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Idealization of Some Topological Concepts

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Abstract. We introduce a notion of β-open sets in terms of ideals, which generalizes the usual notion of β-open sets.

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1. Introduction

The notion of ideal topological spaces was studied by Kuratowski [6] and Vaidyanathasmamy [11]. Applications to various fields were further investigated by Jankovic and Hamlett [5]; Dontchev *et al.* [3], Mukherjee *et al.* [9]; Arenos *et al.* [2]; Nasef and Mahmoud [10], etc. The interest in the idealized version of many general topological properties has grown drastically in the past 20 years.

In this paper, we define β-open sets with respect to an ideal I, and also study some of their properties. It turns out that our notion of β-open sets with respect to a given ideal I generalizes both the usual notion of β-openness [1] and the notion of $\beta - I$-openness considered in [4], in particular, $\beta - I$-openness implies the usual β-openness,which in turn implies β-openness in our sense. This paper is related to [8].

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2. Preliminaries

Throughout this paper (X,τ) (or simply X) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, $cl(A)$, $int(A)$ and A^c denote the closure of A, the interior of A and the complement of A, respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 1. *A subset A of a space* (X,τ) *is called a*

(i) preopen set [7] if $A \subseteq int(cl(A))$,

(ii) β*-open set [1] if* $A \subseteq cl(int(cl(A)))$.

The complement of a preopen (resp. β-open) set is called preclosed (resp. β-closed).

Definition 2. *An ideal I on a topological space (X,τ) is a non empty collection of subsets of X which satisfies the following conditions:*

(i) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity),

(ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additinity).

An ideal topological space (X,τ) with an ideal I on X is denoted by (X,τ,I).

Definition 3. *For a subset $A \subseteq X$, $A^*(I) = \{x \in X : G \cap A \notin I$ for each neighborhood G of $x\}$ is called the local function of A with respect to I and τ [11]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion.*

Definition 4. *A subset A of an ideal topological space (X,τ,I) is said to be $\beta - I$-open [4] if $A \subseteq cl(int(cl^*(A)))$.*

3. β-Openness with Respect to an Ideal

Let X be a topological space. Recall that a subset A of X is said to be β-open [1] if there is a preopen set G such that $G \subseteq A \subseteq cl(G)$. This motivates our first definition.

Definition 5. *A subset A of an ideal topological space (X,τ) is said to be β-open with respect to an ideal I (written as $I-\beta$-open) if there exists a preopen set G such that $G \setminus A \in I$ and $A \setminus cl(G) \in I$.*

If $A \in I$,then it is easy to see that A is $I-\beta$-open. Moreover, every open set A is β-open, and every β-open set B is $I-\beta$-open, for any ideal on X.

Example 1. *Consider a topological space (X,τ); $X = \{a,b,c\}$ and $\tau = \{\emptyset,\{a\},\{a,c\},X\}$. Choose $I = \{\emptyset,\{b\},\{c\},\{b,c\}\}$, and observe that $\{b\}$ is $I-\beta$-open, however $\{b\}$ is not β-open in the sense of [1] as there is no preopen set G such that $G \subseteq \{b\} \subseteq cl(G)$. Thus, if a set is $I-\beta$-open, it may not be β-open in the usual sense.*

For an ideal I that is not countably additive, the concepts of β-openness and $I-\beta$-openness coincide in the following case.

Theorem 1. *For an ideal I on a topological space (X,τ), the following are equivalent:*

(i) I is the minimal ideal on X, that is $I = \{\emptyset\}$,

(ii) The concepts of β-openness and $I-\beta$-openness are the same.

