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Existence and Uniqueness of Mittag-Leffler-Ulam Stable Solution for Fractional Integrodifferential Equations with Nonlocal Initial Conditions

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Abstract. In this paper, the existence and uniqueness of mild solution for fractional integrodifferential equations with nonlocal initial conditions are investigated by using Hölder's inequality, $p-$mean continuity and Schauder's fixed point theorem in Banach spaces. The Mittag-Leffler-Ulam stability results are also obtained by using generalized singular Gronwall's inequality.

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1. Introduction

During the past decades, fractional differential equations have attracted many authors (see for instance [10, 11, 14] and [15]). This is mostly because they efficiently describe many phenomena arising in engineering, physics, economy and science. There has been a significant development in nonlocal problems for fractional differential equations or inclusions (see for instance [1, 7, 12] and [22]).

As we all know, the main difficulty to study the fractional evolution equations is how to obtain the suitable fractional resolvent family generated by the infinitesimal generator A in Banach space. In order to solve this problem, some authors introduced an α-resolvent family under the Riemann-Liouville fractional derivative and some constraints, (see, for example, [2, 6]), and the others introduced suitable operator families with the Caputo fractional derivative in terms of some probability density functions and operator semigroup (see, for example, [17, 22] and [23]). For the latter, a pioneering work has been reported by El-Borai [4, 5].

On the other hand, in the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin and Aoki-Rassias (see [3,

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8] and [9]). Recently, J. Wang *et al.* [18, 19] discussed four type Mittag- leffler-Ulam stability of fractional differential equations and obtained some new and interesting stability results. Motivated by the above work, we consider the nonlocal Cauchy problem of the following form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t), I^{\beta}x(t)), \ t \in J := [0, b] \\ x(0) + g(x) = x_0, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$, I^{β} is the Riemann-Liouville fractional integration of order $0 < \beta < 1$, $b > 0$, A is the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ of operators on $\mathbb{E}$, $f : J \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, $g : C(J, \mathbb{E}) \to \mathbb{E}$ are given functions satisfying some assumptions and x_0 is an element of the Banach space $\mathbb{E}$.

For any strongly continuous semigroup (i.e. C_0 semigroup) $\{Q(t)\}_{t\geq 0}$ on $\mathbb{E}$, we define the generator

$$Au = \lim_{t\to 0^+} \frac{Q(t)u - u}{t} \text{ in } \mathbb{E}.$$

The domain $D(A)$ of this linear operator is the set of all $u \in \mathbb{E}$ for which the limit above exists. Then $D(A)$ is dense in $\mathbb{E}$ and A is closed, meaning that for $u_n \in D(A)$, if $u_n \to u$ and $Au_n \to v$ in $\mathbb{E}$, then $u \in D(A)$ and $Au = v$. For more details, we refer the reader to [13].

2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. We assume that $\mathbb{E}$ is a Banach space with the norm $|\cdot|$. Let $C(J, \mathbb{E})$ be the Banach space of continuous functions from J into $\mathbb{E}$ with the norm $\|x\| = \sup_{t\in J} |x(t)|$, where $x \in C(J, \mathbb{E})$.

Let $B(\mathbb{E})$ be the space of all bounded linear operators from $\mathbb{E}$ to $\mathbb{E}$ with the norm $\|Q\|_{B(\mathbb{E})} = \sup\{|Q(u)| : |u| = 1\}$, where $Q(u) \in B(\mathbb{E})$ and $u \in \mathbb{E}$. Throughout this paper, let A be the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ of operators on $\mathbb{E}$. Clearly

$$M := \sup_{t\in[0,b]} \|Q\|_{B(\mathbb{E})} < \infty. \tag{2}$$

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see I. Podlubny [15].

Definition 1. *The fractional integral of order α with the lower limit zero for a function $h \in AC[0, \infty)$ is defined as*

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \ t > 0, \ 0 < \alpha < 1,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. *Riemann-Liouville derivative of order α with the lower limit zero for a function $h \in AC[0, \infty)$ can be written as*

$$^{L}D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{h(s)}{(t-s)^{\alpha}} ds, \ t > 0, \ 0 < \alpha < 1.$$

Definition 3. *The Caputo derivative of order α with the lower limit zero for a function $h \in AC[0,\infty)$ can be written as*

$$^{C}D^{\alpha}h(t) = ^{L}D^{\alpha}(h(t)-h(0)),\ t>0,\ 0<\alpha<1.$$

Remark 1.

- *If $h(t) \in C^1[0,\infty)$, then*

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{h^{'}(s)}{(t-s)^{\alpha}}ds = I^{1-\alpha}h^{'}(t),\ t>0,\ 0<\alpha<1.$$

- *The Caputo derivative of a constant is equal to zero.*
- *If h is an abstract function with values in $\mathbb{E}$, then integrals which appear in Definitions 1-3 are taken in Bochner's sense.*

For measurable functions $m : J \to \mathbb{R}$, define the norm

$$\|m\|_{L^p(J)} = \begin{cases} (\int_J |m(t)|^p dt)^{\frac{1}{p}},\ 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0}\{\sup_{t\in J-\bar{J}} |m(t)|\},\ p=\infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J,\mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m : J \to \mathbb{R}$ with $\|m\|_{L^p(J)} < \infty$.

Lemma 1 (Hölder's inequality). *Assume that $q,p \ge 1$ and $\frac{1}{q}+\frac{1}{p}=1$. If $l \in L^q(J,\mathbb{R})$ and $m \in L^p(J,\mathbb{R})$, then for $1 \le p \le \infty$, $lm \in L^1(J,\mathbb{R})$ and $\|lm\|_{L^1(J)} \le \|l\|_{L^q(J)}.\|m\|_{L^p(J)}$.*

Lemma 2 ([21] p-mean continuity). *For each $\psi \in L^p(J,\mathbb{E})$ with $1 \le p < +\infty$, we have $\lim_{r\to 0}\int_0^b \|\psi(t+r)-\psi(t)\|^p dt = 0$, where $\psi(s)=0$ for s not in J.*

Lemma 3 (Bochner's Theorem). *A measurable function $H : [0,b] \to \mathbb{R}$ is Bochner's integrable if $|H|$ is Lebesgue integrable.*

Lemma 4 (Schauder Fixed Point Theorem). *If B is a closed bounded and convex subset of a Banach space $\mathbb{E}$ and $F : B \to B$ is completely continuous, then F has a fixed point in B.*

We end this section with an important singular type Gronwall inequality.

