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On Fuzzy δ-I-Open Sets and Decomposition of Fuzzy α-I-continuity

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Abstract: We introduce the notions of fuzzy δ-I-open sets and fuzzy semi δ-I-continuos functions in fuzzy ideal topological space and investigate some of their properties. Additionaly, we obtain decompositions of fuzzy semi-I-continuous functions and fuzzy α-I-continuous functions by using fuzzy δ-I-open sets.

Key words: Fuzzy δ-I-open sets, fuzzy semi δ-I-continuity, fuzzy α-I-continuity

Bulanık Delta-I-Açık Kümeler ve Bulanık Alfa-I-Sürekliliğin Dağılımı Üzerine

Özet: Bulanık ideal topolojik uzaylarda bulanık delta-I-açık küme ve bulanık yarı delta-I-sürekli fonksiyon kavramlarını tanımladık ve bunların bazı özelliklerini araştırdık. Ayrıca, bulanık delta-I-açık kümeleri kullanarak bulanık alfa-I-sürekli ve bulanık yarı-I-sürekli fonksiyonların ayrışımını elde ettik.

Anahtar kelimeler: Bulanık delta-I-açık kümeler, bulanık yarı delta-I-süreklilik, bulanık alfa-I- süreklilik

1. Introduction

The fundamental concept of a fuzzy set was introduced by Zadeh [1]. Subsequently, Chang [2] defined the notion of fuzzy topology. An alternative definition of fuzzy topology was given by Lowen [3]. In general topolgy, by introducing the notion of ideal, Kuratowskı [4], Vaıdyanathaswamy [5,6] and several other authors carried out such analyses. There has been an extensive study on the importance of ideal in general topology in the paper of Janković and Hamlet [7]. Recently, in ideal topological spaces, new continuity types have been studied by Acıkgoz [8-10]. Sarkar [11] introduced the notions of fuzzy ideal and fuzzy local function in fuzzy set theory. In Mahmoud [12] and Nasef [13,14], independently presented some of the ideal concepts in the fuzzy trend and studied many of their properties.
In this paper, we define fuzzy δ-I-open set and fuzzy strong β-I-open set via fuzzy ideal. Moreover, we obtain decompositions of fuzzy semi-I-opens set and fuzzy strong β-I-open sets.

2. Preliminaries

Throughout this paper, X represents a nonempty fuzzy set and fuzzy subset A of X, denoted by $A \leq X$, then is characterized by a membership function in the sense of Zadeh [1]. The basic fuzzy sets are the empty set, the whole set and the class of all fuzzy sets of X which will be denoted by 0, 1 and I^X, respectively. A subfamily τ of I^X is called a fuzzy topology due to Chang [2]. Morever, the pair (X,τ) will be meant by a fuzzy topological space, on which no separation axioms are assumed unless explicitly stated. The fuzzy closure, the fuzzy interior and the fuzzy complement of any set A in (X,τ) are denoted by Cl(A), Int(A) and 1-A, respectively. A fuzzy set which is a fuzzy point with support $x\in X$ and value $\lambda\in(0,1]$ will be designated by x^λ [15]. Also, for a fuzzy point x^λ and a fuzzy set A we shall write $x^\lambda \in A$ to mean that $\lambda \leq A(x)$. The value of a fuzzy set A for some $x\in X$ will be denoted by A(x). For any two fuzzy sets A and B in (X,τ), $A\leq B$ if and only if $A(x)\leq B(x)$ for each $x\in X$. A fuzzy set in (X,τ) is said to be quasi-coincident with a fuzzy set B, denoted by AqB, if there exists $x\in X$ such that $A(x) + B(x) > 1$ [16]. A fuzzy set V in (X,τ) is called a q-neighbourhood (q-nbd, for short) of a fuzzy point x^λ if and only if there exists a fuzzy open set U such that $x^\lambda\, qU \leq V$ [16, 17]. We will denote the set of all q-nbd of x^λ in (X,τ) by $N(x^\lambda)$. A nonempty collection of fuzzy sets I of a set X is called a fuzzy ideal on X, [11, 12], if and only if (1) $A\in I$ and $B \leq A$, then $B\in I$ (heredity), (2) if $A\in I$ and $B\in I$, then $A \vee B\in I$ (finite additivity). The triple (X,τ,I) means fuzzy topological space with a fuzzy ideal I and fuzzy topology τ. For (X,τ,I), the fuzzy local function of $A \leq X$ with respect to τ and I is denoted by $A^*(\tau,I)$ (briefly A^*) [11]. The fuzzy local function $A^*(\tau,I)$ of A is the union of all fuzzy points x^λ such that if $U\in N(x^\lambda)$ and $E\in I$ then there is at least one $y\in X$ for which $U(y)+A(y)-1 > E(y)$ [11]. Fuzzy closure operator of a fuzzy set A in (X,τ,I) is defined as $C^*(A) = A \vee A^*$ [11]. In (X,τ,I), the collection $\tau^*(I)$ means an extension of fuzzy topological space than τ via fuzzy ideal which is constructed by considering the class $\beta = \{U-E: U\in\tau, E\in I\}$ as a base [11]. A subset A of a fuzzy ideal topological space (X,τ,I) is called to be fuzzy α-I-open [18] (resp. fuzzy semi-I-open set [19], fuzzy pre-I-open set [14] if $A \leq Int(Cl^*(Int(A)))$ (resp. $A \leq Cl^*(Int(A))$, $A \leq Int (Cl^*(A))$).

