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ON THE VALUED FUNCTION FIELDS WITH GIVEN VALUE GROUP AND RESIDUE FIELD OR WITH GIVEN RESIDUE FIELD AND GENUS

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ABSTRACT

The valued function fields with given value group and residue field or with given genus and residue field are studied.
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DEĞER GRUBU VE REZİDÜ CİSMİ VEYA REZİDÜ CİSMİ VE CİNSİ VERİLEN DEĞERLENMİŞ FONKSİYON CİSİMLERİ HAKKINDA

ÖZET

Bu çalışmada değer grubu ve rezidü cismi veya rezidü cismi ve cinsi verilen değerlenmiş cisimler çalışılmıştır.
Anahtar Sözcükler: Değerlenmiş cisimler, değerlendirmeler, fonksiyon cisimleri, değer grupları, rezidü cisimleri.

1. INTRODUCTION

Let K be a field, v be a nontrivial valuation on K , G_v and k_v be the value group and residue field of v respectively. The aim of this paper is giving the theorems which study the extensions of valuations to simple transcendental extensions with given residue field and value group and the valued function fields with given residue fields and genus.

Firstly the theorem which states that when k/k_v is a finite extension and $G_v \subseteq G_1 \subseteq G$ are ordered groups such that $[G_1;G_0] < \infty$ and also G is the direct sum of G_1 and an infinite cyclic group then there exists an extension of v to $K(x)$ with the value group G and the residue field k is given.

Then the similar theorem is given for the pairwise independent valuations $v_1, v_2, ..., v_n\ (n \geq 1)$ on K . After then the theorem which is stated as follow is given:

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If k is a function field can be expressed as a quadratic extension of some simple transcendental extension of k_v and g is an integer not less than the genus of k/k_v then there exists a function field F of one variable over K having genus g and and there exists an extension w of v to F such that the residue field of w is k_v – isomorphic to k. The theorems are proved by S.K.Khanduja in 1991,1996,1997.

2. PRELIMINARIES

Let K be a field and v be a valuation on K. Throughout the paper G_v denotes the value group of v and k_v denotes the residue field of v.

Let w be an extension of v to $K(x)$ and we suppose that G_w / G_v is not torsion group. We will write that

$N = \min\{\deg f(x) | f(x) \in [x], w(f)$ is not torsion over $G_v\}$

$S = [k_w ; k_v],\ T = [G_1 ; G_v],$

$G_1 = \{g \in G_w | g$ is torsion over $G_v\}$

v^t denotes the valuation which is defined by

$$v(\sum_{i=1}^{m} a_i t^i) = \min_i (v(a_i)),\quad a_i \in K$$

for any $t \in K(x) \setminus K$ and is called Gaussian valuation on $K(t)$.

For any η in the valuation ring of v we shall denote by η^* its $v-$ residue, i.e. the image of η under the canonical homomorphism from the valuation ring of v onto the residue field of v.

3. RESULTS

Theorem 1: Let (K, v) be a non-trivially valued field having value group G_v and residue field k_v Let $G_v \subseteq G_1 \subseteq G$ be an inclusion of totally ordered abelian groups with $[G_1 : G_0] \langle \infty$ such that G is the direct sum of G_1 and an infinite cyclic group, and let k be a finite extension of k_v. Then there exists an (explicitly constructible) extension w of v to $K(x)$ such that the residue field of w is k, its value group is G, and the equality $N = ST$ holds for the extension w/v.

Moreover there exists a polynomial $P(x)$ in $K[x]$ of degree $[G_1 : G_v]\ [k : k_v]$ with the property that if v_1 is any prolongation of v to $K(x)$ with $v_1(P(x)) > 0$, then the value group of v_1 contains G_1 and its residue field contains k.

