterative formula for the approximate solution of (12), subject to the boundary conditions (13),

$$u_{n+1}(x) =$$
$$u_0(x) + \int_0^x \frac{(x-\varepsilon)^5}{120}\left((1+c)u_n^{(4)}(\varepsilon) - cu_n''(\varepsilon) + c\varepsilon\right)d\varepsilon \tag{20}$$

According to above equations, for first order approximation we have:

$$u_1(x) =$$
$$u_0(x) + \int_0^x \frac{(x-\varepsilon)^5}{120}\left((1+c)u_0^{(4)}(\varepsilon) - cu_0''(\varepsilon) + c\varepsilon\right)d\varepsilon \tag{21}$$

With substituting boundary conditions (13) in the iterative formula (20) the unknown constants of the initial approximation (16) will be determined and with pitting them in the result formula the final answer is approached. More the order of the iteration growth more the accuracy of the solution increases. We calculated the first order of approximation for $c = 1000$:

$$\begin{aligned} u_1(x) &= 1 + 0.00001516539800x^{11} \\ &-0.0002562231482x^{10} + 0.00407918271x^9 \\ &+0.02306008083x^8 - 0.09516837381x^7 \\ &+0.09623299760x^6 + x + 0.005044617900x^5 \\ &-0.03874093580x^4 + 0.3476013510x^3 \end{aligned} \tag{22}$$

In Table 1, results obtained from the first order of RVIM are compared with the exact results (4th order Rung-Kutta method) and the Error value is compared with the Error of two other approximate methods called HAM and OHAM. It is revealed that a good accuracy to the exact results is achieved and RVIM is more accurate and rapid than two other methods, in converging to the exact results.

TABLE1. COMPARISON BETWEEN RVIM RESULTS AND NUMERICAL RESULTS AND HAM AND OHAM RESULTS FOR C=1000

x	Results			Errors (|Num. – Obtained Results|)		
	Numerical	1th order RVIM		HAM$_4$ [13]	OHAM$_3$[14]	RVIM$_1$
0	1.100333417	1.100343863		9.1E-06	1.1E-05	1.04E-05
0.1	1.202669336	1.202725442		1.6E-04	3.3E-06	5.61E-05
0.2	1.309020293	1.309134624		4.4E-04	1.4E-05	1.1E-04
0.3	1.421418993	1.421560777		6.8E-04	5.2E-06	1.4E-04
0.4	1.541928638	1.542044445		7.3E-04	4.2E-05	1.1E-04
0.5	1.672653582	1.672706014		5.8E-04	5.7E-05	5.24E-05
0.6	1.815750369	1.815744644		3.2E-04	4.9E-05	5.7E-06
0.7	1.973439315	1.973415403		9.8E-05	4.5E-05	2.39E-05
0.8	2.148016726	2.148008063		4.7E-06	2.4E-05	8.66E-06
0.9	1.100333417	1.100343863		9.1E-06	1.1E-05	1.04E-05

IV. THIN FILM FLOW PROBLEM

A. Governing Equations

The thin film flow of a third grade fluid down an inclined plane of inclination $\alpha \neq 0$ is governed by the following nonlinear boundary value problem [15]

$$\frac{d^2u}{dy^2}+\frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}+\frac{\rho g\sin\alpha}{\mu}=0 \tag{23}$$

$$u(0)=0,\ \frac{du}{dy}=0 \text{ at } y=\delta. \tag{24}$$

Introducing the parameters

$$y=\delta y^*,\ u=\frac{\delta^2\rho g\sin\alpha}{\mu}u^*,\quad \beta^*=\frac{3\delta^2\rho^2 g\sin\alpha}{\mu^3}(\beta_2+\beta_3) \tag{25}$$

