Global existence for a double dispersive sixth order Boussinesq equation

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Abstract. In this paper, existence of global weak solutions for the double dispersive sixth order Boussinesq equation with power type nonlinearity $\beta |u|^p$ and supercritical initial energy will be investigated. A new functional, which includes not only initial displacement but also initial velocity, will be defined for the potential well method, and by the aid of sign invariance of this functional the Cauchy problem for the supercritical initial energy will be considered.

Key words. Double dispersive Boussinesq equation, global existence, supercritical initial energy.

1 Introduction

In the present work, we consider the question of existence of global solutions for the Cauchy problem of a double dispersive sixth order Boussinesq-type equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} + u_{xxxxxtt} = (f (u))_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.1)$$

$$u (x, 0) = u_0 (x), \quad u_t (x, 0) = u_1 (x), \quad x \in \mathbb{R}, \quad (1.2)$$

where $f (u) = \beta |u|^p, \beta > 0$. Equation (1.1) describes the motion of water waves with surface tension and was considered by Schneider and Eugene in [1]. In [1], it was proven that the long wave limit can be described approximately by two decoupled Kawahara-equations for a degenerate case. The model can be also deduced from 2D water wave problem. Boussinesq-type equations have been extensively studied and several results about existence, blow-up, and energy decay of its solutions have been obtained in [2–6].
Recently, Wang and Mu [7], and Wang and Xue [8] studied Cauchy Problem (1.1)-(1.2).

Wang and Mu [7] proved the global existence of small amplitude solutions of Problem (1.1)-(1.2). Blow up and scattering of the solutions were also investigated in [7]. Even though the results in [7] were obtained for a general nonlinear term, they cannot give the global existence for \( f(u) = \beta |u|^p \), even for the original physical model with \( f(u) = u^2 \).

Some conclusions regarding the well-posedness of Problem (1.1)-(1.2) have been deduced in [8]. The existence of global solutions and finite time blow up were proved by the potential well method for the nonlinear term \( f(u) = |u|^p \). However, they proved the global existence of solutions for the critical and subcritical initial energy, i.e., for \( 0 < E(0) \leq d \), and it is natural to ask whether or not the solutions exist globally in the supercritical initial energy case, i.e., for \( E(0) > d \).

In the present paper, we investigate Problem (1.1)-(1.2) and give the global existence of solutions for \( f(u) = \beta |u|^p \) with supercritical initial energy. Global well-posedness with this type of nonlinearity and supercritical initial energy for some Boussinesq-type equations have been considered very recently in [9–12], but for Problem (1.1)-(1.2) the nonlinear term \( f(u) = \beta |u|^p \) in the case \( E(0) > d \) have not been considered together yet. We use potential well method [8,13–16] for the proof of the global existence. In order to prove global existence in the case of supercritical initial energy, it is also necessary to take into account the initial velocity [11]. For this purpose, we define a new functional that includes both the initial displacement and the initial velocity.

The outline of the paper is as follows. An abbreviated description of the global existence theory for \( E(0) \leq d \) [8] will be given in Section 2. Also it contains our first new functional to prove the global existence for \( E(0) > d \) and some theorems about sign preserving property of this functional. The global existence theory for supercritical initial energy is presented in Section 3. For this purpose, we introduce the second new functional and give the invariance of this functional under the flow of (1.1), (1.2).

The following notations will be used in the paper. \( H^s = H^s(\mathbb{R}) \) will denote the \( L^2 \) Sobolev space on \( \mathbb{R} \) with norm \( \|f\|_{H^s} = \left\| (I - \partial_x^2)^{\frac{s}{2}} f \right\| = \left\| (1 + k^2)^{\frac{s}{2}} \hat{f} \right\| \), where \( s \) is a real number, \( I \) is unitary operator. \( \dot{H}^s \) denotes the homogeneous space corresponding \( H^s \) with the semi-norm \( \|f\|_{\dot{H}^s} = \left\| k^s \hat{f} \right\| \). The notation \( \|f\|_p, \|f\| \) and \( \|f\|_\infty \) will be used subsequently instead of \( L^p(\mathbb{R}), L^2(\mathbb{R}) \) and \( L^\infty(\mathbb{R}) \).
2 Preliminaries

In this section, we present an overview of the basic aspects of global existence of solutions for Problem (1.1)-(1.2) in the case of \( E(0) \leq d \), then we introduce a new functional, and prove global well-posedness for supercritical initial energy by the aid of the new functional.

