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Kinematics of Dual Quaternion Involution Matrices

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Abstract: Rigid-body (screw) motions in three-dimensional Euclidean space $\mathbb{R}^3$ can be represented by involution (resp. anti-involution) mappings obtained by dual-quaternions which are self-inverse and homomorphic (resp. anti-homomrphic) linear mappings. In this paper, we will represent four dual-quaternion matrices with their geometrical meanings; two of them correspond to involution mappings, while the other two correspond to anti-involution mappings.

***Key words*:** Real-quaternion, dual-quaternion, (anti)-involution, rigid-body (screw) motion.

***Mathematics Subject Classifications (2010)*:** 11R52, 53A25, 53A35, 70B10.

Dual Kuaterniyon İnvolüsyon Matrislerin Kinematiği

Özet: Lineer bir dönüşüm aynı zamanda self-inverse (tersi kendisine eşit) ve anti-homomorfik ise involüsyon; self-inverse ve homomorfik ise anti-involüsyondur. Üç-boyutlu Öklid uzayı $\mathbb{R}^3$ teki vida hareketleri dual-kuaterniyonlar ile elde edilen (anti)-involüsyon dönüşümleri ile verilebilir. Biz bu çalışmada, dual-kuaterniyonları kullanarak ikisi involüsyon dönüşüme diğer ikisi ise anti-involüsyon dönüşüme karşılık gelen dört tane matrisi geometrik yorumlarıyla birlikte ele aldık.

***Anahtar Kelimeler*:** Reel-kuaterniyon, dual-kuaterniyon, (anti)-involüsyon, vida hareketi.

***Matematik Konu Sınıflandırması (2010)*:** 11R52, 53A25, 53A35, 70B10.

1. Introduction

Real-quaternions are non-commutative division algebra over the field real numbers $\mathbb{R}$, and are invented by Irish mathematician Sir William Rowan Hamilton in 1843. Hamilton tried to formalize three points in three-dimensional Euclidean space $\mathbb{R}^3$ in the same way that two points can be formalized in the complex field $\mathbb{C}$. But, there exist a problem by multiplying real-quaternions. He overcame with this problem by using the three imaginary parts $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ satisfying the non-commutative multiplication rules

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1,$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Quaternions are widely used in computer graphic technology, physics, kinematics, etc., since they are useful to perceive rotations, reflections and rigid-body (screw) motions. For instance, a reflection of a vector in a plane can be represented by an involution or anti-involution mapping obtained by real-quaternions, see [1]. In this paper, firstly the basic concepts of real- and dual-quaternions will be given. Afterwards, we will represent four (anti)-involution matrices obtained by dual-quaternions. The geometry of these matrices will be given as reflections in four-dimensional dual space $\mathbb{D}^4$, and as rigid-body (screw) motions in $\mathbb{R}^3$ by restricting ourselves to unit pure dual-quaternions.

2. Preliminaries

In this section, a brief summary of the concepts real-quaternions, dual-quaternions and rigid-body (screw) motion will be given.

Real-quaternion algebra

$$\mathbb{H} = \{q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} : \ w, x, y, z \in \mathbb{R}\}$$

is a four dimensional vector space over the field of real-numbers $\mathbb{R}$ with a basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ satisfying the non-commutative *multiplication* rules

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1,$$
$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \ \ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \ \ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}\ .$$

A real-quaternion $q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ consists of a *scalar part* $S(q) = w \in \mathbb{R}$ and *vector part* $\boldsymbol{V}(q) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \in \mathbb{R}^{\boldsymbol{3}}$. The *quaternionic-conjugate* of $q = S(q) + \boldsymbol{V}(q)$ is defined by $\bar{q} = S(q) - \boldsymbol{V}(q)$. If $S(q) = 0$, then q is said to be a *pure*. The set of pure real-quaternions will be denoted by

$$\widehat{\mathbb{H}} = \{q = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} : x, y, z \in \mathbb{R}\}.$$

The *norm* of q is

$$N(q) = \parallel q \parallel = \ q\bar{q} \ = \ \bar{q}q = \ w^2 + x^2 + y^2 + z^2 \in \mathbb{R}.$$

If $N(q) = 1$ then q is said to be a *unit*.

The *multiplicative inverse* of q is valid only when q is non-zero and is given by

$$q^{-1} = \frac{\bar{q}}{\parallel q \parallel}\,.$$

Thus, the algebra $\mathbb{H}$ is a division algebra.

The *complex form* of $q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is defined by

$$\boldsymbol{q} = a + \boldsymbol{\mu} b$$

where $a = w,\ b = \sqrt{x^2 + y^2 + z^2}$ and $\boldsymbol{\mu} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{b}$ for $b \neq 0$.
The algebra $\mathbb{H}$ is isomorphic to the Clifford algebra $Cl_{0,2}$ (i.e. $\mathbb{H} \cong Cl_{0,2}$) in dimension 2 by defining the quaternionic units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, respectively, with the standard anti-commuting generators e_1, e_2, $e_{12}(= e_1 e_2)$ in $Cl_{0,2}$ where

$$e_1^2 = e_2^2 = (e_1 e_2)^2 = -1 \ \text{ and } \ e_1 e_2 = -e_2 e_1.$$

For more details about real-quaternions see $[2 - 4]$.

Dual-number algebra

$$\mathbb{D} = \{A = a + \boldsymbol{\varepsilon} a^* : a, a^* \in \mathbb{R}\}$$

is a two dimensional vector space over the field of real-numbers $\mathbb{R}$ with a basis $\{1, \boldsymbol{\varepsilon}\}$, where a is the *non-dual part*, a^* is the *dual part* and $\boldsymbol{\varepsilon}$ is the *dual unit* satisfying $\boldsymbol{\varepsilon} \neq 0$, $\boldsymbol{\varepsilon} r = r\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^2 = 0$ for all $r \in \mathbb{R}$. The *dual conjugate* of a A is defined by $A^\star = a - \boldsymbol{\varepsilon} a^*$.