The problem in Equations (23) and (24), after omitting asterisks, takes the following form

$$\frac{d^2u}{dy^2}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}+1=0, \tag{26}$$

$$u(0)=0,\ \frac{du}{dy}=0 \text{ at } y=1, \tag{27}$$

where μ is the dynamic viscosity, g is the gravity, ρ is the fluid density and $\beta>0$ is the material constant of a third grade fluid. We note that Equation (26) is a second order nonlinear and inhomogeneous differential equation with two boundary conditions; therefore, it is a well-posed problem.
Through integration of Equation (26) we have

$$\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3+y=C_1, \tag{28}$$

where C_1 is a constant of integration. Employing the second condition of (27) in Equation (28), we obtain $C_1=1$. Thus, the system (26)-(27) can be written as

$$\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3+(y-1)=0, \tag{29}$$

$$u(0)=0 \tag{30}$$

B. Implementation of RVIM

In this section, we employ RVIM to solve Equation (29). The initial guess is in the following form:

$$u_0(y)=ay, \tag{31}$$

Rewriting equations (29), based on selective linear operator we have

$$\frac{du}{dy}=-2\beta\left(\frac{du}{dy}\right)^3+(1-y), \tag{32}$$

Now Laplace transform is implemented with respect to independent variable yon both sides of Eqs. (32). Using the new artificial boundary conditions (which all of them are zero) we have

$$sU(s)=L\left\{-2\beta\left(\frac{du}{dy}\right)^3+(1-y)\right\}, \tag{33}$$

By using the Laplace inverse Transform and convolution theorem, it is concluded that

$$u(y)=\int_0^y\left(-2\beta\left(\frac{du}{d\varepsilon}\right)^3+(1-\varepsilon)\right)d\varepsilon \tag{34}$$

Hence, we arrive at the following iterative formula for the approximate solution of (29), subject to the boundary condition (30),

$$u_{n+1}(y)=u_0(y)+\int_0^y\left(-2\beta\left(\frac{du_n}{d\varepsilon}\right)^3+(1-\varepsilon)\right)d\varepsilon \tag{35}$$

According to above equations, for first order approximation we have:

$$u_1(y)=u_0(y)+\int_0^y\left(-2\beta\left(\frac{du_0}{d\varepsilon}\right)^3+(1-\varepsilon)\right)d\varepsilon \tag{36}$$

With substituting boundary condition (30) in the iterative formula (35) the unknown constant of the initial approximation, *a*, will be determined and with pitting it in the result formula the final answer is approached. We calculated the second and third order of approximations when $\beta=0.1$:

$$u_1(y)=y-0.5x^2 \tag{37}$$

$$u_2(y)=0.9216606811y+0.02499999999y^4 -0.09216606814y^3-0.3725812384y^2 \tag{38}$$

$$\begin{aligned}u_3(y)&=0.9216989942y-0.0000099999999984y^{10}\\&+0.000009216989946y^9-0.000007288066267y^8\\&-0.00185904286y^7+0.002620995626y^6\\&+0.01322497812y^5-0.02451410684y^4\\&-0.02768570224y^3-0.40504871258y^2\end{aligned} \tag{39}$$

The third and fourth orders are calculated but not mentioned for brevity. All the results presented in figures and table in the next section are obtained from the fourth order of RVIM solution.

C. Results and Discussion

The effectiveness of Reconstruction of Variational Method (RVIM) is depicted in Table 2 and Figs 1 and 2. The results are well matched with the results carried out by numerical solution (Runge–Kutta). In Table 2, error is introduced as follow:

$$Error = \left| u(y)_{NM} - u(y)_{RVIM} \right| \quad (40)$$

Errors of first four orders of the approximation are presented in Table 2. It is obvious that the solution rapidly converges to the exact results and just in the 4th order of the iteration we got efficient accuracy.

Figs. 1 and 2 describes the comparison between the RVIM and numerical results for β =0.05 and β =0.1. An excellent agreement is observed. This accuracy gives high confidence in the validity of this problem, and reveals an excellent agreement in engineering accuracy.