Theorem 1 ([16] Theorem 1.4). *For any $t \in [0,b)$, if*

$$u(t) \le a(t) + \sum_{i=1}^{n} b_i(t)\int_0^t (t-s)^{\beta_i-1}u(s)ds,$$

where all the functions are not negative and continuous. The constants $\beta_i > 0$.

b_i $(i=1,2,\ldots,n)$ are the bounded and monotonic increasing functions on $[0,b)$, then

$$u(t) \le a(t) + \sum_{k=1}^{\infty}\left(\sum_{1',2',\cdots,k'=1}^{n} \frac{\prod_{i=1}^{k}[b_{i'}(t)\Gamma(\beta_{i'})]}{\Gamma(\sum_{i=1}^{k}\beta_{i'})}\int_0^t (t-s)^{\sum_{i=1}^{k}\beta_i-1}a(s)ds\right).$$

Remark 2. *For $n = 2$, if the constants $b_1, b_2 \geq 0$, $\beta_1, \beta_2 > 0$, $a(t)$ is nonnegative and locally integrable on $0 \leq t < b$ and $u(t)$ is nonnegative and locally integrable on $0 \leq t < b$ with*

$$u(t) \leq a(t) + b_1 \int_0^t (t-s)^{\beta_1 - 1} u(s) ds + b_2 \int_0^t (t-s)^{\beta_2 - 1} u(s) ds,$$

then

$$u(t) \leq a(t) + \sum_{k=1}^{\infty} \left[\frac{(b_1 \Gamma(\beta_1))^k}{\Gamma(k\beta_1)} \int_0^t (t-s)^{k\beta_1 - 1} a(s) ds + \frac{(b_2 \Gamma(\beta_2))^k}{\Gamma(k\beta_2)} \int_0^t (t-s)^{k\beta_2 - 1} a(s) ds \right].$$

Remark 3. *Under the hypotheses of Remark 2, let $a(t)$ is a nondecreasing function on $0 \leq t < b$. Then we have*

$$u(t) \leq a(t) \left(E_{\beta_1} \left[b_1 \Gamma(\beta_1) t^{\beta_1} \right] + E_{\beta_2} \left[b_2 \Gamma(\beta_2) t^{\beta_2} \right] \right),$$

where E_α is the Mittag-Leffler function [15] defined by $E_\alpha[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in \mathbb{C}$.

3. Existence and Uniqueness of Mild Solutions

We recall the following definition of a mild solution for the nonlocal Cauchy problem (1). For more details, one can see [20, 24].

Definition 4. *By the mild solution of the nonlocal Cauchy problem* (1)*, we mean that the function $x \in C(J, \mathbb{E})$ which satisfies*

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, x(s), I^\beta x(s)) ds, \quad t \in [0, b],$$

where

$$\begin{aligned}
S(t) &= \int_0^\infty \xi_\alpha(\theta) Q(t^\alpha \theta) d\theta, \\
T(t) &= \alpha \int_0^\infty \theta \xi_\alpha(\theta) Q(t^\alpha \theta) d\theta, \\
\xi_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \rho_\alpha(\theta^{-\frac{1}{\alpha}}), \\
\rho_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha), \; \theta \in (0, \infty),
\end{aligned}$$

where ξ_α is the probability density function defined on $(0, \infty)$, which has properties $\xi_\alpha(\theta) \geq 0$ for all $\theta \in (0, \infty)$ and

$$\int_0^\infty \xi_\alpha(\theta) d\theta = 1, \quad \int_0^\infty \theta \xi_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{3}$$

Before stating and proving the main results, we introduce the following hypotheses.

(H1) $Q(t)$ is a compact operator for every $t > 0$.

(H2) The function $f : J \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ satisfies that $f(.,x,y) : J \to \mathbb{E}$ is measurable for all $x, y \in C(J,\mathbb{E})$ and $f(t,.,.) : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$ is continuous for all $t \in J$ and there exists a positive function $\mu(\cdot) \in L^p(J,\mathbb{R}^+)$ for some p with $1 < p < \infty$ such that

$$|f(t,x,y)| \leq \mu(t)(\|x\| + \|y\|),\ x,y \in C(J,\mathbb{E}),\ t \in J.$$

(H3) The function $g : C(J,\mathbb{E}) \to \mathbb{E}$ is completely continuous and there exist constants $L > 0$, $L' > 0$ such that:

$$|g(x)| \leq L\|x\| + L^{'},\ x \in C(J,\mathbb{E}).$$

For each positive constant k, let $B_k = \{x \in C(J,\mathbb{E}) : \|x\| \leq k\}$. Then B_k is clearly a bounded closed and convex subset in $C(J,\mathbb{E})$.

The following existence result for nonlocal Cauchy problem (1) is based on Schauder fixed point theorem.

Theorem 2. *If assumptions* $(H1)-(H3)$ *are satisfied, then the nonlocal Cauchy problem* (1) *has a mild solution provided that*

$$M\left[L - \frac{Mkb^{\alpha-\frac{1}{p}}}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{b^{\beta+\frac{p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)\right] < 1.$$

Proof. For any positive constant k and $x \in B_k$, according to (2) and $(H3)$, it follows that

$$\left|\int_0^\infty \xi_\alpha(\theta)Q(t^\alpha\theta)(x_0 - g(x))d\theta\right| \leq M(|x_0| + Lk + L^{'}). \tag{4}$$

Therefore, the function $\int_0^\infty \xi_\alpha(\theta)Q(t^\alpha\theta)(x_0 - g(x))d\theta$ exists.