We define the kinetic energy at time \( t \geq 0 \) as
\[
E_{\text{kin}}(t) = E_{\text{kin}}(u_t(t)) = \frac{1}{2} \left[ \left\| \left(-\partial_x^2\right)^{-\frac{1}{2}} u_t \right\|^2 + \|u_t\|^2 + \|u_{xt}\|^2 \right].
\]

The potential energy is
\[
E_{\text{pot}}(t) = E_{\text{pot}}(u(t)) = J(u) = \frac{1}{2} \|u\|^2_{H^1} + \frac{\beta}{p+1} \int_{\mathbb{R}} |u|^p \, dx,
\]

the total energy is
\[
E(t) = E(u(t), u_t(t)) = E_{\text{kin}}(u_t(t)) + E_{\text{pot}}(u(t)) = E(0),
\]

\[
I(u) = \|u\|^2_{H^1} + \beta \int_{\mathbb{R}} |u|^p \, dx,
\]

and the potential well depth associated with \( J(u) \) is
\[
d = \inf_{u \in N} J(u), \quad N = \{ u \in H^1 : I(u) = 0, \|u\|_{H^1} \neq 0 \},
\]

which are all well defined.

Note that \( d \) can also be characterized as follows.
\[
d = \frac{p-1}{2(p+1)} \left( \beta S_p^{p+1} \right)^{-2/(p-1)},
\]

where \( S_p \) is the imbedding constant from \( H^1(\mathbb{R}) \) into \( L^{p+1}(\mathbb{R}) \) given by
\[
S_p = \sup_{u \in H^1} \frac{\|u\|_{p+1}}{\|u\|_{H^1}}.
\]

In the case of \( E(0) \leq d \), the behavior of solutions of (1.1), (1.2) can be determined by the functional \( I(u_0) \). Namely, if \( I(u_0) > 0 \), then the problem has a global solution, and if \( I(u_0) < 0 \), then the solution blows up in finite time. However, the global existence for the supercritical case can not be proved by the sign invariance of \( I(u_0) \) [9]. Due to the choice of initial velocity, for \( I(u_0) > 0 \), there are some solutions that blow up in finite time, while the same contrast occurs for \( I(u_0) < 0 \), i.e., global solutions exist in the case of \( I(u_0) < 0 \). It will be proven in the rest of this section that even by taking a more general functional than \( I \), which only includes the initial velocity, global well-posedness can not be proved.
Let us define the first new functional
\[
I_\sigma(u) = (1 - \sigma) \|u\|_{H^1}^2 + \beta \int_\mathbb{R} |u|^p \, dx = I(u) - \sigma \|u\|_{H^1}^2,
\]
for \(\sigma > -\frac{p-1}{2}\). The depth \(D_\sigma\) and \(N_\sigma\) are as follows
\[
D_\sigma = \inf_{u \in N_\sigma} J(u), \quad N_\sigma = \{u \in H^1 : I_\sigma(u) = 0, \|u\|_{H^1} = 0\}. \tag{2.5}
\]
Obviously, taking \(\sigma = 0\), \(I_\sigma\) corresponds to the functional \(I(u)\). Moreover, if \(\sigma < -\frac{p-1}{2}\) then \(D_\sigma < 0\). In this case, all weak solutions of (1.1), (1.2) blow up in a finite time.