Dual-quaternion (also known as *dual number coefficient-quaternion*) algebra

$$\mathbb{H}_{\mathbb{D}} = \{Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} : W, X, Y, Z \in \mathbb{D}\}$$

is a four dimensional vector space over the field of dual-numbers $\mathbb{D}$ with the same basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of real-quaternions, namely $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$ and $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$. The multiplication of the dual unit $\boldsymbol{\varepsilon}$ with the basis elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is commutative that is $\boldsymbol{i\varepsilon} = \boldsymbol{\varepsilon i}$, $\boldsymbol{j\varepsilon} = \boldsymbol{\varepsilon j}$, $\boldsymbol{k\varepsilon} = \boldsymbol{\varepsilon k}$. A dual-quaternion $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ consists of a *scalar part* $S(Q) = W \in \mathbb{D}$ and *vector part* $\boldsymbol{V}(Q) = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \in \mathbb{D}^3$. If $S(Q) = 0$, then Q is called a *pure*. Pure dual-quaternions set will be denoted by

$$\widehat{\mathbb{H}}_{\mathbb{D}} = \{Q = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} : X, Y, Z \in \mathbb{D}\}.$$

The *quaternionic-multiplication* of dual-quaternions $Q_1 = W_1 + X_1\boldsymbol{i} + Y_1\boldsymbol{j} + Z_1\boldsymbol{k}$ and $Q_2 = W_2 + X_2\boldsymbol{i} + Y_2\boldsymbol{j} + Z_2\boldsymbol{k}$ is

$$Q_1Q_2 = S(Q_1)S(Q_2) - \langle \boldsymbol{V}(Q_1), \boldsymbol{V}(Q_2)\rangle + S(Q_1)\boldsymbol{V}(Q_2) + S(Q_2)\boldsymbol{V}(Q_1) + \boldsymbol{V}(Q_1)\wedge\boldsymbol{V}(Q_2)$$

where $S(Q_1) = W_1$, $S(Q_2) = W_2$, $\boldsymbol{V}(Q_1) = X_1\boldsymbol{i} + Y_1\boldsymbol{j} + Z_1\boldsymbol{k}$ and $\boldsymbol{V}(Q_2) = X_2\boldsymbol{i} + Y_2\boldsymbol{j} + Z_2\boldsymbol{k}$. Also, $\langle \boldsymbol{V}(Q_1), \boldsymbol{V}(Q_2)\rangle = X_1X_2 + Y_1Y_2 + Z_1Z_2 \in \mathbb{D}$ and $\boldsymbol{V}(Q_1)\wedge\boldsymbol{V}(Q_2) = \boldsymbol{i}(Y_1Z_2 - Y_2Z_1) + \boldsymbol{j}(Z_1X_2 - Z_2X_1) + \boldsymbol{k}(X_1Y_2 - X_2Y_1) \in \mathbb{D}^3$ denotes, respectively, the usual *inner* and *vector products* of $\boldsymbol{V}(Q_1)$ and $\boldsymbol{V}(Q_2)$ in $\mathbb{D}^3$.

The following three conjugate types can be given for Q:

1. *Quaternion-conjugate*: $\bar{Q} = W - X\boldsymbol{i} - Y\boldsymbol{j} - Z\boldsymbol{k}$
2. *Dual-conjugate*: $Q^\star = W^\star + X^\star\boldsymbol{i} + Y^\star\boldsymbol{j} + Z^\star\boldsymbol{k}$
3. *Total-conjugate*: $\overline{Q^\star} = W^\star - X^\star\boldsymbol{i} - Y^\star\boldsymbol{j} - Z^\star\boldsymbol{k}$

For dual-quaternions P and Q the following conjugation rules can be given:

1. $\overline{\mathcal{P}Q} = \bar{Q}\,\bar{\mathcal{P}}$, $(\mathcal{P}Q)^\star = \mathcal{P}^\star Q^\star$, $\overline{(\mathcal{P}Q)^\star} = (\overline{\mathcal{P}Q})^\star = \overline{Q^\star}\;\overline{P^\star}$.
2. $\overline{\mathcal{P} \pm Q} = \bar{\mathcal{P}} \pm \bar{Q} = \bar{Q} \pm \bar{\mathcal{P}}$, $(\mathcal{P} \pm Q)^\star = \mathcal{P}^\star \pm Q^\star = Q^\star \pm \mathcal{P}^\star$, $\overline{(\mathcal{P} \pm Q)^\star} = (\overline{\mathcal{P} \pm Q})^\star = \overline{\mathcal{P}^\star} \pm \overline{Q^\star} = \overline{Q^\star} \pm \overline{\mathcal{P}^\star}$.
3. $Q\bar{Q} = \bar{Q}Q$ and in general $QQ^\star \neq Q^\star Q$, $Q\,\overline{Q^\star} \neq \overline{Q^\star}Q$.

The *norm* of Q is

$$N(Q) = \parallel Q \parallel = Q\bar{Q} = \bar{Q}Q = W^2 + X^2 + Y^2 + Z^2 \in \mathbb{D}.$$

If $N(Q) = 1$, then Q is said to be a *unit*.

A dual-quaternion $Q = W + X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ can be represented in different forms. Three of them are shown below:

1. *Dual form*:

$$Q = \mathfrak{Re}(Q) + \boldsymbol{\varepsilon}\,\mathfrak{Du}(Q),$$

where $\mathfrak{Re}(Q) = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = a + \boldsymbol{\mu}b$ and $\mathfrak{Du}(Q) = w^* + x^*\boldsymbol{i} + y^*\boldsymbol{j} + z^*\boldsymbol{k} = \mathrm{c} + \boldsymbol{\nu}d$ are real-quaternions.

2. *Complex form*:

$$Q = A + \boldsymbol{\delta} B$$

provided $\Re\mathrm{e}(\boldsymbol{\delta}) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \neq 0$. Here $\boldsymbol{\delta} = (X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k})/\sqrt{X^2 + Y^2 + Z^2}$ is a unit pure dual-quaternion; $A = W$ and $B = \sqrt{X^2 + Y^2 + Z^2}$ are dual-numbers.

3. *Polar form*:

$$Q = \sqrt{N_Q}\,(cos\phi + \boldsymbol{Q} sin\phi)$$

provided $\Re\mathrm{e}(Q) = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \neq 0$ and $\Re\mathrm{e}(\boldsymbol{\delta}) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \neq 0$. Here $\phi \in \mathbb{D}$, $cos\phi = W/\sqrt{N_Q}$, $sin\emptyset = \sqrt{X^2 + Y^2 + Z^2}/\sqrt{N_Q}$ and $\boldsymbol{Q} = \boldsymbol{\delta}$.

The *multiplicative inverse* of Q is valid only if $\Re\mathrm{e}(Q) \neq 0$ and is given by

$$Q^{-1} = \frac{\bar{Q}}{N(Q)}.$$

According to *E. Study map.*, all the oriented lines in $\mathbb{R}^3$ are in one-to-one correspondence with the points of unit dual sphere $\mathbb{D}^3$. In other words, to each oriented line in $\mathbb{R}^3$ corresponds a unit pure dual-quaternion, also to each unit pure dual-quaternion corresponds an oriented line in $\mathbb{R}^3$. For more details about dual-quaternions see $[5 - 7]$.