Proof: Suppose that the finite extension k is generated by non zero elements $\beta_0,...,\beta_{s-1} (s \geq 1)$ over k_v. Define

$n_0 = 1 \qquad n_1 = [k_v(\beta_0),: k_v],..., \qquad n_s = [k_v(\beta_0,...,\beta_{s-1}) : k_v(\beta_0,...,\beta_{s-2})]$,

so that $[k : k_v] = n_0 n_1 ... n_s$

Since G_1 / G_v is a finite abelian group and hence is a direct sum of cyclic groups, there exist $\mu_0,...,\mu_m$ in G_1 and integers $n_{s+1},...,n_{s+m}$ such that $G_1 = G_v + Z\mu_1 + ... + Z\mu_m$; $n_{s+1}\mu_1,...,n_{s+m}\mu_m$ are in G_v and that $n_{s+1}...n_{s+m} = [G_1 : G_v]$. We choose $a_1,...,a_m$ in K so that $n_{s+i}\mu_i = v(a_i)$ for $1 \leq i \leq m$.

We shall define polynomials $g_0(x), g_1(x),..., g_{s+m}(x)$ in $K[x]$ of degree $n_0, n_0 n_1,..., n_0 n_1 ... n_{s+m}$ respectively. Let $X_0,..., X_{s-1}$ be indeterminates. First define polynomials $\varphi_1(X_0), \varphi_2(X_0, X_1),..., \varphi_s(X_0,..., X_{s-1})$ over k_v by specifying that $\varphi_1(X_0)$ is the minimal polynomial of β_0 over k_v; $\varphi_i(\beta_0,...,\beta_{i-2}, X_{i-1}) \quad (2 \leq i \leq s)$ is the minimal polynomial of β_{i-1} over $k_v(\beta_0,...,\beta_{i-2})$ and that

$$\deg_{X_j} \varphi_i(X_0,..., X_{i-1}) < n_{j+1} \text{ for } j = 0,1,..., i-2$$

Taking preimages of non-zero coefficients of φ_i and taking the multiplicative identity of K as preimage of the multiplicative identity of k_v (with respect to the canonical homomorphism from the valuation ring V_v of v onto k_v), we obtain a polynomial $f_i(X_0,..., X_{i-1})$ in $V_v[X_0,..., X_{i-1}]$ with $X_{i-1}^{n_i}$ occurring as a monomial in $f_i(X)$ such that $\deg_{X_j} f_i(X) < n_{j+1}$ for $j = 0,..., i-2$, and $f_i^*(X) = \varphi_i(X)$.

Fix any non-zero elements $b_1,..., b_s$ of K with $v_0(b_i) > 0$ for all i. We define elements $g_0, g_1,..., g_s$ of $K[x]$ by

$g_0 = x, \quad g_1 = f_1(g_0)/b_1, \quad g_2 = f_2(g_0, g_1)/b_2,..., \quad g_s = f_s(g_0,..., g_{s-1})/b_s$.

Note that $\deg g_0 = 1 = n_0$ and $\deg g_1 = \deg \varphi_1 = n_1$. Since every term in $f_2(g_0, g_1)$ other than $g_1^{n_2}$ has degree

$$\leq \deg(g_0^{n_1-1} g_1^{n_2-1}) = n_1 - 1 + n_1(n_2 - 1) = n_1 n_2 - 1,$$

it follows that $\deg g_2 = n_1 n_2$. This argument can be repeated to prove that

$$\deg g_i = n_0 ... n_1 \qquad (0 \leq i \leq s) \tag{3.1}$$

Recall that $a_1,\ldots,a_m$ are elements of K satisfying $v(a_i)=n_{s+i}\mu_i$ for $1\le i\le m$. We now define polynomials $g_{s+1},\ldots,g_{s+m}$ by

$$g_{s+1}=\frac{g_s^{n_{s+1}}}{a_1}-1,\ldots,g_{s+m}=\frac{g_{s+m-1}^{n_{s+m}}}{a_m}-1.$$