In view of (2), (3) and $(H2)$, we get

$$\begin{aligned}
&\int_0^t \left|\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta\right| ds\\
\leq& M\int_0^t\int_0^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}\left|f(s,x(s),I^\beta x(s))\right| d\theta ds\\
\leq& \frac{M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)(\|x\| + \|I^\beta x\|)ds\\
\leq& \frac{M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)(k + \frac{k}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}d\tau)ds
\end{aligned}$$

$$
\begin{aligned}
&=\frac{Mk}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)\left(1+\frac{s^\beta}{\beta\Gamma(\beta)}\right)ds\\
&=\frac{Mk}{\Gamma(1+\alpha)}\left(\int_0^t (t-s)^{\alpha-1}\mu(s)ds+\frac{1}{\Gamma(1+\beta)}\int_0^t (t-s)^{\alpha-1}s^\beta\mu(s)ds\right)\\
&\leq \frac{Mk}{\Gamma(1+\alpha)}\left(\left(\int_0^t (t-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t (\mu(s))^p ds\right)^{\frac{1}{p}}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\left(\int_0^t (t-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^t (s^{\beta p}\mu(s))^p ds\right)^{\frac{1}{p}}\right)\\
&\leq\frac{Mk}{\Gamma(1+\alpha)}\left(\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} t^{\frac{p+(\alpha-1)p-1}{p}}\|\mu\|_{L^p(J,\mathbb{R}^+)}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\beta)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} t^{\frac{p+(\alpha-1)p-1}{p}}\left(\int_0^t s^{\frac{\beta p^2}{p-1}}ds\right)^{\frac{p-1}{p^2}}\left(\int_0^t (\mu(s))^{p^2}ds\right)^{\frac{1}{p^2}}\right)\\
&\leq\frac{Mk}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} t^{\frac{p+(\alpha-1)p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\left(\frac{1}{1+\frac{\beta p^2}{p-1}}\right)^{\frac{p-1}{p^2}}\right.\\
&\quad\left.\times t^{\frac{\beta p^2+p-1}{p^2}}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)\\
&\leq\frac{Mkb^{\alpha-\frac{1}{p}}}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{b^{\beta+\frac{p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right),
\end{aligned}
\tag{5}
$$

for all $t\in J$. Thus, $\left|\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta\right|$ is Lebesgue integrable with respect to $s\in[0,t]$ for all $t\in[0,b]$. From Lemma 3 (Bochner's theorem), it follows that $\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta$ is Bochner's integrable with respect to $s\in[0,t]$ for all $t\in J$.

For each positive constant k, define an operator F on B_k by the formula

$$
(Fx)(t)=S(t)(x_0-g(x))+\alpha\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,x(s),I^\beta x(s))ds,\ t\in[0,b], \tag{6}
$$

where $x \in B_k$. Let

$$k = \frac{M\left(|x_0| + L'\right)}{1 - M\left[L - \frac{Mkb^{\alpha-\frac{1}{p}}}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{b^{\beta+\frac{p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)\right]}. \tag{7}$$

In the following, we will prove that F has a fixed point on B_k . Our proof will be divided into two steps.

Step I. $\|Fx\| \leq k$ whenever $x \in B_k$.
For each $x \in B_k$ and $t \in J$, by using the similar method as we did in (4) and (5), we have

$$\begin{aligned}
|(Fx)(t)| =& \left| S(t)(x_0 - g(x)) + \alpha \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, x(s), I^\beta x(s)) ds \right| \\
\leq& \left| \int_0^\infty \xi_\alpha(\theta) Q(t^\alpha \theta)(x_0 - g(x)) d\theta \right| \\
&+ \alpha \left| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^\alpha \theta) f(s, x(s), I^\beta x(s)) d\theta ds \right| \\
\leq& M(|x_0| + Lk + L^{'}) \\
&+ \frac{Mkb^{\alpha-\frac{1}{p}}}{\Gamma(1+\alpha)} \left(\frac{p-1}{p+(\alpha-1)p-1} \right)^{\frac{p-1}{p}} \left(\|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{b^{\beta+\frac{p-1}{p^2}}}{\Gamma(1+\beta)} \|\mu\|_{L^{p^2}(J,\mathbb{R}^+)} \right) \\
=& k.
\end{aligned} \tag{8}$$

Hence $\|Fx\| \leq k$ for each $x \in B_k$.

Step II. F is a completely continuous operator.

Firstly, we will prove that F is continuous on B_k. For any $x_n, x \subseteq B_k, n = 1, 2, \ldots$ with $\lim_{n\to\infty} \|x_n - x\| = 0$, we get

$$\lim_{n\to\infty} x_n(t) = x(t), \text{ for } t \in J.$$

Thus by condition $(H2)$, we have

$$\lim_{n\to\infty} f(t, x_n(t), I^\beta x_n(t)) = f(t, x(t), I^\beta x(t)),$$

for $t \in J$.

So, we can conclude that

$$\sup_{s\in[0,b]} |f(s, x_n(s), I^\beta x_n(s)) - f(t, x(s), I^\beta x(s))| \to 0, \text{ as } n \to \infty.$$

On the other hand, for $t \in J$

$$|Fx_n(t) - Fx(t)|$$

$$=\left|S(t)(g(x_n)-g(x))+\alpha\int_0^t (t-s)^{\alpha-1}T(t-s)\left(f(s,x_n(s),I^\beta x_n(s))-f(s,x(s),I^\beta x(s))\right)ds\right|$$
$$\leq\left|\int_0^\infty \xi_\alpha(\theta)Q(t^\alpha\theta)(g(x_n)-g(x))d\theta\right|$$
$$+\alpha\left|\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)\left(f(s,x_n(s),I^\beta x_n(s))-f(s,x(s),I^\beta x(s))\right)d\theta ds\right|$$
$$\leq M\|g(x_n)-g(x)\|+\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|f(s,x_n(s),I^\beta x_n(s))-f(s,x(s),I^\beta x(s))|ds$$
$$\leq M\|g(x_n)-g(x)\|+\frac{Mb^\alpha}{\Gamma(\alpha+1)}\sup_{s\in[0,b]}|f(s,x_n(s),I^\beta x_n(s))-f(t,x(s),I^\beta x(s))|,$$

which implies

$$\|Fx_n-Fx\|\leq M\|g(x_n)-g(x)\|+\frac{Mb^\alpha}{\Gamma(\alpha+1)}\sup_{s\in[0,b]}|f(s,x_n(s),I^\beta x_n(s))-f(t,x(s),I^\beta x(s))|.$$

Hence, by condition $(H3)$ we get $\|Fx_n-Fx\|\to 0$, as $n\to\infty$. This means that F is continuous.