For \(\sigma > -\frac{p-1}{2}\), we have the following lemmas.

**Lemma 2.1** Assume that \(u \in H^1(R)\). If \(I_\sigma(u) < 0\), then \(\|u\|_{H^1} > r(\sigma)\). If \(I_\sigma(u) = 0\), then \(\|u\|_{H^1} \geq r(\sigma)\) or \(\|u\|_{H^1} = 0\), where \(r(\sigma) = \left(\frac{1-\sigma}{\beta S^p_{p+1}}\right)^{1/(p-1)}\).

**Proof.** First, from \(I_\sigma(u) < 0\), we have \(\|u\|_{H^1} > r(\sigma)\).

If \(\|u\|_{H^1} = 0\), then \(I_\sigma(u) = 0\), if \(I_\sigma(u) = 0\) and \(\|u\|_{H^1} \neq 0\), then from
\[
(1 - \sigma) \|u\|_{H^1}^2 = -\beta \int_\mathbb{R} |u|^p \, dx \leq \beta S_{p+1}^p \|u\|_{H^1}^{p+1},
\]
we have \(\|u\|_{H^1} \geq r(\sigma)\).

It follows that \(\|u\|_{H^1} \geq r(\sigma)\). ■

**Lemma 2.2** If \(\|u\|_{H^1} < r(\sigma)\), then \(I_\sigma(u) \geq 0\).

**Proof.** By \(\|u\|_{H^1} < r(\sigma)\), we obtain
\[
-\beta \int_\mathbb{R} |u|^p \, dx \leq \beta S_{p+1}^p \|u\|_{H^1}^{p+1} < (1 - \sigma) \|u\|_{H^1}^2,
\]
from which follows \(I_\sigma(u) \geq 0\). ■

**Theorem 2.3** Let \(D_\sigma\) be defined as above. Then for \(\sigma > -\frac{p-1}{2}\), we have
\[
D_\sigma = b(\sigma) r^2(\sigma),
\]
where \(b(\sigma) = \frac{1}{2} - \frac{(1-\sigma)}{p+1}\). If we write \(D_\sigma\) in terms of \(d\), we obtain
\[
D_\sigma = b(\sigma) (1-\sigma)^{2/(p-1)} \frac{2(p+1)}{p-1} d. \tag{2.7}
\]

63
Proof. If \( u \in N_\sigma \), by Lemma 2.1 we have \( \|u\|_{H^1} \geq r(\sigma) \). In the proof of Lemma 2.1 the inequality (2.6) is an equality if and only if \( u \) is a minimizer of the imbedding \( H^1 \) into \( L^{p+1} \). Since \( \|u\|_{L^{p+1}} = S_p \|u\|_{H^1} \) is attained only for \( \vec{u} = (\cosh \left( \frac{p-1}{2} x \right)^{- \frac{1}{p+1}} \) [17], and it has constant sign, we have

\[
\inf_{u \in N_\sigma} \|u\|_{H^1} = r(\sigma).
\]

Hence from

\[
\inf_{u \in N_\sigma} J(u) = \inf_{u \in N_\sigma} \left( \frac{1}{2} \|u\|_{H^1}^2 + \frac{\beta}{p+1} \int_R |u|^p u\,dx \right)
\]

\[
= \inf_{u \in N_\sigma} \left[ \left( \frac{1}{2} - \frac{(1-\sigma)}{p+1} \right) \|u\|_{H^1}^2 + \frac{1}{p+1} I_\sigma(u) \right]
\]

\[
= \left( \frac{1}{2} - \frac{(1-\sigma)}{p+1} \right) \inf_{u \in N_\sigma} \|u\|_{H^1}^2,
\]

and by definition of \( D_\sigma \) we obtain \( D_\sigma = b(\sigma) r^2(\sigma) \). \( \blacksquare \)

We can also state the following properties of \( D_\sigma \), which can be proved easily.

i) \( D_\sigma \) is strictly increasing on \( \sigma \in (-\frac{p-1}{2}, 0) \cup (1, \infty) \) and strictly decreasing on \( (0, 1) \).

ii) \( \lim_{\sigma \to 1} D_\sigma = 0 \), and \( D_{\sigma_0} = 0 \), where \( \sigma_0 = -\frac{p-1}{2} \).