3. Fuzzy δ-I-Open Sets

Definition 1.1. A subset A of a fuzzy ideal topological space (X, τ, I) is called fuzzy δ-I-open (resp.fuzzy strong β-I-open) set if

$$Int(Cl^*(A))\leq Cl^*(Int(A)) \text{ (resp. } A\leq Cl^*(Int(Cl^*(A)))).$$

The family of all fuzzy δ-I-open (resp.fuzzy strong β-I-open) sets of (X, τ, I) is denoted by FδIO(X) (resp. FSβIO(X)). A subset A of a fuzzy ideal topological space (X, τ, I) is said to be fuzzy δ-I-closed (resp.fuzzy strong β-I-closed) if its complement is fuzzy δ-I-open (resp. fuzzy strong β-I-open).

Proposition 1.2. Let (X, τ, I) be a fuzzy ideal topological space. Then a subset of X is fuzzy semi-I-open if and only if it is both fuzzy δ-I-open and fuzzy strong β-I-open.

Proof. Necessity. Let A be a fuzzy semi-I-open set, then we have

$$A\leq Cl^*(Int(A))\leq Cl^*(Int(Cl^*(A))).$$

This shows that A is fuzzy strong β-I-open. Moreover,

$$Int(Cl^*(A)) \leq Cl^*(A) \leq Cl^*(Cl^*(Int(A))) = Cl^*(Int(A)).$$

Therefore, A is fuzzy δ-I-open.

Sufficiency. Let A be fuzzy δ-I-open and fuzzy strong β-I-open, then we have

$$Int(Cl^*(A)) \leq Cl^*(Int(A)).$$

Thus we obtain that

$$Cl^*(Int(Cl^*(A))) \leq Cl^*(Cl^*(Int(A))) = Cl^*(Int(A)).$$

Since A is fuzzy strong β-I-open, we have

$$A \leq Cl^*(Int(Cl^*(A))) \leq Cl^*(Int(A))$$

and

$$A \leq Cl^*(Int(A)).$$

Hence A is a fuzzy semi-I-open set.

Proposition 1.3. Let (X, τ, I) be a fuzzy ideal topological space. Then a subset of X is fuzzy α-I-open if and only if it is both fuzzy δ-I-open and fuzzy pre-I-open.

Proof. Necessity. Let A be a fuzzy α-I-open set. Since every fuzzy α-I-open set is fuzzy semi-I-open, by Proposition 1,2. A is fuzzy δ-I-open set. Now we prove that

$$A \leq Int(Cl^*(A)).$$

Since A is a fuzzy α-I-open, we have

$$A \leq Int(Cl^*(Int(A))) \leq Int(Cl^*(A)).$$

Hence A is a fuzzy pre-I-open set.

Sufficiency. Let A be fuzzy δ-I-open and fuzzy pre-I-open set. Then we have

$$Int(Cl^*(A)) \leq Cl^*(Int(A))$$

and hence

$$Int(Cl^*(A)) \leq Int(Cl^*(Int(A))).$$

Since A is fuzzy pre-I-open, we have $A \leq Int(Cl^*(A))$. Therefore we obtain that

$$A \leq Int(Cl^*(Int(A)))$$

and hence A is fuzzy α-I-open set.