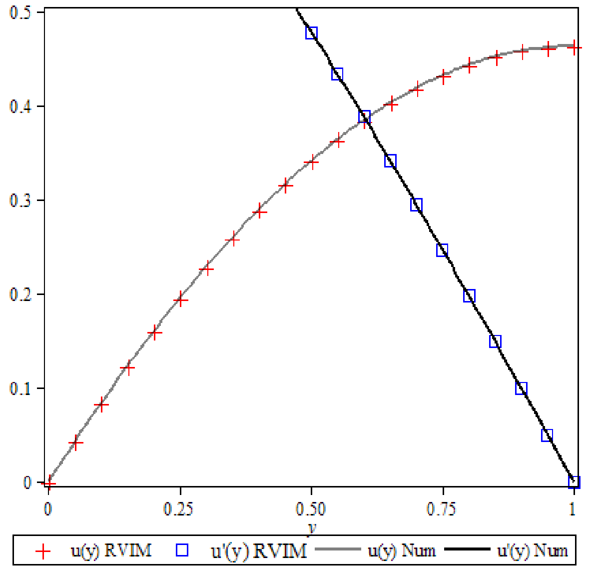

Fig. 2.Comparison between RIVM and Numerical results for $\beta = 0.1$

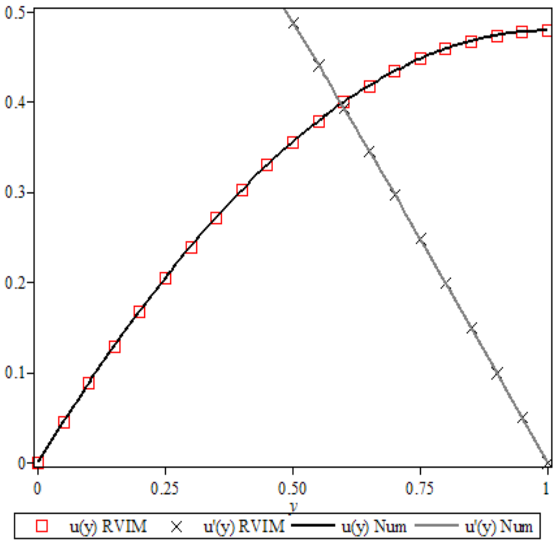

Fig. 1.Comparison between RIVM and Numerical results for $\beta = 0.05$

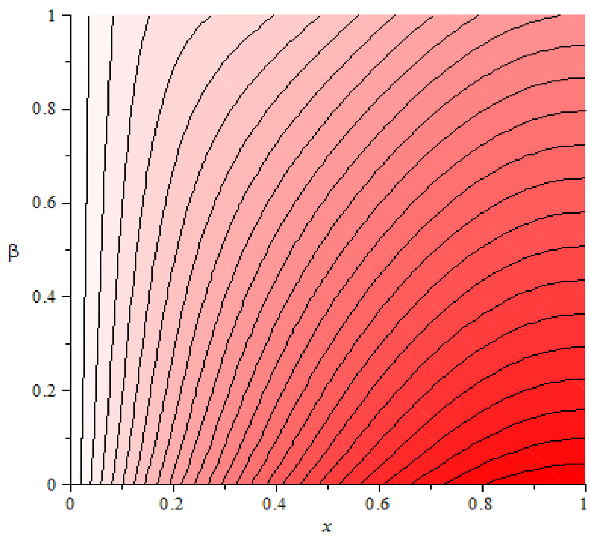

Fig. 3.Contour of *u(y)* for different values of material constant from 0 to

TABLE2. COMPARISON BETWEEN NUMERICAL RESULTS AND FIRST 4 ORDERS OF RVIM SOLUTION

x	Result		Error			
	Numerical	4th order RVIM	1th order	2nd order	3rd order	4th Order
0	0	0	0	0	0	0
0.1	0.088138419	0.088149704	0.006861581	0.000212171	4.89931E-05	1.12857E-05
0.2	0.168025973	0.168055406	0.011974027	0.000705585	0.000144392	2.94331E-05
0.3	0.239379703	0.239423264	0.015620297	0.001300206	0.000236682	4.35612E-05
0.4	0.301920879	0.301972623	0.018079121	0.001871766	0.000305136	5.17436E-05
0.5	0.355381573	0.355437002	0.019618427	0.002345199	0.000346851	5.54283E-05
0.6	0.399512222	0.399568956	0.020487778	0.00268707	0.000367566	5.67343E-05
0.7	0.434090632	0.434147761	0.020909368	0.002896576	0.000375465	5.71286E-05
0.8	0.45893138	0.458988537	0.02106862	0.002996145	0.000377559	5.71573E-05
0.9	0.473894277	0.473951351	0.021105723	0.003022969	0.000377905	5.7074E-05
1	0.478891578	0.478948856	0.021108422	0.003021797	0.000377705	5.72784E-05

In figure 3 the effects of varying β have been investigated. As it is shown by contour in this figure the concentration of *u(y)* occurs in $x=1$ for small amount of material constant. It is illustrated that with increasing β the amount of *u(y)* decreases generally. In low values of x these changes are not sensible, while in greater x values, the changes of *u(y)* are more noticeable.