Next, we will show that $\{Fx, x\in B_k\}$ is equicontinuous. For any $x\in B_k$ and $0\leq t_1\leq t_2\leq b$, we get

$$|(Fx)(t_2)-(Fx)(t_1)|$$
$$=\Big|[S(t_2)-S(t_1)](x_0-g(x))$$
$$+\alpha\int_0^{t_2}(t_2-s)^{\alpha-1}T(t_2-s)f(s,x(s),I^\beta x(s))ds$$
$$-\alpha\int_0^{t_1}(t_1-s)^{\alpha-1}T(t_1-s)f(s,x(s),I^\beta x(s))ds\Big|$$
$$\leq\left|\int_0^\infty \xi_\alpha(\theta)[Q(t_2^\alpha\theta)-Q(t_1^\alpha\theta)](x_0-g(x))d\theta\right|$$
$$+\alpha\Big|\int_0^{t_2}\int_0^\infty \theta(t_2-s)^{\alpha-1}\xi_\alpha(\theta)Q((t_2-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds$$
$$-\int_0^{t_1}\int_0^\infty \theta(t_1-s)^{\alpha-1}\xi_\alpha(\theta)Q((t_1-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\Big|$$
$$\leq\left|\int_0^\infty \xi_\alpha(\theta)[Q(t_2^\alpha\theta)-Q(t_1^\alpha\theta)](x_0-g(x))d\theta\right|$$
$$+\alpha\left|\int_{t_1}^{t_2}\int_0^\infty \theta(t_2-s)^{\alpha-1}\xi_\alpha(\theta)Q((t_2-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|$$

$$+\alpha\left|\int_0^{t_1}\int_0^{\infty}\theta[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\xi_\alpha(\theta)Q((t_2-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|$$
$$+\alpha\left|\int_0^{t_1}\int_0^{\infty}\theta(t_1-s)^{\alpha-1}\xi_\alpha(\theta)[Q((t_2-s)^\alpha\theta)-Q((t_1-s)^\alpha\theta)]f(s,x(s),I^\beta x(s))d\theta ds\right|$$
$$=\left|\int_0^{\infty}\xi_\alpha(\theta)[Q(t_2^\alpha\theta)-Q(t_1^\alpha\theta)](x_0-g(x))d\theta\right|+\alpha(I_1+I_2+I_3),$$

where

$$I_1=\alpha\left|\int_{t_1}^{t_2}\int_0^{\infty}\theta(t_2-s)^{\alpha-1}\xi_\alpha(\theta)Q((t_2-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|,$$
$$I_2=\alpha\left|\int_0^{t_1}\int_0^{\infty}\theta[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\xi_\alpha(\theta)Q((t_2-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|,$$
$$I_3=\alpha\left|\int_0^{t_1}\int_0^{\infty}\theta(t_1-s)^{\alpha-1}\xi_\alpha(\theta)[Q((t_2-s)^\alpha\theta)-Q((t_1-s)^\alpha\theta)]f(s,x(s),I^\beta x(s))d\theta ds\right|.$$

By using analogous argument performed in (4) and (5), we can conclude that

$$\begin{aligned}
I_1\leq&\frac{\alpha Mk}{\Gamma(\alpha+1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mu(s)(1+\frac{1}{\Gamma(\beta+1)}s^\beta)ds\\
\leq&\frac{Mk}{\Gamma(1+\alpha)}\left(\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_1}^{t_2}(\mu(s))^p ds\right)^{\frac{1}{p}}\right.\\
&\left.+\frac{1}{\Gamma(1+\beta)}\left(\int_{t_1}^{t_2}(t_s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_1}^{t_2}(s^{\beta p}\mu(s))^p ds\right)^{\frac{1}{p}}\right)\\
\leq&\frac{Mk}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}(t_2-t_1)^{\frac{p+(\alpha-1)p-1}{p}}\\
&\times\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\left(\int_{t_1}^{t_2}s^{\frac{\beta p^2}{p-1}}ds\right)^{\frac{p-1}{p^2}}\left(\int_{t_1}^{t_2}(\mu(s))^{p^2}ds\right)^{\frac{1}{p^2}}\right)\\
\leq&\frac{Mk}{\Gamma(1+\alpha)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}(t_2-t_1)^{\alpha-\frac{1}{p}}\\
&\times\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\left(\frac{1}{1+\frac{\beta p^2}{p-1}}\right)^{\frac{p-1}{p^2}}(t_2^{\frac{\beta p^2}{p-1}+1}-t_1^{\frac{\beta p^2}{p-1}+1})^{\frac{p-1}{p^2}}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right).
\end{aligned}$$

One can deduce that $\lim_{t_1\to t_2} I_1 = 0$. Also, note that

$$\begin{aligned}
I_2 \leq& \frac{\alpha M k}{\Gamma(\alpha+1)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\mu(s)(1+\frac{1}{\Gamma(\beta+1)}s^{\beta})ds\\
\leq& \frac{\alpha M k}{\Gamma(\alpha+1)}\left(\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\right)^{\frac{1}{p}}\left(\int_0^{t_1}(\mu(s))^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}\right.\\
&\left.+\frac{1}{\Gamma(\beta+1)}\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\right)^{\frac{1}{p}}\left(\int_0^{t_1}(s^{\beta}\mu(s))^{\frac{p}{p-1}}ds\right)^{\frac{p-1}{p}}\right)\\
\leq& \frac{\alpha M k}{\Gamma(\alpha+1)}\left(\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\right)^{\frac{1}{p}}\|\mu\|_{L^{\frac{p}{p-1}}(J,\mathbb{R}^+)}\right.\\
&+\frac{1}{\Gamma(\beta+1)}\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\right)^{\frac{1}{p}}\left(\int_0^{t_1}s^{\frac{\beta p^2}{(p-1)^2}}ds\right)^{\frac{(p-1)^2}{p^2}}\\
&\left.\times\left(\int_0^{t_1}(\mu(s))^{\frac{p^2}{p-1}}ds\right)^{\frac{p-1}{p^2}}\right)\\
\leq& \frac{\alpha M k}{\Gamma(\alpha+1)}\left(\|\mu\|_{L^{\frac{p}{p-1}}(J,\mathbb{R}^+)}+\frac{b^{\beta+\frac{(p-1)^2}{p^2}}}{\Gamma(\beta+1)}\left(\frac{1}{1+\frac{\beta p^2}{(p-1)^2}}\right)^{\frac{(p-1)^2}{p^2}}\|\mu\|_{L^{\frac{p^2}{p-1}}(J,\mathbb{R}^+)}\right)\\
&\times\left(\int_0^{b}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Using Lagrange mean value theorem, one can obtain $(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\to 0$ as $t_1\to t_2$, for $s\in J$. By Lemma 2, we can deduce that $\int_0^b[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^p ds\to 0$ as $t_1\to t_2$. Thus we deduce that $\lim_{t_1\to t_2} I_2 = 0$.