Proof. First suppose that $I = \{\emptyset\}$. It suffices to show that whenever a set A is $I-\beta$-open, then it is β-open in the usual sense. Indeed, if A is $I-\beta$-open, then there is a preopen set G such that $G \setminus A$,and $A \setminus cl(G) \in I = \{\emptyset\}$, and so $G \subseteq A \subseteq cl(G)$, proving that A is β-open. Conversely, suppose that whenever a set A is $I-\beta$-open, then it is β-open. Let $B \in I$. Then, B is $I-\beta$-open, and by assumption, B is β-open. Thus, there is a preopen set H_1, such that $H_1 \subseteq B \subseteq cl(H_1)$. Since $B \in I$ and $H_1 \subseteq B$,we have that $H_1 \in I$, and so $B \cup H_1 \in I$. As $B \cup H_1$ is $I-\beta$-open, it is β-open, so that there is a preopen set H_2 for which $H_2 \subseteq (B \cup H_1) \subseteq cl(H_2)$. Similarly, there is a preopen set H_3 such that $H_3 \subset (B \cup H_1 \cup H_2) \subset cl(H_3)$. Continuing in this way, we have an infinite collection of preopen sets $H_1, H_2, H_3, \ldots$, such that $B \cup H_1 \cup H_2 \cup H_3 \cup \ldots \in I$, which is impossible, as the ideal I is not closed under countable additivity. Thus, it must be the case that $H_1 = \emptyset$ (similarly for the other $H_i' s$, $i = 1,2,3,\ldots$), therefore, $cl(H_1) = \emptyset$, and the relations $H_1 \subseteq B \subseteq cl(H_1)$, then give $B = \emptyset$, proving that $I = \{\emptyset\}$. □

Proposition 1. *Let I and Jbe two ideals on a topological space (X,τ),*

(i) If $I \subseteq J$, then every $I-\beta$-open set A is $J-\beta$-open;

(ii) If A is $(I \cap J)-\beta$-open, then it is simultaneously $I-\beta$-open and $J-\beta$-open.

Corollary 1. *For a subset A of an ideal topological space (X,τ), recall that $I_A = \{A \cap E : E \in I\}$ is also an ideal on X.*

(i) If a set B is $I_A-\beta$-open,then it is $I-\beta$-open,

(ii) If $A = \emptyset$,then $I_A = I_\emptyset = \{\emptyset\}$,vthe minimal ideal. Thus, if a set C is $I_\emptyset-\beta$-open, then C is also $I-\beta$-open.

Proposition 2. *If A and B are both $I-\beta$-open, then so is their union $A \cup B$.*

Proof. Let the given conditions hold. To show that $A \cup B$ is $I-\beta$-open, we need to produce a preopen set G such that $G \setminus (A \cup B) \in I$ and $(A \cup B) \setminus cl(G) \in I$. Since A and B are both $I-\beta$-open, there are two preopen sets G_1 and G_2 such that $G_1 \setminus A \in I$, $A \setminus cl(G_1) \in I$, $G_2 \setminus B \in I$, $B \setminus cl(G_2) \in I$. Choose $G = G_1 \cup G_2$, and observe that $(G_1 \cup G_2) \setminus (A \cup B) = ((G_1 \setminus A) \setminus B) \cup ((G_2 \setminus B) \setminus A) \in I$. Also, $(A \cup B) \setminus cl(G_1 \cup G_2) = ((A \setminus cl(G_1)) \setminus cl(G_2)) \cup ((B \setminus cl(G_2)) \setminus cl(G_1)) \in I$. Therefore, by definition, $A \cup B$ is $I-\beta$-open. □

Proposition 3. *Let (X,τ) be a topological space in which there is a preopen singleton subset $\{a\}$ satisfying $cl(\{a\}) = X$.For any ideal I on X with $\{a\} \in I$, we have that:*

(i) Every singleton subset of X is $I-\beta$-open;

(ii) Every finite subset of X is $I-\beta$-open.