Clearly (3.1) is satisfied for all $i\le s+m$; in particular

$$\deg g_{s+m}=[G_1:G_v][k:k_v].$$

Let $P(x)=g_{s+m}$ and w be any prolongation of v to K with $w(P(x))>0$. It follows from the defining relation between g_{s+m},g_{s+m-1} and the strong triangle law that

$$n_{s+m}w(g_{s+m-1})=w(a_m)=n_{s+m}\mu_m>0.$$

Applying this argument to $g_{s+m-1},\ldots,g_{s+1}$ respectively, we see that

$$n_{s+i}w(g_{s+i-1})=w(a_i)=n_{s+i}\mu_i \qquad (1\le i\le m). \tag{3.2}$$

It is immediate from (3.2) that the value group of w contains $\mu_1,\ldots,\mu_m$ and hence contains $G_1=G_0+\mathrm{Z}\mu_1+\ldots+\mathrm{Z}\mu_m$.

We next prove that the residue field of w contains a k_v isomorphic copy of $k=k_v(\beta_0,\ldots,\beta_{s-1})$. Using the fact that $w(g_s)>0$ and proceeding exactly as in the proof of [9,Lemma p.595] one can prove that $w(g_i)\ge 0$ for $0\le i\le s-1$. Consequently, on recalling that $g_i=f_i(g_0,\ldots,g_{i-1})/b_i$, we have

$$w(f_1(g_0))\ge w(b_1)>0,\ldots,w(f_s(g_0,\ldots,g_{s-1}))\ge w(b_s)>0,$$

which on passing to the residue field of v_1 yields

$$f_1^*(g_0^*)=0,\ldots,f_s^*(g_0^*,\ldots,g_{s-1}^*)=0 \tag{3.3}$$

Let $f_i^*(X_0,\ldots,X_{i-1})=\varphi_i(X_0,\ldots,X_{i-1})$. It now follows from (3.3) that the minimal polynomial of g_0^* over k_v is $\varphi_1(X_0)$ and that of g_{i-1}^* over $k_v(g_0^*,\ldots,g_{i-2}^*)$ is $\varphi i(g_0^*,\ldots,g_{i-2}^*,X_{i-1})$ for $2\le i\le s$. It was assumed that no β_i is zero; the same is therefore true of $g_0^*,\ldots,g_{s-1}^*$, i.e.,

$$w(g_i)=0,\qquad 0\le i\le s-1 \tag{3.4}$$

According to [9,Remark3.7] $k_v[X_0,\ldots,X_{s-1}]/(\varphi_1,\ldots,\varphi_s)$ is k_v-isomorphic to $k_v[\beta_0,\ldots,\beta_{s-1}]$ under the map taking X_i to β_i. For the same reason there exists an isomorphism between $k_v[X_0,...,X_{s-1}]/(\varphi_1,...,\varphi_s)$ and $k_v[g_0^*,...,g_{s-1}^*]$. Hence the subfield $k_v[g_0^*,...,g_{s-1}^*]$ of the residue field of v_1 is k_v which is isomorphic to k. The proof of second part of theorem complete.

Let n_i, a_i, g_i and $g_{s+m} = P(x)$ be as above. Suppose now that $G = G_1 \oplus Z\theta$ with $\theta > 0$. Then θ is not torsion mod G_v. Let v_2 denote the valuation of the field $K(P(x)) \subseteq K\ (x)$ defined on the ring $K[P(x)]$ by

$$v_2\left(\sum_i c_i P(x)^i\right) = \min_i\{v(c_i) + i\theta\}.$$

Let w be any extension of v_2 to $K(x)$. (This extension will turn out to be unique.) We show that v is a desired valuation.

Let $N = N(w/v),\ S$ and T be as defined in introduction. Since $w(P(x)) = \theta$ is not torsion mod G_v

$$N \le \deg P(x) = [k : k_v][G_1 : G_v] \tag{3.5}$$

Also $w(P(x)) > 0$, so by above considerations the value group of w contains G_1 and its residue field contains k; in particular

$$[k : k_v] \le S, \quad [G1 : Gv] \le T, \tag{3.6}$$

in view of [4, Thm. 1.3]. $N \ge ST$ always holds. It now follows from (3.5) and (3.6) that

$$[k : k_v] = S,\ \ [G_1 : G_0] = T,\ \ N = ST;$$

and that the residue field of w is k.