For $t_1 = 0$, $0 < t_2 \leq b$, it is easy to see that $I_3 = 0$. For $t_1 > 0$ and $\varepsilon > 0$ be enough small, we have

$$\begin{aligned}
I_3 \leq& \alpha\int_0^{t_1-\varepsilon}\int_0^{\infty}\theta(t_1-s)^{\alpha-1}\xi_\alpha(\theta)\|Q((t_2-s)^{\alpha}\theta)-Q((t_1-s)^{\alpha}\theta)\|_{B(E)}|f(s,x(s),I^{\beta}x(s))|d\theta ds\\
&+\alpha\int_{t_1-\varepsilon}^{t_1}\int_0^{\infty}\theta(t_1-s)^{\alpha-1}\xi_\alpha(\theta)\|Q((t_2-s)^{\alpha}\theta)-Q((t_1-s)^{\alpha}\theta)\|_{B(E)}|f(s,x(s),I^{\beta}x(s))|d\theta ds\\
\leq& \frac{\alpha k}{\Gamma(\alpha+1)}\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\mu(s)(1+\frac{s^{\beta}}{\Gamma(\beta+1)})\sup_{s\in[0,t_1-\varepsilon]}\|Q((t_2-s)^{\alpha}\theta)-Q((t_1-s)^{\alpha}\theta)\|_{B(E)}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha Mk}{\Gamma(\alpha+1)}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})ds\\
\leq&\frac{\alpha k}{\Gamma(\alpha+1)}\left(\left(\int_0^{t_1-\varepsilon}(t_1-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^{t_1-\varepsilon}(\mu(s))^p ds\right)^{\frac{1}{p}}\right.\\
&\left.+\frac{1}{\Gamma(1+\beta)}\left(\int_0^{t_1-\varepsilon}(t_1-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_0^{t_1-\varepsilon}(s^{\beta p}\mu(s))^p ds\right)^{\frac{1}{p}}\right)\\
&\times\sup_{s\in[0,t_1-\varepsilon]}\|Q((t_2-s)^\alpha\theta)-Q((t_1-s)^\alpha\theta)\|_{B(E)}\\
&+\frac{2\alpha Mk}{\Gamma(\alpha+1)}\left(\left(\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_1-\varepsilon}^{t_1}(\mu(s))^p ds\right)^{\frac{1}{p}}+\frac{1}{\Gamma(1+\beta)}\right.\\
&\left.\times\left(\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\frac{(\alpha-1)p}{p-1}}ds\right)^{\frac{p-1}{p}}\left(\int_{t_1-\varepsilon}^{t_1}(s^{\beta p}\mu(s))^p ds\right)^{\frac{1}{p}}\right)\\
\leq&\frac{\alpha k}{\Gamma(\alpha+1)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\left(t_1^{\frac{p+(\alpha-1)p-1}{p-1}}-\varepsilon^{\frac{p+(\alpha-1)p-1}{p-1}}\right)^{\frac{p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\right.\\
&\left.\times\left(\int_0^{t_1-\varepsilon}s^{\frac{\beta p^2}{p-1}}ds\right)^{\frac{p-1}{p^2}}\left(\int_0^{t_1-\varepsilon}(\mu(s))^{p^2}ds\right)^{\frac{1}{p^2}}\right)\sup_{s\in[0,t_1-\varepsilon]}\|Q((t_2-s)^\alpha\theta)-Q((t_1-s)^\alpha\theta)\|_{B(E)}\\
&+\frac{2\alpha Mk}{\Gamma(\alpha+1)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\varepsilon^{\frac{p+(\alpha-1)p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\left(\int_{t_1-\varepsilon}^{t_1}s^{\frac{\beta p^2}{p-1}}ds\right)^{\frac{p-1}{p^2}}\right.\\
&\left.\times\left(\int_{t_1-\varepsilon}^{t_1}(\mu(s))^{p^2}ds\right)^{\frac{1}{p^2}}\right)\\
\leq&\frac{\alpha k}{\Gamma(\alpha+1)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\left(t_1^{\frac{p+(\alpha-1)p-1}{p-1}}-\varepsilon^{\frac{p+(\alpha-1)p-1}{p-1}}\right)^{\frac{p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\right.\\
&\left.\times\left(\frac{1}{1+\frac{\beta p^2}{p-1}}\right)^{\frac{p-1}{p^2}}\left(t_1-\varepsilon\right)^{\frac{\beta p^2+p-1}{p^2}}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)\sup_{s\in[0,t_1-\varepsilon]}\|Q((t_2-s)^\alpha\theta)-Q((t_1-s)^\alpha\theta)\|_{B(E)}\\
&+\frac{2\alpha Mk}{\Gamma(\alpha+1)}\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}}\varepsilon^{\frac{p+(\alpha-1)p-1}{p}}
\end{aligned}
$$

$$\times\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{1}{\Gamma(1+\beta)}\left(\frac{1}{1+\frac{\beta p^2}{p-1}}\right)^{\frac{p-1}{p^2}}\varepsilon^{\frac{\beta p^2+p-1}{p^2}}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)$$

Since $(H1)$ implies the continuity of $\{Q(t)\}_{t\geq 0}$ in t in the uniform operator topology, it is easy to see that I_3 tends to zero independently of $x \in B_k$ as $t_2 - t_1 \to 0$, $\varepsilon \to 0$. Thus $(Fx)(t_2)-(Fx)(t_1)$ tends to zero independently of $x \in B_k$ as $t_2 - t_1 \to 0$, which means that $\{Fx, x \in B_k\}$ is equicontinuous.