Remark 1.4. By the Example 1.4.1 and Example 1.4.2, we obtain the following results.

(1) Fuzzy δ-I-openness and fuzzy strong β-I-openness are independent of each other,
(2) Fuzzy δ-I-openness and fuzzy pre-I-openness are independent of eac other.

Example 1.4.1. Let X={a,b,c} and A, B be fuzzy sets of X defined as follows:

$$A(a) = 0.2,\ A(b) = 0.7,\ A(c) = 0.4$$
$$B(a) = 0.7,\ B(b) = 0.9,\ B(c) = 0.1$$

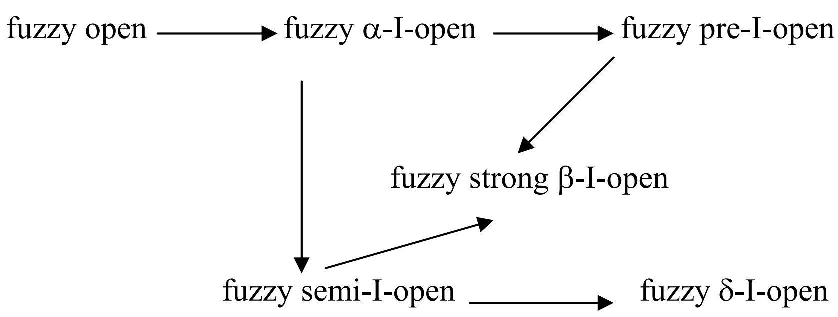
We put τ = {0, A, 1} and I = {0}. Then B is fuzzy pre-I-open and fuzzy strong β-I-open, but B is not fuzzy δ-I-open.

Example 1.4.2. Let X={a,b,c} and A, B be fuzzy sets of X defined as follows:

$$A(a) = 0.7,\ A(b) = 0.3,\ A(c) = 0.4$$
$$B(a) = 0.8,\ B(b) = 0.4,\ B(c) = 0.5$$

We put $\tau = \{0, A, 1\}$ and $I = \wp(X)$. Then B is fuzzy δ-I-open, but B is neither fuzzy strong β-I-open and nor fuzzy pre-I-open.

Remark 1.5. By Proposition 1.2, Remark 1.4 and [18], we have the following diagram:



Proposition 1.6. Let A, B be subsets of a fuzzy ideal topological space (X, τ, I). If $A \le B \le Cl^*(A)$ and $A \in F\delta IO(X)$, then $B \in F\delta IO(X)$.

Proof. Suppose that $A \le B \le Cl^*(A)$ and $A \in F\delta IO(X)$. Then, since $A \in F\delta IO(X)$, we have

$$Int(Cl^*(A)) \le Cl^*(Int(A)).$$

Since $A \le B$, we have

$$Cl^*(Int(A)) \le Cl^*(Int(B))$$

and

$$Int(Cl^*(A)) \le Cl^*(Int(B)).$$

Since $B \le Cl^*(A)$, we have

$$Cl^*(B) \le Cl^*(Cl^*(A)) = Cl^*(A)$$

and

$$Int(Cl^*(B)) \le Int(Cl^*(A)).$$

Therefore, we obtain that $Int(Cl^*(B)) \le Cl^*(Int(B))$. This shows that B is fuzzy δ-I-open.

Definition 1.7. A subset A of a fuzzy ideal topological space (X, τ, I) is called fuzzy τ^*-dense set if $Cl^*(A) = X$.

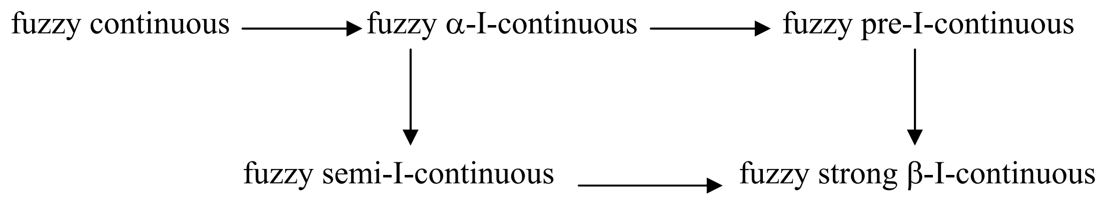
Corollary 1.8. Let (X, τ, I) be a fuzzy ideal topological space. If $A \le X$ is fuzzy δ-I-open and fuzzy τ^*-dense, then every subset of X containing A is fuzzy δ-I-open.