We now determine v explicitly on $K[x]$; the assertion about the value group of v will follow as an immediate consequence.

Let $f(x)$ be any non-zero element of $K[x]$. By successive division by powers of $P(x)$ it can be uniquely written in the form

$$f(x) = f_0(x) + f_1(x)P(x) + ... + f_r(x)P(x)^r,$$

where the polynomial $f_i(x) \in K[x]$ is either zero or has degree less than that of $P(x)$. By [4, Lemma 3.8] any polynomial over K of degree less than $\deg P(x) = n_0 n_1 ... n_{s+1}$ can be uniquely written as a finite linear combination with coefficients in K of elements of the type

$$g_0^{j_0} ... g_{s+m-1}^{j_{s+m-1}} \text{ where } 0 \le j_0 \le n_1 - 1, ..., 0 \le j_{s+m-1} \le n_{s+m} - 1.$$

So a non zero polynomial $f(x)$ in $K[x]$ can be uniquely written as a finite sum

$$f(x) = \sum_{\substack{0 \le j_{i-1} \le n_i - 1 \\ 0 \le j_{s+m} < \infty}} a_{j_0,...,j_{s+m}} g_0^{j_0} ... g_{s+m-1}^{j_{s+m-1}} P(x)^{j_{s+m}}, \tag{3.7}$$

with coefficients $a_{j_0,\dots js+m}$ in K. In view of equations (3.4) and (3.2) (which holds for any prolongation v_1 of v to $K(x)$ with $v_1(P(x))>0)$, $v(g_i)=0$ for $0\le i\le s-1$ and $v(g_{s+i-1})=\mu_i$ for $1\le i\le m$. So by the triangle law, we have

$$v(f(x))\ge \min_{(j_0,\dots,j_{s+m})}\{v_0(a_{j_0,\dots,js+m})+j_s\mu_1+\dots+j_{s+m-1}\mu_m+j_{s+m}\theta\} \tag{3.8}$$

Our claim is that the equality holds in (3.8). This is so because any two distinct terms in the sum on the right hand side of (3.7) have distinct v-valuations; the last assertion can be easily verified using the fact that θ is non-torsion mod G_0 together with the fact that G_1/G_0 is the direct sum of its m cyclic subgroups generated by the elements $G_0+\mu_1,\dots,G_0+\mu_m$ respectively.

Theorem 2: Let $v_1,\dots,v_n$ $(n\ge 1)$ be (non-trivial) pairwise independent valuations of a field K having value groups $G_{v_1},\dots G_{v_n}$ and residue fields $k_{v_1},\dots,k_{v_n}$ respectively. For $1\le i\le n,$ let G_i' be a totally ordered abelian group containing G_{v_1} as an ordered subgroup with $\lfloor G_i':G_{v_i}\rfloor$ finite and k_i' be a finite extension of k_{v_i}. Then there exist valuations $v_1',\dots,v_n'$ of $K(x)$ together with a polynomial $t\in K[x]$ of degree max $\{\lfloor G_i':G_{v_i}\rfloor\lfloor k_i':k_{v_i}\rfloor: 1\le i\le n\}$ such that for $1\le i\le n$,

(i) v_i' extends the valuation w_i of $K(t)$ where w_i is the Gauss ewtension of v_i to $K(t)$

(ii) the value group of v_i' is G_i' and its residue field is $k_i'(z_i)$ where z_i is the v_i'-residue of t