V. CONCLUSION

In this paper, a strong analytical method called Reconstruction of Variational Iteration Method (RVIM) has been successfully applied to find explicit solutions of two boundary value problems, which may occur in different fields of science and engineering. Both of illustrating examples confirm the convenience, reliability and efficiency of this method. RVIM can be introduced to overcome the limitations and difficulties existing in other approximate method, e.g. large computation need, use of small parameters, convergence in high orders of approximations. It is predicted that RVIM can be used widely in mathematical, physical and engineering problems, due to its simplicity and efficiency.

REFERENCES

[1] G. Adomian, Nonlinear dissipative wave equations, Appl. Math. Lett.11 (1998) 125_126.

[2] G. Adomian, R. Rach, Modified decomposition solution of linear and nonlinear boundary-value problems. Nonlinear Analysis: Theory, Methods & Applications Volume 23 Issue 5, Sept. 1994 Pages 615 – 619.

[3] Abdul-Majid Wazwaz, A comparison between the Adomian decomposition method and the Taylor series method in the series solutions, Applied Mathematics and Computation, 97(1998), 37 - 44.

[4] Abdul-Majid Wazwaz, The variational iteration method (VIM) for solving linear and nonlinear Volterra integral and integro differential equations, International Journal of Com. Math., 87(5) (2010) 1131-1141.

[5] D.D.Ganji, Hafez Tari, H.Babazadeh, The application of He’s variational iteration method to nonlinear equations arising in heat transfer.Physics Letters A 363 (3)(2007)213.

[6] M.Rafei, D.D.Ganji, H.Daniali,H.Pashaei,The variational iteration method for nonlinear oscillators with discontinuities .Journal of Soundand Vibration 305 (2007) 614.

[7] A. Sadighi, D.D. Ganji, Analytic treatment of linear and nonlinear Schrödinger equations: A study with Homotopy perturbation and Adomian decomposition methods. Phys. Lett. A 372 (2008) 465.

[8] S.H. HoseinNia, H. Soltani, J. Ghasemi, A.N. Ranjbar, D.D. Ganji, maintaining the stability of nonlinear differential equations by the enhancement of HPM. Phys.Lett. A 372 (2008) 2855-2861.

[9] M. Esmaeilpour and D. D. Ganji, “Application of He’s Homotopy perturbation method to boundary layer flow and convection heat transfer over a flat plate,” Physics Letters A, vol. 372, no. 1, pp. 33–38,2007.

[10] D.D. Ganji, G.A. Afrouzi, H. Hosseinzadeh, R.A. Talarposhti, Fourth order Volterraintegro-differential equations using modified Homotopy perturbation method. Phys. Lett. A 371 (2007) 20.

[11] A.A. Imani , D.D. Ganji , Houman B. Rokni , H. Latifizadeh , E.Hesameddini , M. HadiRafiee. Approximate traveling wave solution for shallow water wave equation. Applied Mathematical Modeling 36 (2012) 1550–1557

[12] E.Hesameddini, H.Latifizadeh, Reconstruction of Variational Iteration Algorithms using the Laplace Transform, Int. J. Nonlinear Sci. Numerical Simulation 10 (2009)1377-1382

[13] Siraj-Ul-Islam, Sirajul-Haq and Javed Ali, Numerical solution of special 12th-order boundary value problems using differential transform method, Comm.Nonl.Sc.Nu.Sim., 14, 4, pp. 1132-1138, 2009.

[14] Javed ALI, Saeed ISLAM, Hamid KHAN, Gul ZAMAN, The solution of a parameterized sixth order boundary value problem by the optimal asymptotic method, Proceedings of the Romanian Academy S.A, Vol. 12, pp.167-172.

[15] A. M. Siddiqui, R.Mahmood and Q. K. Ghori, “Homotopy Perturbation Method for Thin Film Flow of a Third Grade Fluid Down an Inclined Plane,” Chaos, Solitons& Fractals, Vol. 35, No. 1, 2008, pp. 140-147.

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