2.1. Screw Operator

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be unit pure dual-quaternions and the angle between them $\varphi = \phi + \boldsymbol{\varepsilon}\phi^* \in \mathbb{D}$. The *quaternionic-multiplication* of these unit pure dual-quaternions can be given as:

$$\boldsymbol{AB} = -\langle \boldsymbol{A}, \boldsymbol{B} \rangle + \boldsymbol{A} \wedge \boldsymbol{B} = -cos\varphi + \frac{\boldsymbol{A} \wedge \boldsymbol{B}}{\sqrt{\| \boldsymbol{A} \wedge \boldsymbol{B} \|}}\sqrt{\| \boldsymbol{A} \wedge \boldsymbol{B} \|} = -cos\varphi + \boldsymbol{S} sin\varphi$$

where $\boldsymbol{S} = \boldsymbol{A} \wedge \boldsymbol{B} / \sqrt{\| \boldsymbol{A} \wedge \boldsymbol{B} \|}$ that means $\boldsymbol{S} \perp \boldsymbol{A}$, $\boldsymbol{S} \perp \boldsymbol{B}$ (so, $\boldsymbol{S}$ is parallel to $\boldsymbol{A} \wedge \boldsymbol{B}$) and $\| \boldsymbol{S} \| = 1$. Thus,

$$\boldsymbol{BA} = -\langle \boldsymbol{B}, \boldsymbol{A} \rangle + \boldsymbol{B} \wedge \boldsymbol{A} = -cos\varphi - \boldsymbol{S} sin\varphi = -(cos\varphi + \boldsymbol{S} sin\varphi)$$

and

$$\boldsymbol{B}\,\boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}\,\boldsymbol{A} = cos\varphi + \boldsymbol{S} sin\varphi.$$

By taken $\boldsymbol{BA}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A} = Q$, it can be written $\boldsymbol{B} = Q\boldsymbol{A}$ and $\boldsymbol{A} = \boldsymbol{B}Q$.

The geometric interpretation of the product $Q\boldsymbol{A}$ in $\mathbb{R}^3$ can be given as rotating the line $d_{\boldsymbol{A}}$ (corresponding to $\boldsymbol{A}$) by angle $\phi \in \mathbb{R}$ in positive direction about the line $d_{\boldsymbol{S}}$ (corresponding to $\boldsymbol{S}$), and translating by magnitude $\phi^* \in \mathbb{R}$ along the line $d_{\boldsymbol{S}}$, see Fig. 1.

As a result he following two special cases can be given:

1. If $\phi \neq 0$ and $\phi^* = 0$, then the operator $Q = cos\varphi + \boldsymbol{S} sin\varphi$ describes a rotation.
2. If $\phi^* \neq 0$ and $\phi = 0$, then the operator $Q = cos\varphi + \boldsymbol{S} sin\varphi = 1 + \boldsymbol{\varepsilon}\phi^*\boldsymbol{S}$ describes a translation.

Since a rotation about an axis and translation along the same axis describes a *rigid-body (screw) motion*, every unit dual-quaternion $Q = cos\varphi + \boldsymbol{S}sin\varphi$ can be handled as a screw operator.

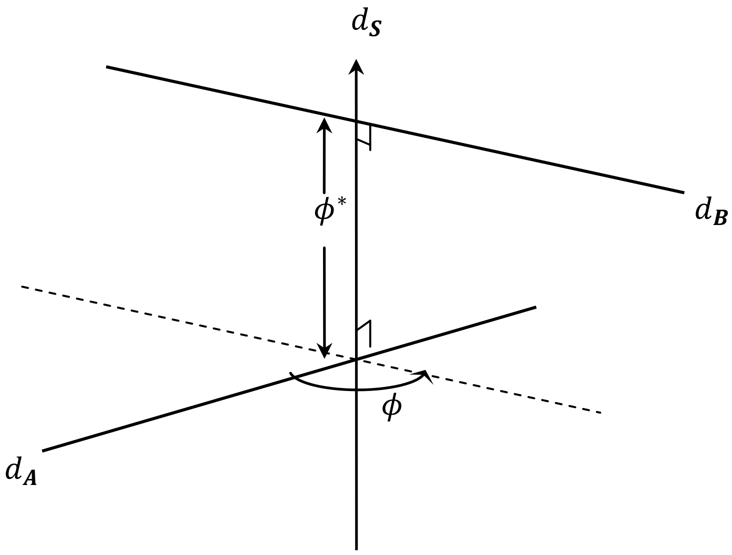

Figure 1. Geometry of the rigid-body (screw) motion where $d_{\boldsymbol{A}}$, $d_{\boldsymbol{B}}$ and $d_{\boldsymbol{S}}$ denotes the lines corresponding to unit pure dual-quaternions $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{S}$, respectively.

Proposition 1. Let $\boldsymbol{A}$ *be a unit pure dual-quaternion and* $Q = cos(\varphi/2) + \boldsymbol{S}sin(\varphi/2)$ *be a unit dual-quaternion where* $\varphi/2 = (\phi/2) + \boldsymbol{\varepsilon}(\phi^*/2) \in \mathbb{D}$*. Then the product* $Q\boldsymbol{A}\bar{Q}$ *represents a rotation of the line* $d_{\boldsymbol{A}}$ *(corresponding to* $\boldsymbol{A}$*) by angle* $\phi \in \mathbb{R}$ *in positive direction about the line* $d_{\boldsymbol{S}}$ *(corresponding to* $\boldsymbol{S}$*), afterwards a translation with magnitude* $\phi^* \in \mathbb{R}$ *along* $d_{\boldsymbol{S}}$*.*

3. (Anti)-Involution Matrices Of Dual Quaternions

In this section, two dual-quaternion matrices corresponding to dual-quaternion involution transformations and another two dual-quaternion matrices corresponding to dual-quaternion anti-involution transformations will be given. These matrices will be presented with their geometrical meanings as reflections in $\mathbb{D}^4$, and the matrices corresponding to dual-quaternion involution and anti-involution transformations' vector parts will be given with their geometrical meanings as rigid-body (screw) motions in $\mathbb{R}^3$.

A linear transformation f is an *involution* if it is *self-inverse* (i.e., $f\big(f(x)\big) = x$) and *anti-homomorphic* (i.e., $f(x_1x_2) = f(x_2)f(x_1)$). Also, f is said to be *anti-involution* if it is *self-inverse* and *homomorphic* (i.e., $f(x_1x_2) = f(x_1)f(x_2)$), see [8].

3.1. Involution Matrices of Dual Quaternions

Proposition 2. Let $Q = A + \boldsymbol{\delta}B$ *be an arbitrary dual-quaternion, then the transformation*

$$f_{\boldsymbol{V}}: \mathbb{H}_{\mathbb{D}} \to \mathbb{H}_{\mathbb{D}}\,; \quad Q \mapsto f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\bar{Q}\boldsymbol{V} = A + \boldsymbol{V}\boldsymbol{\delta}\boldsymbol{V}B$$

is an involution for a choosen unit pure dual-quaternion $\boldsymbol{V}$*, see* [7].