Proof. Let the given conditions hold. Suppose that $\{b\}$ is a singleton subset of X. Since ($\{a\}$ is preopen), $\{a\}\setminus\{b\}=\{a\}\in I$, and $\{b\}\setminus cl(\{a\})=\{b\}\setminus X=\emptyset\in I$, it follows that $\{b\}$ is $I-\beta$-open, this proves that (i). To see that (ii) holds, let $A=\{b_1,b_2,b_3,\dots,b_n\}$ be a finite subset of X. Since $A=\{b_1\}\cup\{b_2\}\cup\{b_3\}\cup\ldots\cup\{b_n\}$, the result follows from the fact that each singleton subset $\{b_i\},(i=1,2,3,\dots,n)$ is $I-\beta$-open and a repeated use of Proposition 2 above. □

Proposition 3 does not hold for any choice of ideal.

Example 2. *Consider $X=\{a,b,c\}$ and $\tau=\{\emptyset,\{a\},\{a,c\},X\}$ and observe that $cl(\{a\})=X$. If we choose the minimal ideal $I=\{\emptyset\}$ on X, then the singleton subset $\{b\}$ is not $I-\beta$-open, as there is no preopen set G satisfying $G\setminus\{b\}\in I$ and $\{b\}\setminus cl(G)\in I$, simultaneously.*

Proposition 4. *Let A and B be subsets of a topological space (X,τ) such that A is preopen, $A\subseteq B$, and A is dense in B (that is, $B\subseteq cl(A)$). Then B is $I-\beta$-open for any ideal I on X. In particular, the conclusion holds in the special case when $B=cl(A)$.*

Remark 1. *If A and B are two $I-\beta$-open sets, then their intersection $A\cap B$ need not be $I-\beta$-open. For example, let $X=\{a,b,c\}$ be equipped with a topology $\tau=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$. Note that $cl(\{a\})=\{a,b\}$ and $cl(\{c\})=\{b,c\}$; moreover, the subsets $\{a,b\}$ and $\{b,c\}$ are β-open with respect to the minimal ideal $I=\{\emptyset\}$, in view of Proposition 4 above. However, the singleton subset $\{b\}=\{a,b\}\cap\{b,c\}$ is not β-open with respect to the minimal ideal $I=\{\emptyset\}$. Obviously, if $A,B\in I$, then $A\cap B$ will be β-open with respect to the ideal I. For those subsets that are not members of the ideal I, one rather strong condition for their intersection to be β-open with respect to the ideal I is given below.*

Proposition 5. *Let I be an ideal on a topological space (X,τ), where every non-empty preopen subset of X is dense and the collection of preopen subsets of X satisfies the finite intersection property:*

(i) If A is $I-\beta$-open and $A\subseteq B$, then B is $I-\beta$-open,

(ii) If A is $I-\beta$-open, then so is $A\cup B$, for any subset B of X,

(iii) If both A and B are $I-\beta$-open, then so is their intersection $A\cap B$,

Proof. (i) Suppose that A is $I-\beta$-open and that $A\subseteq B$. There is a preopen set G such that $G\setminus A\in I$ and $A\setminus cl(G)\in I$. Notice that such a preopen set G is necessarily non-empty, since we are dealing with those subsets of X that do not belong to the ideal I. Since $A\subseteq B$, we have that $G\setminus B\subseteq G\setminus A\in I$; moreover, $B\setminus cl(G)=B\setminus X=\emptyset\in I$. Thus, B is $I-\beta$-open.
(ii) Follows immediately from (i) and the observation since $A\subseteq B\Leftrightarrow A\cup B=B$.
(iii) Suppose that both A and B are $I-\beta$-open. Without loss of generality, suppose that $A\cap B\neq\emptyset$; otherwise, $A\cap B$ will be trivially $I-\beta$-open. By assumption, there are preopen sets G,H such that $G\setminus A$, $A\setminus cl(G)\in I$ and $H\setminus B$, $B\setminus cl(H)\in I$. Consider the preopen set $G\cap H$, which is non-empty (by the finite intersection property). Since

$$(G\cap H)\setminus(A\cap B)=((G\setminus A)\cap H)\cup(G\cap(H\setminus B))\in I$$

and $(A \cap B) \setminus cl(G \cap H) = (A \cap B) \setminus X = \emptyset \in I$, it follows that $A \cap B$ is $I - \beta$-open. □