It remains to prove that for $t \in [0,b]$, the set $V(t) = \{(Fx)(t), x \in B_k\}$ is relatively compact in $\mathbb{E}$. Obviously, $V(0)$ is relatively compact in $\mathbb{E}$. Let $0 < t \leq b$ be fixed. For $\forall\varepsilon \in (0,t)$ and $\forall\delta > 0$, define an operator $F_{\varepsilon,\delta}$ on B_k by the formula

$$\begin{aligned}(F_{\varepsilon,\delta}x)(t) =&\int_\delta^\infty \xi_\alpha(\theta)Q(t^\alpha\theta)(x_0-g(x))d\theta\\
&+\alpha\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\\
=&Q(\varepsilon^\alpha\delta)\left(\int_\delta^\infty \xi_\alpha(\theta)Q(t^\alpha\theta-\varepsilon^\alpha\delta)(x_0-g(x))d\theta\right.\\
&\left.+\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta-\varepsilon^\alpha\delta)f(s,x(s),I^\beta x(s))d\theta ds\right),\end{aligned}$$

where $x \in B_k$. Then from the compactness of $Q(\varepsilon^\alpha\delta)(\varepsilon^\alpha\delta > 0)$, we obtain that the set $V_{\varepsilon,\delta}(t) = \{(F_{\varepsilon,\delta}x)(t), x \in B_k\}$ is relatively compact in $\mathbb{E}$. Obviously, $V(0)$ is relatively compact in $\mathbb{E}$ for $\forall\varepsilon \in (0,t)$ and $\forall\delta > 0$. Moreover, for every $x \in B_k$, we have

$$\begin{aligned}|(Fx)(t)-(F_{\varepsilon,\delta}x)(t)| \leq&\left|\int_0^\delta \xi_\alpha(\theta)Q(t^\alpha\theta)(x_0-g(x))d\theta\right|\\
&+\alpha\left|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|\\
&+\left|\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right.\\
&\left.-\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,x(s),I^\beta x(s))d\theta ds\right|\\
\leq&M(x_0+Lk+L')\int_0^\delta \xi_\alpha(\theta)d\theta\\
&+\alpha Mk\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})d\theta ds\end{aligned}$$

$$
\begin{aligned}
&+\alpha M k\int_{t-\varepsilon}^{t}\int_{\delta}^{\infty}\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})d\theta ds\\
\leq & M(x_0+Lk+L^{'})\int_0^\delta \xi_\alpha(\theta)d\theta\\
&+\alpha M k\left(\int_0^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})ds\right)\int_0^\delta \theta\xi_\alpha(\theta)d\theta\\
&+\alpha M k\left(\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})\right)\int_0^\infty \theta\xi_\alpha(\theta)d\theta ds\\
&\leq M(x_0+Lk+L^{'})\int_0^\delta \xi_\alpha(\theta)d\theta\\
&+\alpha M k\left(\int_0^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})ds\right)\int_0^\delta \theta\xi_\alpha(\theta)d\theta\\
&+\alpha M k\left(\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})\right).
\end{aligned}
$$

Using again the same analogous performed in (5), we have

$$
\begin{aligned}
\int_0^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})ds \leq & \left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} t^{\frac{p+(\alpha-1)p-1}{p}}\\
&\times\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{t^{\frac{\beta p^2+p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\mu(s)(1+\frac{s^\beta}{\Gamma(\beta+1)})ds \leq & \left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} \varepsilon^{\frac{p+(\alpha-1)p-1}{p}}\\
&\times\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{\varepsilon^{\frac{\beta p^2+p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
|(Fx)(t)-(F_{\varepsilon,\delta}x)(t)| \leq & M(x_0+Lk+L^{'})\int_0^\delta \xi_\alpha(\theta)d\theta\\
&+\alpha M k\left(\frac{p-1}{p+(\alpha-1)p-1}\right)^{\frac{p-1}{p}} t^{\frac{p+(\alpha-1)p-1}{p}}\left(\|\mu\|_{L^p(J,\mathbb{R}^+)}+\frac{t^{\frac{\beta p^2+p-1}{p^2}}}{\Gamma(1+\beta)}\|\mu\|_{L^{p^2}(J,\mathbb{R}^+)}\right)
\end{aligned}
$$

$$\times \int_0^{\delta} \theta \xi_\alpha(\theta) d\theta + \alpha M k \left(\frac{p-1}{p+(\alpha-1)p-1} \right)^{\frac{p-1}{p}}$$
$$\times \varepsilon^{\frac{p+(\alpha-1)p-1}{p}} \left(\|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{\varepsilon^{\frac{\beta p^2+p-1}{p^2}}}{\Gamma(1+\beta)} \|\mu\|_{L^{p^2}(J,\mathbb{R}^+)} \right).$$

Therefore, there are relatively compact sets $\{(F_{\varepsilon,\delta}x)(t), x \in B_k\}$ arbitrarily close to the set $\{(Fx)(t), x \in B_k\}$ for $t \in (0, b]$. Hence, $\{(Fx)(t), x \in B_k\}$ is relatively compact in $\mathbb{E}$. Moreover, $\{(Fx)(t), x \in B_k\}$ is uniformly bounded by (8). Therefore, $\{(Fx)(t), x \in B_k\}$ is relatively compact by Ascoli-Arzèla Theorem.

Also, Since F is continuous on B_K. Then F is a completely continuous operator. Obviously F maps B_k into itself. Hence, Schauder fixed point theorem shows that F has a fixed point $x \in B_k$, which means that the nonlocal Cauchy problem (1) has at least one mild solution on J. The proof is complete. □

The following existence and uniqueness result for the nonlocal Cauchy problem (1) is based on Banach contraction principle. We will need the following assumption.

(H4) There exists a positive constant L_f such that for any $x, x^*, y, y^* \in C(J, B_k)$, we have

$$|f(t, x, x^*) - f(t, y, y^*)| \le L_f(\|x - y\| + \|x^* - y^*)\|,$$

for $t \in J$, where k is defined as in (7).

Theorem 3. *If assumptions* $(H2)-(H4)$ *are satisfied, then the nonlocal Cauchy problem (1) has a unique mild solution provided that*

$$\left(ML + \frac{\alpha M L_f}{\Gamma(\alpha+1)} \left(\frac{b^\alpha}{\alpha} + \frac{\Gamma(\alpha) b^{\alpha+\beta}}{\Gamma(\alpha+\beta)} \right) \right) < 1. \tag{9}$$

Proof. It is easy to see that $\int_0^\infty \xi_\alpha(\theta) Q(t^\alpha \theta)(x_0 - g(x)) d\theta$ exists and

$$\int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^\alpha \theta) f(s, x(s), I^\beta x(s)) d\theta ds$$

Bochner's integrable with respect to $s \in [0, t]$ for all $t \in J$. For $x \in B_k$, Consider the operator F on B_k which is given by (6).

Obviously, it is sufficient to proof that F has a unique fixed point on B_k.