The geometry of the product $\boldsymbol{V\delta V}$ in $\mathbb{R}^3$ can be given as: Let $\boldsymbol{\delta} = \vec{a} + \boldsymbol{\varepsilon}\left(\vec{A} \wedge \vec{a}\right)$ and $\boldsymbol{V} = \vec{b} + \boldsymbol{\varepsilon}\left(\vec{B} \wedge \vec{b}\right)$ where the points A and B are the closest points to each other on the lines $d_{\boldsymbol{\delta}}$ and $d_{\boldsymbol{V}}$, respectively. The line passing through the points A and B is the axis of the motion. By taken $\overrightarrow{s_0} = \overrightarrow{AB} / | \overrightarrow{AB} |$, the direction of the translation of the motion is in the same direction of the vector $\overrightarrow{s_0}$ with magnitude $2d = 2\left|\overrightarrow{AB}\right|$. Denote by M the plane that includes the line $d_{\boldsymbol{V}}$ and is perpendicular to $\overrightarrow{s_0}$ and denote by $\overrightarrow{a_1}$ the unit orthogonal projection vector of $\vec{a}$ on M. Define $\theta \in \mathbb{R}^+$ as $\langle \overrightarrow{a_1}, \vec{b} \rangle = cos\theta$, then the rotation of the motion occurs by angle $\pi - 2\theta \in \mathbb{R}$ in negative direction in both cases if $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a right-handed set (i.e. $\overrightarrow{s_0} = \overrightarrow{a_1} \wedge \vec{b}$) and if $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a left-handed set (i.e. $\overrightarrow{s_0} = \vec{b} \wedge \overrightarrow{a_1}$), see Fig. 2. Thus, $\boldsymbol{V\delta V}$ describes a rigid-body (screw) motion in $\mathbb{R}^3$. It is important to emphasize that this motion can also be given as a reflection of the line $d_{\boldsymbol{\delta}}$ about the line $d_{\boldsymbol{V}}$.

Now, we will give two screw operator formulas corresponding to the product $\boldsymbol{V\delta V}$; one if $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a right-handed set and one if $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a left-handed set.

1. If $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a right-handed set and if we take

$$Q = cos\left(\left(\frac{\pi}{2} - (-\theta)\right) + \boldsymbol{\varepsilon} d\right) + \boldsymbol{S} sin\left(\left(\frac{\pi}{2} - (-\theta)\right) + \boldsymbol{\varepsilon} d\right)$$
$$= (-sin\theta - \boldsymbol{\varepsilon} d cos\theta) + \boldsymbol{S}(cos\theta - \boldsymbol{\varepsilon} d sin\theta)$$

then

$$\bar{Q} = (-sin\theta - \boldsymbol{\varepsilon} d cos\theta) - \boldsymbol{S}(cos\theta - \boldsymbol{\varepsilon} d sin\theta).$$

In this case, the operator $Q\boldsymbol{\delta}\bar{Q}$ will rotate the line $d_{\boldsymbol{\delta}}$ by angle $\pi - 2\theta \in \mathbb{R}$ in negative direction about the axis $\boldsymbol{S}$ and translate with magnitude $2d$ about the same axis $\boldsymbol{S}$. The translation is in the same direction with the axis vector $\boldsymbol{S}$. Thus, we can give the equation $Q\boldsymbol{\delta}\bar{Q} = \boldsymbol{V\delta V}$.

If unit pure dual-quaternions $\boldsymbol{\delta}$ and $\boldsymbol{V}$ are parallel, then by taken

$$Q = cos\left(2\left(\frac{\pi}{2} - (-\theta)\right) + 2\boldsymbol{\varepsilon} d\right) + \boldsymbol{S}\, sin\left(2\left(\frac{\pi}{2} - (-\theta)\right) + 2\boldsymbol{\varepsilon} d\right)$$
$$= cos\big((\pi + 2\theta) + 2\boldsymbol{\varepsilon} d\big) + \boldsymbol{S}\, sin\big((\pi + 2\theta) + 2\boldsymbol{\varepsilon} d\big),$$

we can give the equation $Q\boldsymbol{\delta}\bar{Q} = \boldsymbol{V\delta V}$ as $Q\boldsymbol{\delta} = \boldsymbol{V\delta V}$.