The following theorems show the invariance of \( I_\sigma \) under the flow of (1.1), (1.2) for \( 0 < E(0) < d \) and \( E(0) = d \), respectively, and can be proved by contradiction as in [10].

**Theorem 2.4** Assume that \( u_0 \in H^1(\mathbb{R}) \), \( u_1 \in H^1(\mathbb{R}) \cap H^{-1}(\mathbb{R}) \). Let \( 0 < E(0) < d \). Then the sign of \( I_\sigma \) is invariant under the flow of (1.1), (1.2) for \( \sigma \in (\sigma_1, \sigma_2] \), where \( (\sigma_1, \sigma_2] \) is the maximal interval such that \( D_\sigma = E(0) \).

**Theorem 2.5** Let all the assumptions of Theorem 2.4 hold and suppose that \( E(0) = d \). Then, the sign of \( I_0 \) (recall that when \( E(0) = d \), we have \( \sigma_1 = \sigma_2 = 0 \)) is invariant with respect to (1.1), (1.2) for every \( t \in [0, \infty) \).

Now, we give a lemma for \( \sigma > 1 \), which states similar results to Lemmas 2.1, 2.2, and can be proved similarly.

**Lemma 2.6** Assume that \( u \in H^1(\mathbb{R}) \). Let \( \sigma > 1 \). If \( I_\sigma(u) > 0 \), then \( \|u\|_{H^1} > s(\sigma) \). If \( I_\sigma(u) = 0 \), then \( \|u\|_{H^1} \geq s(\sigma) \) or \( \|u\|_{H^1} = 0 \), where \( s(\sigma) = \left( \frac{\sigma-1}{2 \sigma^2} \right)^{1/(p-1)} \). Moreover, if \( \|u\|_{H^1} < s(\sigma) \), then \( I_\sigma(u) \leq 0 \) and \( I_\sigma(u) = 0 \) if and only if \( \|u\|_{H^1} = 0 \).
\textbf{Theorem 2.7} Assume that \( u_0 \in H^1(\mathbb{R}) \), \( u_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}) \). If \( E(0) > 0 \), then \( I_\sigma(u(t)) \leq 0 \) for every \( t > 0 \) and \( \sigma \geq \sigma_m \), where \( \sigma_m \) is the maximal positive root of \( D_{\sigma} = E(0) \).

\textbf{Proof.} We give the proof of the theorem for \( \sigma = \sigma_m \) and \( \sigma > \sigma_m \) separately. First, we prove the theorem for \( \sigma = \sigma_m \). By contradiction, assume that there exists some \( t' > 0 \) such that \( I_{\sigma_m}(u(t')) > 0 \). By Lemma 2.1, we have \( \|u\|_{H^1} > 0 \) and there exists a value \( \sigma > \sigma_m \) such that \( I_{\sigma}(u(t')) = 0 \). Then, by (2.1), \( D_{\sigma_m} = E(0) \geq J(u(t')) \geq \inf_{u \in N_{\sigma}} J(u) = D_{\sigma} \). By definition of \( D_{\sigma} \), for \( \sigma > \sigma_m > 1 \) we have \( D_{\sigma} > D_{\sigma_m} \). A contradiction occurs, which proves the theorem for \( \sigma = \sigma_m \). For \( \sigma \geq \sigma_m \), \( I_{\sigma_m}(u(t)) \geq I_{\sigma}(u(t)) \) implies that the theorem is true for every \( \sigma \geq \sigma_m \).

The following corollary gives a more precise result for subcritical initial energy.