According to (8), we know that F is an operator from B_k into itself. For any $x, y \in B_k$ and $t \in J$, according to $(H3)$, $(H4)$ and (2), we have

$$|(Fx)(t) - (Fy)(t)| \le \left| \int_0^\infty \xi_\alpha(\theta) Q(t^\alpha \theta)(g(y) - g(x)) d\theta \right|$$

$$+\alpha\left|\int_0^t\int_0^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^\alpha\theta)[f(s,x(s),I^\beta x(s))-f(s,y(s),I^\beta y(s))]d\theta ds\right|$$
$$\leq ML\|x-y\|$$
$$+\frac{\alpha ML_f}{\Gamma(\alpha+1)}\left(\int_0^t(t-s)^{\alpha-1}|x(s)-y(s)|ds+\int_0^t(t-s)^{\alpha-1}\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}|x(\tau)-y(\tau)|d\tau ds\right).$$

By changing the order of the second integral, we get

$$\begin{aligned}\int_0^t\int_0^s(t-s)^{\alpha-1}(s-\tau)^{\beta-1}|x(\tau)-y(\tau)|d\tau ds=&\int_0^t\int_\tau^t(t-s)^{\alpha-1}(s-\tau)^{\beta-1}|x(\tau)-y(\tau)|dsd\tau\\ =&\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^t(t-\tau)^{\alpha+\beta-1}|x(\tau)-y(\tau)|d\tau\\ =&\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}|x(s)-y(s)|ds.\end{aligned}$$

Therefore, we get

$$\begin{aligned}|(Fx)(t)-(Fy)(t)|\leq&ML\|x-y\|\\ &+\frac{\alpha ML_f}{\Gamma(\alpha+1)}\left(\int_0^t(t-s)^{\alpha-1}ds+\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}ds\right)\|x-y\|\\ \leq&ML\|x-y\|+\frac{\alpha ML_f}{\Gamma(\alpha+1)}\left(\frac{b^\alpha}{\alpha}+\frac{\Gamma(\alpha)b^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\right)\|x-y\|.\end{aligned}$$

Thus
$$\|Fx-Fy\|\leq\left(ML+\frac{\alpha ML_f}{\Gamma(\alpha+1)}\left(\frac{b^\alpha}{\alpha}+\frac{\Gamma(\alpha)b^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\right)\right)\|x-y\|.$$

which means that F is a contraction according to (9). By applying Banach contraction principle, we know that F has a unique fixed point on B_k. The proof is complete. □

4. Mittag-Leffler-Ulam Stabilities

In this section, we consider the Mittag-Leffler-Ulam stability of the nonlocal Cauchy problem (1). Let ϵ be a positive real number and $\varphi : J \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities

$$\left|{}^CD^\alpha y(t)-Ay(t)-f\left(t,y(t),I^\beta y(t)\right)\right|\leq\epsilon,\ t\in J \tag{10}$$
$$\left|{}^CD^\alpha y(t)-Ay(t)-f\left(t,y(t),I^\beta y(t)\right)\right|\leq\varphi(t),\ t\in J \tag{11}$$
$$\left|{}^CD^\alpha y(t)-Ay(t)-f\left(t,y(t),I^\beta y(t)\right)\right|\leq\epsilon\varphi(t),\ t\in J \tag{12}$$

Definition 5. *Eq.* (1) *is Mittag-Leffler-Ulam-Hyers stable, with respect to* E_α *if there exists a real number* $c > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $y \in C^1(J, E)$ *of the inequality* (10)*, there exists a mild solution* $x \in C(J, E)$ *of Eq.* (1) *with*

$$\left|y(t) - x(t)\right| \le c\epsilon E_\alpha[t],\ t \in J.$$

Definition 6. *Eq.* (1) *is generalized Mittag-Leffler-Ulam-Hyers stable, with respect to* E_α *if there exists* $\theta \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta(0) = 0$ *such that for each solution* $y \in C^1(J, E)$ *of the inequality* (10)*, there exists a mild solution* $x \in C(J, E)$ *of Eq.* (1) *with*

$$\left|y(t) - x(t)\right| \le \theta(\epsilon) E_\alpha[t],\ t \in J.$$

Definition 7. *Eq.* (1) *is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to* φE_α *if there exists* $C_\varphi > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $y \in C^1(J, E)$ *of the inequality* (12)*, there exists a mild solution* $x \in C(J, E)$ *of Eq.*(1) *with*

$$\left|y(t) - x(t)\right| \le C_\varphi \epsilon \varphi(t) E_\alpha[t],\ t \in J.$$

Definition 8. *Eq.* (1) *is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to* φE_α *if there exists* $C_\varphi > 0$ *such that for each solution* $y \in C^1(J, E)$ *of the inequality* (11)*, there exists a mild solution* $x \in C(J, E)$ *of Eq.*(1) *with*

$$\left|y(t) - x(t)\right| \le C_\varphi \varphi(t) E_\alpha[t],\ t \in J.$$

Remark 4. *It is clear that:* (i) *Definition 5* $\Longrightarrow$ *Definition 6;* (ii) *Definition 7* $\Longrightarrow$ *Definition 8.*

Remark 5. *A function* $y \in C^1(J, E)$ *is a solution of the inequality* (10) *if and only if there exist a function* $h \in C(J, E)$ *(which depend on y) such that*

(i) $|h(t)| \le \epsilon,\ t \in J,$

(ii) ${}^C D^\alpha y(t) = Ay(t) + f\left(t, y(t), I^\beta y(t)\right) + h(t),\ t \in J.$

One can have similar remarks for the inequalities (11) and (12).

Remark 6. *If* $y \in C^1(J, E)$ *i a solution of the inequality* (10)*, then* y *is a solution of the following integral inequality*

$$\left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y(s), I^\beta y(s)) ds \right| \le \frac{M b^\alpha}{\Gamma(\alpha+1)}\, \epsilon.$$

We have similar remarks for the solutions of inequalities (11) and (12).