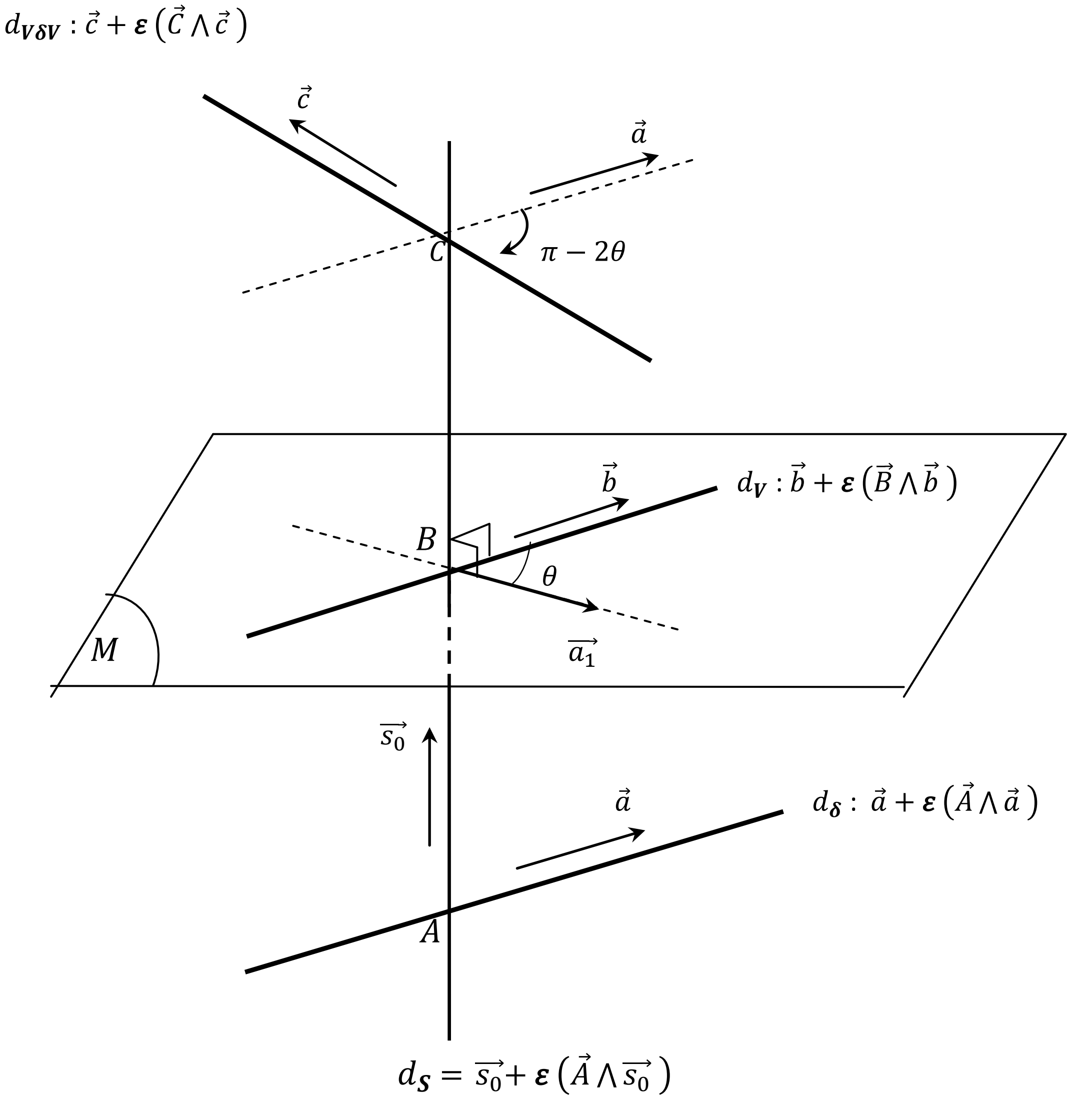

Figure 2. Screw motion of the product $\boldsymbol{V\delta V}$ where the line d_S, which corresponds to unit pure dual-quaternion $\boldsymbol{S} = \overrightarrow{s_0} + \boldsymbol{\varepsilon}(\vec{A} \wedge \overrightarrow{s_0})$, denotes the axis of the motion and $|\overrightarrow{AB}| = |\overrightarrow{BC}| = d$.

2. If $\{\overrightarrow{a_1}, \vec{b}, \overrightarrow{s_0}\}$ is a left-handed set and if we take

$$Q = cos\left(\left(\frac{\pi}{2} - \theta\right) + \boldsymbol{\varepsilon}d\right) + \boldsymbol{S}\, sin\left(\left(\frac{\pi}{2} - \theta\right) + \boldsymbol{\varepsilon}d\right)$$

$$= (sin\theta - \boldsymbol{\varepsilon}dcos\theta) + \boldsymbol{S}(cos\theta + \boldsymbol{\varepsilon}dsin\theta),$$

then

$$\bar{Q} = (sin\theta - \boldsymbol{\varepsilon}dcos\theta) - \boldsymbol{S}(cos\theta + \boldsymbol{\varepsilon}dsin\theta).$$

In this case, the operator $Q\boldsymbol{\delta}\bar{Q}$ will rotate the line $d_{\boldsymbol{\delta}}$ by angle $\pi - 2\theta \in \mathbb{R}$ in negative direction about the axis $\boldsymbol{S}$ and translate with magnitude $2d$ about the same axis $\boldsymbol{S}$. The translation is in the same direction with the axis vector $\boldsymbol{S}$. Thus, we can give the equation $Q\boldsymbol{\delta}\bar{Q} = \boldsymbol{V\delta V}$.

If unit pure dual-quaternions $\boldsymbol{\delta}$ and $\boldsymbol{V}$ are parallel, then by taken

$$Q = cos\left(2\left(\frac{\pi}{2} - \theta\right) + 2\boldsymbol{\varepsilon} d\right) + \boldsymbol{S}\, sin\left(2\left(\frac{\pi}{2} - \theta\right) + 2\boldsymbol{\varepsilon} d\right)$$
$$= cos\big((\pi - 2\theta) + 2\boldsymbol{\varepsilon} d\big) + \boldsymbol{S}\, sin\big((\pi - 2\theta) + 2\boldsymbol{\varepsilon} d\big),$$

we can give the equation $Q\boldsymbol{\delta}\bar{Q} = \boldsymbol{V}\boldsymbol{\delta}\boldsymbol{V}$ as $Q\boldsymbol{\delta} = \boldsymbol{V}\boldsymbol{\delta}\boldsymbol{V}$.

Now, the matrix representation will be obtained corresponding to the involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\bar{Q}\boldsymbol{V}$ for a choosen unit pure dual-quaternion $\boldsymbol{V} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$:

$f_{\boldsymbol{V}}(1) = -\boldsymbol{V}\bar{1}\boldsymbol{V} = 1,$
$f_{\boldsymbol{V}}(\boldsymbol{i}) = -\boldsymbol{V}\bar{\boldsymbol{i}}\boldsymbol{V} = (1 - 2X^2)\boldsymbol{i} - 2XY\boldsymbol{j} - 2XZ\boldsymbol{k}\,,$
$f_{\boldsymbol{V}}(\boldsymbol{j}) = -\boldsymbol{V}\bar{\boldsymbol{j}}\boldsymbol{V} = -2XY\boldsymbol{i} + (1 - 2Y^2)\boldsymbol{j} - 2YZ\boldsymbol{k}\,,$
$f_{\boldsymbol{V}}(\boldsymbol{k}) = -\boldsymbol{V}\bar{\boldsymbol{k}}\boldsymbol{V} = -2XZ\boldsymbol{i} - 2YZ\boldsymbol{j} + (1 - 2Z^2)\boldsymbol{k}\,.$

Thus, the matrix product of the involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\bar{Q}\boldsymbol{V}$ can be given as

$$TQ := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2X^2 & -2XY & -2XZ \\ 0 & -2XY & 1 - 2Y^2 & -2YZ \\ 0 & -2XZ & -2YZ & 1 - 2Z^2 \end{bmatrix} \begin{bmatrix} A \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

where $Q = A + \boldsymbol{\delta}B$, $\boldsymbol{\delta}B = (\eta_1, \eta_2, \eta_3)$, and the 4×1 matrix corresponds to Q while the 4×4 matrix corresponds to T. It can be easily checked that T is orthogonal, symmetric and $\det(T) = -1$ that means $f_{\boldsymbol{V}}(Q)$ represents a reflection in $\mathbb{D}^4$. Another geometric interpretation of the linear transformation $f_{\boldsymbol{V}}(Q)$ can be given as: It leaves the scalar part A of Q invariant, and in $\mathbb{R}^3$ it reflects the line $d_{\boldsymbol{\delta}}$ (corresponding to $\boldsymbol{\delta} = \boldsymbol{\delta}B/\sqrt{\boldsymbol{\delta}B}$) about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$), and afterwards changes the direction of the line (obtained after the reflection) oppositely.