Proof: Let $\lfloor G_i':G_{v_i}\rfloor=e_i$ and $\lfloor k_i:k_{v_i}\rfloor=f_i$. In view of [8, Theo. 3.1, Cor. 3.2], there exists a polynomial $t_i\in K[x]$ of degree e_if_i and a valuation v_i' of $K(x)$ extending the Gaussian valuation $v_i^{t_i}$ of $K(t_i)$ such that the value group of v_i' is G_i' and its residue field is a simple transcendental extension $k_i'(\xi_i)$ (say) of k_i'. Let denote the v_i'-residue of t_i, so that the residue field of $v_i^{t_i}$ is $k_{v_i}(y_i)$; our claim is that $k_i'(\xi_i)=k_i'(y_i)$ for $1\le i\le n$. In view of the fundamental inequality [8, Cor. 13.10] applied to the extension $(K(x),v_i')/(K(t_i),v_i^{t_i})$, we have

$$[K(x):K(t_i)]\ge\lfloor G_i':G_{v_i}\rfloor\lfloor k_i'(\xi_i):k_{v_i}(y_i)\rfloor$$

i.e.,

$$e_if_i\ge e_if_i[k_i'(\xi_i):k_i'(y_i)]$$

which proves the claim.

Fix any i, $1\le i\le n$. Since $\lfloor G_j':G_{v_j}\rfloor<\infty$ and G_{v_j} is a non-trivial group, one can choose $g_{ij}\in G_{v_j}$ such that $v_i'(t_i)>g_{ij}$ if $1\le j\le n$, $j\ne i$. By Independence

Theorem [8, Cor. 11.17] applied to pairwise independent valuations $v_1,\ldots,v_n$, there exists $\beta_i \in K$ satisfying $v_i(\beta_i)=0$ and $v_j(\beta_j)=g_{ij}<v'_j(t_i)$ for $1\le j\le n$, $j\ne i$.

Observe that t_i/β_i is residually transcendental for v_i'/v_i and is chosen so that $v'_j(t_i/\beta_i)>0$ if $i\ne j$.

Set

$$t=\frac{t_1}{\beta_1}+\ldots+\frac{t_n}{\beta_n}.$$

The v'_i-residue of t, being the same as that of t_i/β_i, is transcendental over the residue field k_{v_i} of v_i and hence v'_i coincides with the Gaussian valuation w_i on $K(t)$ in view of [2, Chap. VIA, 10.1, Prop.2]. This proves assertion (i) of the theorem.

Since the w_i-residue z_i (say) of t differs from the v'_i-residue y_i of t_i by an element of k_i (in fact by the v_i-residue of β_i), it follows from the claim proved above that the residue field $k'_i(y_i)$ of v'_i equals $k'_i(z_i)$, which proves (ii).

Recall that t_i is a polynomial in x of degree e_if_i so the polynomial t is of degree $\le \max\limits_{1\le i\le n} e_if_i=d$ say. The theorem is proved as soon as it is shown that $\deg t\ge d$. Let i be an index such that $d=e_if_i$. Using the fact that the residue field of v'_i is $k_{v_i}(z_i)$, we have in view of the fundamental inequality [3, Cor. 13.10] that

$$[K(x):K(t)]\ge [G'_i:G_{v_i}][k'_i(z_i):k_{v_i}(z_i)]=e_if_i=d.$$

This proves the desired assertion.

Theorem 3: Let v be a non-trivial valuation of arbitrary rank of an algebraically closed field K having residue field k_v. Let k be any function field which can be expressed as a quadratic extension of some simple transcendental extension of k_v and g be an integer not less than the genus of k/k_v. Then there exist a function field F of 1 variable over K having genus g and a prolongation w of v to F such that the residue field of w is k_v isomorphic to k.

Proof: We retain the notations introduced in the beginning of the previus section and shall denote the genus of k/k_v by g_0 Two cases are distinguished.