Theorem 4. *If assumptions* $(H3)$ *and* $(H4)$ *are satisfied, then the nonlocal Cauchy problem (1) is Mittag-Leffler-Ulam-Hyers stable.*

Proof. Let $y \in C^1(J,E)$ is a solution of the inequality (10). Let us denote by $x \in C(J,E)$ the unique mild solution of the nonlocal Cauchy problem

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + f(t, x(t), I^\beta x(t)), \ t \in J \\ x(0) = y(0). \end{cases} \tag{13}$$

We have

$$x(t) = S(t)(y_0 - g(y)) + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y(s), I^\beta y(s)) ds.$$

Then we get

$$\begin{aligned}
&\left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y(s), I^\beta y(s)) ds \right| \\
&\leq \left| \int_0^t (t-s)^{\alpha-1} T(t-s) h(s) ds \right| \\
&\leq \alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^{\alpha-1}\theta) |h(s)| d\theta ds \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)} \, \epsilon \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{M b^\alpha}{\Gamma(\alpha+1)} \, \epsilon.
\end{aligned}$$

From these relations, we have

$$\begin{aligned}
|y(t) - x(t)| &= \left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, x(s), I^\beta x(s)) ds \right| \\
&\leq \left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y(s), I^\beta y(s)) ds \right| \\
&\quad + \left| \int_0^t (t-s)^{\alpha-1} T(t-s) [f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))] ds \right| \\
&\leq \frac{M b^\alpha}{\Gamma(\alpha+1)} \, \epsilon \\
&\quad + \alpha \left| \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^{\alpha-1}\theta) [f(s, y(s), I^\beta y(s)) - \right. \\
&\quad \left. - f(s, x(s), I^\beta x(s))] d\theta ds \right| \\
&\leq \frac{M b^\alpha}{\Gamma(\alpha+1)} \, \epsilon + \\
&\quad + \frac{\alpha M}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} |f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))| ds
\end{aligned}$$

$$\leq \frac{Mb^{\alpha}}{\Gamma(\alpha+1)}\,\epsilon + \frac{\alpha M L_f}{\Gamma(\alpha+1)}\Bigg(\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds$$
$$+\frac{1}{\Gamma(\beta)}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\beta-1}|x(\tau)-y(\tau)|d\tau ds\Bigg).$$

Using a similar manner of the second integral as in Proof of Theorem 3, we get

$$|y(t)-x(t)| \leq \frac{Mb^{\alpha}}{\Gamma(\alpha+1)}\,\epsilon + \frac{\alpha M L_f}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds$$
$$+\frac{M L_f}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|x(s)-y(s)|ds.$$

An application of Remark 2 and Remark 3 (with $b_1 = \frac{\alpha M L_f}{\Gamma(\alpha+1)}$, $b_2 = \frac{M L_f}{\Gamma(\alpha+\beta)}$, $\beta_1 = \alpha$ and $\beta_2 = \alpha+\beta$) to the last inequality yields the desired estimation

$$|y(t)-x(t)| \leq \frac{Mb^{\alpha}}{\Gamma(\alpha+1)}\,\left(E_{\alpha}\left[ML_f\ t^{\alpha}\right] + E_{\alpha+\beta}\left[ML_f\ t^{\alpha+\beta}\right]\right)\epsilon.$$

Thus, the conclusion of our theorem hold. □

The following theorem gives generalized Mittag-Leffler-Ulam-Hyers stability.

Theorem 5. *If assumptions* $(H3)$ *and* $(H4)$ *are satisfied. Suppose there exist* $\lambda > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds \leq \lambda\varphi(t),$$

for all $t \in J$*, where* $\varphi \in C(J,\mathbb{R}^+)$ *is nondecreasing. Then the nonlocal Cauchy problem (1) is generalized Mittag-Leffler-Ulam-Hyers stable with respect to* φE_{α}*.*

Proof. Let $y \in C^1(J,E)$ is a solution of the inequality (11). By Remark 5, we have for $t \in J$ that y is a solution of the following integral inequality

$$\left|y(t)-S(t)(y_0-g(y))-\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s),I^{\beta}y(s))ds\right| \leq M\lambda\ \varphi(t).$$

Let us denote by $x \in C(J,E)$ the unique mild solution of the nonlocal Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + f(t,x(t),I^{\beta}x(t)),\ t\in J \\ x(0) = y(0). \end{cases} \tag{14}$$

We have

$$x(t) = S(t)(y_0-g(y)) + \int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s),I^{\beta}y(s))ds.$$

Then we get

$$
\begin{aligned}
&\left|y(t)-S(t)(y_0-g(y))-\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s),I^\beta y(s))ds\right|\\
\leq&\left|\int_0^t (t-s)^{\alpha-1}T(t-s)\varphi(s)ds\right|\\
\leq&\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^{\alpha-1}\theta)\varphi(s)d\theta ds\\
\leq&\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\\
\leq&M\lambda\ \varphi(t).
\end{aligned}
$$

Again, from these relations, we have

$$
\begin{aligned}
|y(t)-x(t)|=&\left|y(t)-S(t)(y_0-g(y))-\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,x(s),I^\beta x(s))ds\right|\\
\leq&\left|y(t)-S(t)(y_0-g(y))-\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y(s),I^\beta y(s))ds\right|\\
&+\left|\int_0^t (t-s)^{\alpha-1}T(t-s)[f(s,y(s),I^\beta y(s))-f(s,x(s),I^\beta x(s))]ds\right|\\
\leq&M\lambda\ \varphi(t)+\alpha\left|\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)Q((t-s)^{\alpha-1}\theta)\right.\\
&\left.\times[f(s,y(s),I^\beta y(s))-f(s,x(s),I^\beta x(s))]d\theta ds\right|\\
\leq&M\lambda\ \varphi(t)+\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s),I^\beta y(s))-f(s,x(s),I^\beta x(s))|ds\\
\leq&M\lambda\ \varphi(t)+\frac{\alpha M L_f}{\Gamma(\alpha+1)}\left(\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds\right.\\
&\left.+\frac{1}{\Gamma(\beta)}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{\beta-1}|x(\tau)-y(\tau)|d\tau ds\right).
\end{aligned}
$$

Using a similar manner of the second integral as in Proof of Theorem 3, we get

$$
\begin{aligned}
|y(t)-x(t)|\leq&\frac{Mb^\alpha}{\Gamma(\alpha+1)}\ \epsilon+\frac{\alpha M L_f}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds\\
&+\frac{M L_f}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|x(s)-y(s)|ds.
\end{aligned}
$$

According to Remark 2 and Remark 3, we obtain our required assertion

$$|y(t)-x(t)| \leq M\lambda \left(E_\alpha\left[ML_f\ t^\alpha\right]+E_{\alpha+\beta}\left[ML_f\ t^{\alpha+\beta}\right]\right)\varphi(t).$$

The conclusion of our theorem hold. □

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