Corollary 1. Let $Q = \boldsymbol{\delta}B = \eta_1\boldsymbol{i} + \eta_2\boldsymbol{j} + \eta_3\boldsymbol{k}$ *be a pure and* $\boldsymbol{V} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ *be a choosen unit pure dual-quaternions. Then, the matrix product*

$$\begin{bmatrix} 1 - 2X^2 & -2XY & -2XZ \\ -2XY & 1 - 2Y^2 & -2YZ \\ -2XZ & -2YZ & 1 - 2Z^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

reflects the line $d_{\boldsymbol{\delta}}$ *(corresponding to* $\boldsymbol{\delta} = \boldsymbol{\delta}B/\sqrt{\boldsymbol{\delta}B}$*) about the line* $d_{\boldsymbol{V}}$ *(corresponding to* $\boldsymbol{V}$*) and afterwards changes the direction of the line (obtained after reflection) oppositely.*

Example 1. Let

$$\boldsymbol{P} = \left(\frac{1 - \boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{i} + \left(\frac{1 + \boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{j} + \frac{1}{\sqrt{3}}\boldsymbol{k}, \quad \boldsymbol{V} = \boldsymbol{j}$$

be dual-quaternions. The matrix product of the involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\bar{\boldsymbol{P}}\boldsymbol{V}$ can be given as

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - \boldsymbol{\varepsilon} \\ 1 + \boldsymbol{\varepsilon} \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 - \boldsymbol{\varepsilon} \\ -1 - \boldsymbol{\varepsilon} \\ 1 \end{bmatrix}$$

that is $f_{\boldsymbol{V}}(Q)$ reflects the line corresponding to $\boldsymbol{P}$ about the line corresponding to $\boldsymbol{V}$, and afterwards changes its direction oppositely in $\mathbb{R}^3$. Furthermore, since $\boldsymbol{S} = -\boldsymbol{k}$ and $\{\boldsymbol{P}, \boldsymbol{V}, \boldsymbol{S}\}$ is a left-handed set, the product $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\bar{\boldsymbol{P}}\boldsymbol{V}$ can be given as a screw operator $Q\boldsymbol{P}\bar{Q}$, where

$$Q = \left(sin\frac{\pi}{4} - \boldsymbol{\varepsilon} cos\frac{\pi}{4}\right) - \boldsymbol{k}\left(cos\frac{\pi}{4} + \boldsymbol{\varepsilon} sin\frac{\pi}{4}\right)$$
$$= \left(\frac{\sqrt{2}}{2} - \boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right) - \boldsymbol{k}\left(\frac{\sqrt{2}}{2} + \boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right).$$

Proposition 3. Let $Q = (a + \boldsymbol{\mu}_1 b) + \boldsymbol{\varepsilon}(c + \boldsymbol{\omega}_1 e)$ *be an arbitrary dual-quaternion and* $\boldsymbol{V} = (\mp\boldsymbol{\mu}) + \boldsymbol{\varepsilon}(\boldsymbol{\omega} d)$ *be a choosen unit pure dual-quaternion and* $\boldsymbol{\mu} \perp \boldsymbol{\omega}$ *. Then, the transformation*

$$f_{\boldsymbol{V}}: \mathbb{H}_{\mathbb{D}} \to \mathbb{H}_{\mathbb{D}}\,; \quad Q \mapsto f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\overline{(Q^{\star})}\boldsymbol{V} = A^{\star} + \boldsymbol{V}\boldsymbol{\delta}^{\star}\boldsymbol{V}B^{\star}$$

is an involution under the following two restrictions, see [3]:

(i) *If* $\boldsymbol{V} = \pm\boldsymbol{\mu}$ *then* $Q = (a + \boldsymbol{\mu}_1 b) + \boldsymbol{\varepsilon}(c + \boldsymbol{\omega}_1 e)$,

(ii) *If* $Q = (a \pm \boldsymbol{\mu\omega} b) + \boldsymbol{\varepsilon}(c \pm \boldsymbol{\mu\omega} e)$ *then* $\boldsymbol{V} = \mp\boldsymbol{\mu} + \boldsymbol{\varepsilon}(\boldsymbol{\omega} d)$ *where* $d \neq 0$.

The matrix product of the involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\overline{(Q^{\star})}\boldsymbol{V}$ can be given as

$$T\overline{(Q^{\star})} \coloneqq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2X^2-1 & 2XY & 2XZ \\ 0 & 2XY & 2Y^2-1 & 2YZ \\ 0 & 2XZ & 2YZ & 2Z^2-1 \end{bmatrix} \begin{bmatrix} \overline{A^{\star}} \\ \overline{(\eta_1{}^{\star})} \\ \overline{(\eta_2{}^{\star})} \\ \overline{(\eta_3{}^{\star})} \end{bmatrix}$$

where $\overline{(Q^{\star})} = A^{\star} - (\boldsymbol{\delta}B)^{\star}$, $(\boldsymbol{\delta}B)^{\star} = \left(\overline{(\eta_1{}^{\star})}, \overline{(\eta_2{}^{\star})}, \overline{(\eta_3{}^{\star})}\right)$, and the 4×1 matrix corresponds to $\overline{(Q^{\star})}$ while the 4×4 matrix corresponds to T. Since T is orthogonal, symmetric and $\det(T) = -1$, the linear transformation $f_{\boldsymbol{V}}(Q)$ represents a reflection in $\mathbb{D}^4$. Another geometric interpretation of the linear transformation $f_{\boldsymbol{V}}(Q)$ can be given as: It reflects the scalar part A of Q about the real-axis of dual-plane, and in $\mathbb{R}^3$ it reflects the line $d_{\boldsymbol{\delta}^*}$ (corresponding to $\boldsymbol{\delta}^* = (\boldsymbol{\delta}B)^{\star}/\sqrt{(\boldsymbol{\delta}B)^{\star}}$) about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$), and afterwards changes the direction of the line (obtained after the reflection) oppositely.