\textbf{Corollary 2.8} Suppose \( u_0 \in H^1(\mathbb{R}) \), \( u_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}) \). Let \( 0 < E(0) < d \) and \( I_0(u_0) \geq 0 \). Then,
\[ 0 < I_0(u(t)) < \sigma_m \|u\|^2_{H^1} \quad (2.8) \]
for every \( t > 0 \).

\textbf{Proof.} We know that for \( I_0(u(t)) > 0 \), the solution \( u(x,t) \) of problem (1.1), (1.2) is globally defined. Since \( E(0) = D_{\sigma_m} \) for some \( \sigma_m > 1 \) then by Theorem 2.7 we have \( I_{\sigma_m}(u(t)) \leq 0 \) for every \( t \in [0, \infty) \). Thus we get the inequality (2.8) from below and from above.

\textbf{Remark 2.9} We tried to characterize the behavior of solutions for \( E(0) > 0 \) in terms of initial displacement. We constituted the new functional \( I_\sigma(u) \) and proved the sign invariance of \( I_\sigma(u) \) for \( 0 < E(0) < d \) and \( E(0) = d \). However, the case \( E(0) > 0 \) is still an open question, because from Theorem 2.7, we concluded that in this case \( I_\sigma(u) \) is always non-positive. Due to numerical results of [9], we know that to prove global existence such a functional must include the initial velocity too. We will introduce this new functional in the next section.

\section{Global existence for supercritical initial energy}

In this section, we will introduce a functional which allows us to prove the existence of global solutions of (1.1), (1.2) in the supercritical initial energy case. The new functional is as follows

\[ \bar{H}(v, \omega) = \|v\|^2_{H^1} + \beta \int_{\mathbb{R}} |v|^{p} v dx - \left\| \left(-\partial_x^2\right)^{-\frac{1}{2}} \omega \right\|^2 - \|\omega\|^2 - \|\omega_x\|^2 \]
\[ = I_0(v) - \left\| \left(-\partial_x^2\right)^{-\frac{1}{2}} \omega \right\|^2 - \|\omega\|^2 - \|\omega_x\|^2. \]
Instead of \( \tilde{H} (u (., t), u_t (., t)) \), we write \( H (u, t) \).

The following theorem which shows the sign invariance of \( H (u, t) \) is essential in the proof of the global existence theorem.

**Theorem 3.1** Assume that \( u_0, u_1 \in H^1 (\mathbb{R}) \cap \dot{H}^{-1} (\mathbb{R}) \), and \( E (0) > 0 \). For some \( \sigma > \sigma_m, \sigma_m \) defined as above, suppose that

\[
\begin{align*}
\left( (-\partial_x^2)^{-\frac{1}{2}} u_1, (-\partial_x^2)^{-\frac{1}{2}} u_0 \right) + (u_1, u_0) + (u_{1x}, u_{0x}) + \frac{1}{2} \left\| (-\partial_x^2)^{-\frac{1}{2}} u_0 \right\|^2 \\
+ \frac{1}{2} \| u_0 \|^2 + \frac{1}{2} \| u_{0x} \|^2 + \frac{(p + 1) \sigma}{p - 1 + (p + 3) \sigma} E (0) \leq 0.
\end{align*}
\]

Then, \( H (u, t) \) is positive provided \( H (u, 0) \) is positive, for every \( t \in [0, \infty) \).