CASE I. If char $k_0\ne 2$, then as in [1, Chapter 16] we can write $k=k_v(t,u)$ where t is transcendental over k_v and $u^2=h(t)$ is a square-free monic polynominal over k_0 of degree $2g_0+1$ or $2g_0+2$. Write

$$h(t)=(t-a_1^*)\ldots(t-a_n^*),\quad a_i\in K_0, v_0(a_i)\ge 0$$

and

$$g = g_0 + r$$

Choose distinct elements $b_1,...,b_{2_r}$ in K such that $v(b_i) > 0$ for each i. Define a square-free polynominal $H(x) \in K[x]$ by

$$H(x) = (x - a_1)...(x - a_n)(b_1 x + 1)...(b_2 x + 1)$$

and set
$$F = K\left(x, \sqrt{H(x)} \right)$$

Then the genus of F being $(n + 2r - 1)/2$ or $(n + 2r - 2)/2$ equals g. Let w be (the) valuation of F which extends the Gaussian valuation v^x of $K(x)$. Since $H(x)$ is chosen so that $H(x)^* = h(x^*)$, it follows from [7, Lem.2.1] that the residue field of w is $k_v(x^*, \sqrt{h(x^*)})$ and hence is k_v isomorphic to k

CASE II. When char $k_v = 2$, then we express k as $k_v(t,u)$ where t is transcendental over k_v and u satisfies an irreducible relation $u^2 + P(t)u + Q(t) = 0$ for some non-zero polynominals $P(t), Q(t) \in k_v[t]$ of degres not exceeding $g_0 + 1$ and $2g_0 + 2$ respectively; this can be done in view of [1, Chapter 16, sections 5,7] for $g_0 \geq 1$ and if $g_0 = 0$, then $k = k_v(u)$ is a simple transcendental extension of k_v, in which case we may take $t = u^2 + u + 1$
We split two subcases.

SUBCASE 1. Char $K = 0$ Choose $A(x) \in K[x]$ of degree $g + 1$ such that

(i) $v^x(A(x)) = 0;$

(ii) the leading coefficient a of $A(x)$ satisfies $0 \leq v_0(a) < v_0(2)$

(iii) $A(x)^* = P(x^*)$

Choose $B(x) \in K[x]$ of degree $\leq 2g + 2$ satisfying $v^x(B(x)) = 0$ and $B(x)^* = Q(x^*)$

Define

$$f(x) = A(x)^2 + 4B(x) + 2^m$$

where an integer $m \geq 3$ is chosen so that no root of the derivative of $A(x)^2 + 4B(x)$ is a root of $f(x)$. It is clear from the choice of A(x) and $B(x)$ that $\deg f(x) = 2g + 2$. Since $f(x)$ is square-free, the function field $F = K\left(x, \sqrt{f(x)} \right)$ has genus g. Let w be (the) valuation of F which extends v^x. It is immediate from [7. Lem. 2.2] that the residue field of w is k_v isomorphic to $k = k_v(t,u)$.

SUBCASE 2. Let char $K = 2$. Arguing as in for the previous subcase, we can choose a square-free polynomial A(x) with coefficients from K of degree $g+1$ such that $v^x(A(x)) = 0$ and $A(x)^* = P(x^*)$. Let $B_1(x)$ be a polynomial of degree $\leq 2g+2$ with $v^x(B_1(x)) = 0$ and $B_1(x)^* = Q(x^*)$. Choose an integer $m \geq 1$ so that

(i) no root of the derivative of $B_1(x)$ is a root of $B(x) = B_1(x) + 2^m$;

(ii) for each root α of $A(x)$, $(B_1'(\alpha)/A'(\alpha))^2$ is different from $B_1(\alpha) + 2^m$.

Define $F = K(x, y)$, where y satisfies

$$y^2 + A(x)y + B(x) = 0$$

Let w be the valuation of F extending v^x. Keeping in view that $F = K(x, \xi)$, where $\xi = y / A(x)$ satisfies the polynomial $Z^2 + Z + B(x)/A(x)^2$ and arguing as in the proof of [7,Lem. 2.2], it can be easily seen that the residue field k_w of w is given by

$$k_w = k_v(x^*, \xi^*) = k_v(x^*, y^*),$$

where y^* satisfies the polynomial $Z^2 + P(x^*)Z + Q(x^*)$. Clearly k_w is k_v-isomorphic to k. This also shows that $K_0(x, y) \neq K_0(x)$. It now follows from [7,Lem. 2.3 and Lem.2.4] that the genus of F is g.

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