Corollary 2. Let $Q = \boldsymbol{\delta}B = \eta_1\boldsymbol{i} + \eta_2\boldsymbol{j} + \eta_3\boldsymbol{k}$ *be a pure and* $\boldsymbol{V} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ *be a choosen unit pure dual-quaternions with the two restrictions given by Proposition 3. Then, the matrix product*

$$\begin{bmatrix} 2X^2-1 & 2XY & 2XZ \\ 2XY & 2Y^2-1 & 2YZ \\ 2XZ & 2YZ & 2Z^2-1 \end{bmatrix} \begin{bmatrix} \overline{(\eta_1{}^{\star})} \\ \overline{(\eta_2{}^{\star})} \\ \overline{(\eta_3{}^{\star})} \end{bmatrix}$$

reflects the line $d_{\boldsymbol{\delta}^*}$ *(corresponding to* $\boldsymbol{\delta}^* = (\boldsymbol{\delta}B)^{\star}/\sqrt{(\boldsymbol{\delta}B)^{\star}}$ *) about the line* $d_{\boldsymbol{V}}$ *(corresponding to* $\boldsymbol{V}$*) and afterwards changes the direction of the line (obtained after reflection) oppositely.*

Example 2. Let

$$\boldsymbol{P} = \left(\frac{-1-\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{i} + \left(\frac{1-\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{j} + \frac{1}{\sqrt{3}}\boldsymbol{k}, \;\; \boldsymbol{V} = \boldsymbol{j}$$

be dual-quaternions. The matrix product of the involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\overline{(\boldsymbol{P}^{\star})}\boldsymbol{V}$ can be given as

$$\frac{1}{\sqrt{3}}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1-\boldsymbol{\varepsilon}\\ -1-\boldsymbol{\varepsilon}\\ -1\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}-1+\boldsymbol{\varepsilon}\\ -1-\boldsymbol{\varepsilon}\\ 1\end{bmatrix}$$

that is $f_{\boldsymbol{V}}(Q)$ reflects the line corresponding to $\boldsymbol{P}^{\star}$ about the line corresponding to $\boldsymbol{V}$ and afterwards changes its direction oppositely in $\mathbb{R}^3$. Furthermore, since $\boldsymbol{S}=\boldsymbol{k}$ and $\{\boldsymbol{P}^{\star},\boldsymbol{V},\boldsymbol{S}\}$ is a left-handed set, the product $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}\overline{(\boldsymbol{P}^{\star})}\boldsymbol{V}$ can be given as a screw operator $Q\boldsymbol{P}^{\star}\bar{Q}$, where

$$\begin{aligned} Q &= \left(sin\frac{\pi}{4}-\boldsymbol{\varepsilon}cos\frac{\pi}{4}\right)+\boldsymbol{k}\left(cos\frac{\pi}{4}+\boldsymbol{\varepsilon}sin\frac{\pi}{4}\right)\\ &=\left(\frac{\sqrt{2}}{2}-\boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right)+\boldsymbol{k}\left(\frac{\sqrt{2}}{2}+\boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right). \end{aligned}$$

3.2. Anti-Involution Matrices of Dual Quaternions

Proposition 4. Let $Q=(a+\boldsymbol{\mu}_1 b)+\boldsymbol{\varepsilon}(c+\boldsymbol{\omega}_1 e)$ be an arbitrary dual-quaternion and $\boldsymbol{V}=(\mp\boldsymbol{\mu})+\boldsymbol{\varepsilon}(\boldsymbol{\omega}d)$ be a choosen unit pure dual-quaternion for $\boldsymbol{\mu}\perp\boldsymbol{\omega}$. Then, the transformation

$$f_{\boldsymbol{V}}:\mathbb{H}_{\mathbb{D}}\to\mathbb{H}_{\mathbb{D}}\,;\quad Q\mapsto f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}Q^{\star}\boldsymbol{V}=A^{\star}-\boldsymbol{V}\boldsymbol{\delta}^{\star}\boldsymbol{V}B^{\star}$$

is an anti-involution under the following two restrictions, see [7]:

(i) *If $\boldsymbol{V}=\pm\boldsymbol{\mu}$ then $Q=(a+\boldsymbol{\mu}_1 b)+\boldsymbol{\varepsilon}(c+\boldsymbol{\omega}_1 e)$,*

(ii) *If $Q=(a\pm\boldsymbol{\mu\omega}b)+\boldsymbol{\varepsilon}(c\pm\boldsymbol{\mu\omega}e)$ then $\boldsymbol{V}=\mp\boldsymbol{\mu}+\boldsymbol{\varepsilon}(\boldsymbol{\omega}d)$ where $d\neq 0$.*

The matrix product of the anti-involution transformation $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}Q^{\star}\boldsymbol{V}$ can be given as

$$TQ^{\star}:=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 2X^2-1 & 2XY & 2XZ\\ 0 & 2XY & 2Y^2-1 & 2YZ\\ 0 & 2XZ & 2YZ & 2Z^2-1\end{bmatrix}\begin{bmatrix}A^{\star}\\ {\eta_1}^{\star}\\ {\eta_2}^{\star}\\ {\eta_3}^{\star}\end{bmatrix}$$

where $Q^{\star}=A^{\star}+(\boldsymbol{\delta}B)^{\star}$, $(\boldsymbol{\delta}B)^{\star}=({\eta_1}^{\star},{\eta_2}^{\star},{\eta_3}^{\star})$, $\boldsymbol{V}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$, and the 4×1 matrix corresponds to $Q^{\star}$ while the 4×4 matrix corresponds to T. Since T is orthogonal, symmetric and $\det(T)=+1$, the linear transformation $f_{\boldsymbol{V}}(Q)$ represents a rotation in $\mathbb{D}^4$. Another geometric interpretation of the linear transformation $f_{\boldsymbol{V}}(Q)$ can be given as: It reflects the scalar part A of Q about the real-axis of dual-plane, and in $\mathbb{R}^3$ it reflects the line $d_{\boldsymbol{\delta}^*}$ (corresponding to $\boldsymbol{\delta}^*=(\boldsymbol{\delta}B)^{\star}/\sqrt{(\boldsymbol{\delta}B)^{\star}}$) about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$).

Corollary 3. Let $Q=\boldsymbol{\delta}B=\eta_1\boldsymbol{i}+\eta_2\boldsymbol{j}+\eta_3\boldsymbol{k}$ be a pure and $\boldsymbol{V}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$ be a choosen unit pure dual-quaternions with the two restrictions given by Propositin 4. Then the matrix product

$$\begin{bmatrix}2X^2-1 & 2XY & 2XZ\\ 2XY & 2Y^2-1 & 2YZ\\ 2XZ & 2YZ & 2Z^2-1\end{bmatrix}\begin{bmatrix}{\eta_1}^{\star}\\ {\eta_2}^{\star}\\ {\eta_3}^{\star}\end{bmatrix}$$

reflects the line $d_{\boldsymbol{\delta}^*}$ (corresponding to $\boldsymbol{\delta}^* = (\boldsymbol{\delta}B)^\star/\sqrt{(\boldsymbol{\delta}B)^\star}$ about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$).