Remark 2. *In Example 2, we saw that the singleton subset $\{b\}$ was not β-open with respect to the minimal ideal $I = \{\emptyset\}$. Notice that the set $\{a, b\} = \{a\} \cup \{b\}$ is β-open with respect to $I = \{\emptyset\}$, simply because the non-empty preopen singleton subset $\{a\}$ is dense in X, this is an instance of Proposition 5, (ii) above.*

Proposition 6. *Under the conditions of Proposition 5, we have that A is $I - \beta$-open if and only if $cl(A)$ is $I - \beta$-open.*

Proof. If A is $I - \beta$-open, then, because $A \subseteq cl(A)$,so is $cl(A)$, by Proposition 5, (ii). Conversely, suppose that $cl(A)$ is $I - \beta$-open. Then there is a preopen set G such that $G \setminus cl(A) \in I$ and $cl(A) \setminus cl(G) \in I$. Notice that G is necessarily non-empty, otherwise, we would have $cl(G) = \emptyset$, which forces $A \in I$, which we do not want (as we are dealing with those subsets that do not belong to the ideal I). To show that A is $I - \beta$-open, consider the preopen set $H = G \setminus cl(A) = G \cap (cl(A))^c \in I$, by assumption. We have that $H \setminus A = G \cap (cl(A))^c \cap A^c \in I$, because of the heredity property, moreover, $A \setminus cl(H) = A \setminus cl(G \cap (cl(A)^c)) = A \setminus X = \emptyset \in I$. This shows that A is $I - \beta$-open. □

Theorem 2. *The following are equivalent for a subset A of an ideal topological space (X, τ):*

(i) $X \setminus A$ *is* $I - \beta$*-open,*

(ii) There exists a preclosed set F such that $int(F) \setminus A \in I$ and $A \setminus F \in I$.

Proof. First suppose that $X \setminus A$ is $I - \beta$-open. Then there exists a preopen set G such that $G \setminus (X \setminus A) \in I$ and $(X \setminus A) \setminus cl(G) \in I$. Since $G \setminus (X \setminus A) = A \setminus (X \setminus G)$ and $(X \setminus A) \setminus cl(G) = int(X \setminus G) \setminus A$, we have (ii) by choosing the closed set $X \setminus G$ as F. Conversely, if we suppose that (ii) holds, then the choice of the preopen set $G = X \setminus F$ shows that $X \setminus A$ is $I - \beta$-open. □

Definition 6. *A subset A of an ideal topological space (X, τ) is said to be β-closed with respect to I (written as $I - \beta$-closed) if and only if $X \setminus A$ is $I - \beta$-open.*

Proposition 7. *If both A and B are $I - \beta$-closed, then so is their intersection $A \cap B$.*

Proof. Let the given conditions hold. There are preclosed sets F_1, F_2 such that $int(F_1) \setminus A$, $(A \setminus F_1) \in I$ and $int(F_2) \setminus B$, $(B \setminus F_2) \in I$. With $F = F_1 \cap F_2$, we have that

$$int(F_1 \cap F_2) \setminus (A \cap B) = ((int(F_1) \setminus A) \cap int(F_2)) \cup ((int(F_1)) \cap (int(F_2) \setminus B)) \in I,$$

and $(A \cap B) \setminus (F_1 \cap F_2) = ((A \setminus F_1) \cap B) \cup (A \cap (B \setminus F_2)) \in I$. Therefore, $A \cap B$ is $I - \beta$-closed. □

4. Conclusion

Topology plays a significant role in quantum physics, high energy physics and superstring theory. In this paper, new types of sets via ideals were investigated and some of their properties were obtained. The notions of the sets and functions in topological and fuzzy topological spaces are highly developed and used extensively in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design research and mathematical sciences.

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