Proof. The proof is obvious by Proposition 1.6.

4. On Decomposition of Fuzzy α-I-continuity and Fuzzy Semi-I-Continuity

Definition 1.9. A function $f : (X, \tau, I) \to (Y, \varphi)$ is called fuzzy strong β-I-continuous (resp. fuzzy α-I-continuous [18], fuzzy semi-I-continuous [19], fuzzy pre-I-continuous [14] if for every $V \in \varphi$, $f^{-1}(V)$ is fuzzy strong β-I-open (resp. fuzzy α-I-open, fuzzy semi-I-open, fuzzy pre-I-open) in (X, τ, I).

Remark 1.10. By Definition 1.9, we have the following diagram in which none of the implications is reversible as shown by Example 1.10.1 and Example 1.10.2.



Example 1.10.1. Let X={a,b,c}, Y={0.1, 0.3, 0.7}, τ={0, A, 1}, φ={0, B, 1} and I={0}. A is a fuzzy set of X and B is a fuzzy set of Y defined as follows:

A(a)=0.2, A(b)=0.7, A(c)=0.4

B(0.1)=0.6, B(0.3)=0.3, B(0.7)=0.8

Let $f : (X, \tau, I) \rightarrow (Y, \varphi)$ be a function defined as follows:

f(a)=0.1, f(b)=0.7, f(c)=0.3.

Then f is fuzzy pre-I-continuous, but it is not fuzzy semi-I-continuous.

(1) For $B\in\varphi$, we have

$$f^{-1}(B)(a)=B(f(a))=B(0.1)=0.6,$$
$$f^{-1}(B)(b)=B(f(b))=B(0.7)=0.8,$$
$$f^{-1}(B)(c)=B(f(c))=B(0.3)=0.3.$$

Set $f^{-1}(B) = D$. Since $D\leq Int(Cl^*(D))$, D is fuzzy pre-I-open.

(2) For $1\in\varphi$, we have $f^{-1}(1) = 1$. It is obvious that 1 is fuzzy pre-I-open.

(3) For $0\in\varphi$, we have $f^{-1}(0) = 0$. It is obvious that 0 is fuzzy pre-I-open.

By (1), (2), (3); f is fuzzy pre-I-continuous. Since Int(D) = 0 and $Cl^*(D) = 1$, D is not fuzzy δ-I-open and hence not fuzzy semi-I-open. Thus f is not fuzzy semi-I-continuous.

Example 1.10.2. Let X={a,b,c}, Y={0.3, 0.5, 0.7}, τ={0, A, 1}, φ={0, B, 1} and I={0}. A is a fuzzy set of X and B is a fuzzy set of Y defined as follows:

A(a)=0.2, A(b)=0.4, A(c)=0.1

B(0.3)=0.6, B(0.5)= 0.4, B(0.7)=0.7

Let $f : (X, \tau, I) \rightarrow (Y, \varphi)$ be a function defined as follows:

f(a)=0.7, f(b)=0.5, f(c)=0.3.

Then f is fuzzy semi-I-continuous, but it is not fuzzy pre-I-continuous.

(1) For $B\in\varphi$, we have

$$f^{-1}(B)(a)=B(f(a))=B(0.7)=0.7,$$
$$f^{-1}(B)(b)=B(f(b))=B(0.5)=0.4,$$
$$f^{-1}(B)(c)=B(f(c))=B(0.3)=0.6.$$

Set $f^{-1}(B) = D$. Since $D \leq Cl^*(Int(D))$, D is fuzzy semi-I-open.

(2) For $1\in\varphi$, we have $f^{-1}(1)=1$. It is obvious that 1 is fuzzy semi-I-open.

(3) For $0\in\varphi$, we have $f^{-1}(0)=0$. It is obvious that 0 is fuzzy semi-I-open.

By (1), (2), (3); f is fuzzy semi-I-continuous. Since $Int(Cl^*(D))=A$ and $A\leq D$, D is not fuzzy pre-I-open. Thus f is not fuzzy pre-I-continuous.