Example 3. Let

$$\boldsymbol{P} = \frac{1}{\sqrt{3}}\boldsymbol{i} + \left(\frac{1+\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{j} + \left(\frac{1-\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{k},\ \ \boldsymbol{V} = \boldsymbol{j}$$

be dual-quaternions. The matrix product of the anti-involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\boldsymbol{P}^\star\boldsymbol{V}$ can be given as

$$\frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1-\boldsymbol{\varepsilon} \\ 1+\boldsymbol{\varepsilon} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1-\boldsymbol{\varepsilon} \\ -1-\boldsymbol{\varepsilon} \end{bmatrix}$$

that is $f_{\boldsymbol{V}}(Q)$ reflects the line corresponding to $\boldsymbol{P}^\star$ about the line corresponding to $\boldsymbol{V}$. Furthermore, since $\boldsymbol{S} = -\boldsymbol{i}$ and $\{\boldsymbol{P}^\star, \boldsymbol{V}, \boldsymbol{S}\}$ is a right-handed set, the product $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}\overline{(\boldsymbol{P}^\star)}\boldsymbol{V}$ can be given as a screw operator $-Q\boldsymbol{P}^\star\bar{Q}$, where

$$\begin{aligned} Q &= \left(-sin\frac{\pi}{4} - \boldsymbol{\varepsilon} cos\frac{\pi}{4}\right) - \boldsymbol{i}\left(cos\frac{\pi}{4} - \boldsymbol{\varepsilon} sin\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{2}}{2} - \boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right) - \boldsymbol{i}\left(\frac{\sqrt{2}}{2} - \boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right). \end{aligned}$$

Proposition 5. Let $Q = A + \boldsymbol{\delta}B$ be an arbitrary dual-quaternion, then the transformation

$$f_{\boldsymbol{V}}: \mathbb{H}_{\mathbb{D}} \to \mathbb{H}_{\mathbb{D}}\,; \quad Q \mapsto f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}Q\boldsymbol{V} = A - \boldsymbol{V}\boldsymbol{\delta}\boldsymbol{V}B$$

is an anti-involution for a choosen unit pure dual-quaternion $\boldsymbol{V}$, see [7].

The matrix product of the anti-involution transformation $f_{\boldsymbol{V}}(Q) = -\boldsymbol{V}Q\boldsymbol{V}$ can be given as

$$TQ := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2X^2-1 & 2XY & 2XZ \\ 0 & 2XY & 2Y^2-1 & 2YZ \\ 0 & 2XZ & 2YZ & 2Z^2-1 \end{bmatrix}\begin{bmatrix} A \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

where $Q = A + \boldsymbol{\delta}B$, $\boldsymbol{\delta}B = (\eta_1, \eta_2, \eta_3)$, and the 4×1 matrix corresponds to Q while the 4×4 matrix corresponds to T. It can be easily checked that T is orthogonal, symmetric and $\det(T) = +1$ that means $f_{\boldsymbol{V}}(Q)$ represents a rotation in $\mathbb{D}^4$. Also, the geometry of the linear transformation $f_{\boldsymbol{V}}(Q)$ can be given in $\mathbb{R}^3$ as: It leaves the scalar part A of Q invariant, and in $\mathbb{R}^3$ it reflects the line $d_{\boldsymbol{\delta}}$ (corresponding to $\boldsymbol{\delta} = \boldsymbol{\delta}B/\sqrt{\boldsymbol{\delta}B}$) about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$).

Corollary 4. Let $Q = \boldsymbol{\delta}B = \eta_1\boldsymbol{i} + \eta_2\boldsymbol{j} + \eta_3\boldsymbol{k}$ be a pure and $\boldsymbol{V} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ be a choosen unit pure dual-quaternions. Then, the matrix product

$$\begin{bmatrix} 2X^2-1 & 2XY & 2XZ \\ 2XY & 2Y^2-1 & 2YZ \\ 2XZ & 2YZ & 2Z^2-1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}$$

reflects the line $d_{\boldsymbol{\delta}}$ (corresponding to $\boldsymbol{\delta} = \boldsymbol{\delta}B/\sqrt{\boldsymbol{\delta}B}$) about the line $d_{\boldsymbol{V}}$ (corresponding to $\boldsymbol{V}$).

Example 4. Let

$$\boldsymbol{P}=\left(\frac{1-\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{i}+\left(\frac{1+\boldsymbol{\varepsilon}}{\sqrt{3}}\right)\boldsymbol{j}+\frac{1}{\sqrt{3}}\boldsymbol{k},\ \ \boldsymbol{V}=\boldsymbol{j}$$

be dual-quaternions. The matrix product of the involution transformation $f_{\boldsymbol{V}}(Q)=-\boldsymbol{VPV}$ can be given as

$$\frac{1}{\sqrt{3}}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1-\boldsymbol{\varepsilon}\\ 1+\boldsymbol{\varepsilon}\\ 1\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}-1+\boldsymbol{\varepsilon}\\ 1+\boldsymbol{\varepsilon}\\ -1\end{bmatrix}$$

that is $f_{\boldsymbol{V}}(Q)$ reflects the line corresponding to $\boldsymbol{P}$ about the line corresponding to $\boldsymbol{V}$. Furthermore, since $\boldsymbol{S}=-\boldsymbol{k}$ and $\{\boldsymbol{P},\boldsymbol{V},\boldsymbol{S}\}$ is a left-handed set, the product $f_{\boldsymbol{V}}(Q)=-\boldsymbol{VPV}$ can be given as a screw operator $-Q\boldsymbol{P}\bar{Q}$, where

$$\begin{aligned}Q &= \left(sin\frac{\pi}{4}-\boldsymbol{\varepsilon}cos\frac{\pi}{4}\right)-\boldsymbol{k}\left(cos\frac{\pi}{4}+\boldsymbol{\varepsilon}sin\frac{\pi}{4}\right)\\ &=\left(\frac{\sqrt{2}}{2}-\boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right)-\boldsymbol{k}\left(\frac{\sqrt{2}}{2}+\boldsymbol{\varepsilon}\,\frac{\sqrt{2}}{2}\right).\end{aligned}$$

4. Conclusion

The matrices of the dual-quaternion involution transformations $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}\bar{Q}\boldsymbol{V}$ and $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}\overline{(Q^{\star})}\boldsymbol{V}$ represent reflections in $\mathbb{D}^4$, however they represent reflections in $\mathbb{R}^3$. Also, the matrices of the anti-involution transformations $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}Q^{\star}\boldsymbol{V}$ and $f_{\boldsymbol{V}}(Q)=-\boldsymbol{V}Q\boldsymbol{V}$ represent rotations in $\mathbb{D}^4$, however they represent reflection in $\mathbb{R}^3$.

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