Definition 1.11. A function f:(X, τ, I)→(Y, φ) is called fuzzy semi-δ-I-continuous if for every V∈φ, $f^{-1}(V)$∈FδIO(X).

Theorem 1.12. For a fuction f:(X, τ, I)→(Y, φ), the following properties are equivalent:
(a) f is fuzzy semi-I-continuous,
(b)f is fuzzy strong β-I-continuous and fuzzy semi-δ-I-continuous.

Proof. The proof is obvious by Proposition 1.2.

Theorem 1.13. For a fuction f:(X, τ, I)→(Y, φ), the following properties are equivalent:
(a) f is fuzzy α-I-continuous.
(b) f is fuzzy pre-I-continuous and fuzzy semi-I-continuous.
(c) f is fuzzy pre-I-continuous and and fuzzy semi-δ-I-continuous.

Proof. The proof is obvious by Proposition 1.2. and Proposition 1.3.

Remark 1.14. By Example 1.14.1. and Example 1.14.2. we can realize the following properties:
(a) fuzzy strong β-I-continuity and fuzzy semi-δ-I-continuity are independent of each other.
(b)fuzzy pre-I-continuity and and fuzzy semi-δ-I-continuity are independent of each other.

Example 1.14.1. Let (X, τ, I) be the same fuzzy ideal topological space and A the subset of X as in Example 1.10.2. We obtain that A is a fuzzy pre-I-open set which is not fuzzy semi-I-open. Thus f is a fuzzy pre-I-continuous function which is not fuzzy semi-δ-I-continuous.

Example 1.14.2. Let X={a,b,c}, Y={0.1, 0.5, 0.7}, τ={0, A, 1}, φ={0, B, 1} and I=℘(X). A is a fuzzy set of X and B is a fuzzy set of Y defined as follows:

A(a)=0.8, A(b)=0.2, A(c)=0.4

B(0.1)=0.9, B(0.5)=0.4, B(0.7)=0.7

Let f:(X, τ, I)→(Y, φ) be a function defined as follows:

f(a)=0.1, f(b)=0.5, f(c)=0.7.

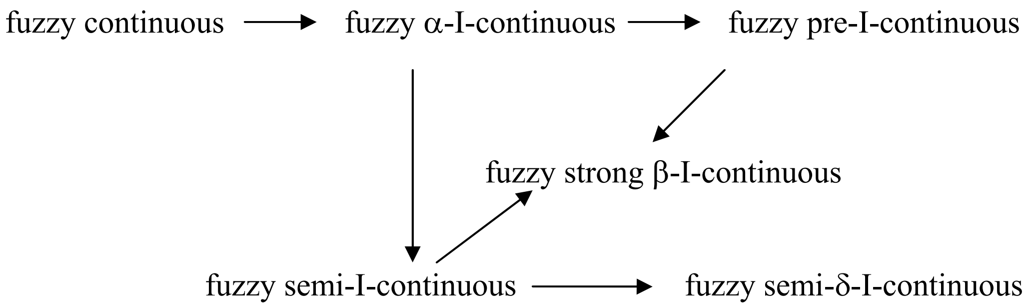
Then f is fuzzy semi-δ-I-continuous, but it is not fuzzy strong β-I-continuous.

(1) For B∈φ, we have

$f^{-1}(B)(a)=B(f(a))=B(0.1)=0.9,$
$f^{-1}(B)(b)=B(f(b))=B(0.5)=0.4,$
$f^{-1}(B)(c)=B(f(c))=B(0.7)=0.7.$

Set $f^{-1}(B)$ = D. Since $Int(Cl^*(D))\leq Cl^*(Int(D))$, D is fuzzy δ-I-open.
(2) For 1∈φ, we have $f^{-1}(1) = 1$. It is obvious that 1 is fuzzy δ-I-open.
(3) For 0∈φ, we have $f^{-1}(0) = 0$. It is obvious that 0 is fuzzy δ-I-open.
By (1), (2), (3); f is fuzzy semi-δ-I-continuous. Since Int(D)=A and A≤D, D is not fuzzy strong β-I-open. Thus f is not fuzzy strong β-I-continuous.

Remark 1.15. By Definition 1.9, Definition 1.11. and Remark 1.14., we have the following diagram:



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