**Proof.** The main tool used to prove the theorem is a modified blow up technique [18]. To this end, we define

\[
\theta (t) = \left\| (-\partial_x^2)^{-\frac{1}{2}} u \right\|^2 + \| u \|^2 + \| u_x \|^2.
\]

Differentiating \( \theta (t) \) gives

\[
\theta' (t) = 2 \left( (-\partial_x^2)^{-\frac{1}{2}} u_t, (-\partial_x^2)^{-\frac{1}{2}} u \right) + 2 (u_t, u) + 2 (u_{xt}, u_x),
\]

and

\[
\theta'' (t) = 2 \left\| (-\partial_x^2)^{-\frac{1}{2}} u_{tt} \right\|^2 + 2 \left( (-\partial_x^2)^{-\frac{1}{2}} u_{tt}, (-\partial_x^2)^{-\frac{1}{2}} u \right) \\
+ 2 (u_{tt}, u) + 2 \| u_t \|^2 + 2 (u_{xt}, u_x) + 2 \| u_{xt} \|^2 \\
= 2 \left\| (-\partial_x^2)^{-\frac{1}{2}} u_t \right\|^2 + 2 \| u_t \|^2 + 2 \| u_{xt} (t) \|^2 \\
+ 2 \left( (-\partial_x^2)^{-1} u_{tt}, u \right) + 2 (u_{tt}, u) - 2 (u_{xt}, u) \\
= 2 \left\| (-\partial_x^2)^{-\frac{1}{2}} u_t \right\|^2 + 2 \| u_t \|^2 + 2 \| u_{xt} (t) \|^2 - 2 I_0 (u) \\
= -2 H (u, t).
\]

To get a contradiction, let us assume that there exists some \( t' > 0 \) such that \( H (u, t') = 0 \). Since \( \theta'' (t) < 0 \), we conclude that \( \theta' (t) \) is strictly decreasing on \([0, t']\). Moreover, (3.1) implies \( \theta' (0) < 0 \) and therefore \( \theta' (t) < 0 \) in \([0, t']\), from which follows that \( \theta (t) \) is strictly decreasing.
on \([0, t']\). By the energy identity and \(H(u, t') = 0\), we have

\[
E(0) = \frac{1}{2} \left( \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t(t') \right\|^2 + 2 \left\| u_t(t') \right\|^2 + \left\| u_{xt}(t') \right\|^2 \right) + \frac{p-1}{2(p+1)} \left\| u(t') \right\|^2_{H^1} + \frac{1}{p+1} I(u(t'))
\]

\[
= \left( \frac{1}{2} + \frac{1}{p+1} \right) \left( \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t(t') \right\|^2 + 2 \left\| u_t(t') \right\|^2 + \left\| u_{xt}(t') \right\|^2 \right) + \frac{p-1}{2(p+1)} \left\| u(t') \right\|^2_{H^1}.
\]

(3.2)

Theorem 2.7 and \(H(u, t') = 0\) yield

\[
\|u\|^2_{H^1} \geq \sigma_m^{-1} J_0(u(t')) \geq \sigma^{-1} \left( \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t(t') \right\|^2 + 2 \left\| u_t(t') \right\|^2 + \left\| u_{xt}(t') \right\|^2 \right).
\]

The use of this inequality in (3.2) gives

\[
E(0) \geq \left( \frac{1}{2} + \frac{1}{p+1} + \frac{p-1}{2(p+1)} \right) \left( \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t(t') \right\|^2 + 2 \left\| u_t(t') \right\|^2 + \left\| u_{xt}(t') \right\|^2 \right).
\]

This can be rephrased in terms of \(\theta(t)\) and \(\theta'(t)\) as

\[
E(0) \geq \frac{(p+3)\sigma + p-1}{2(p+1)\sigma} \left[ \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} (u_t(t') + u(t')) \right\|^2 + \left\| u_t(t') + u(t') \right\|^2 \right]
\]

\[
+ \left\| u_{xt}(t') + u(t') \right\|^2 - 2 \left( \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t(t'), \left( -\partial_x^2 \right)^{-\frac{1}{2}} u(t') \right\|^2 \right) - 2 (u_t, u)
\]

\[
- 2 (u_{xt}(t'), u_x(t')) - \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u(t') \right\|^2 - \left\| u(t') \right\|^2 - \left\| u_{x}(t') \right\|^2 \right].
\]

By the monotonicity of \(\theta(t)\) and \(\theta'(t)\), we have

\[
E(0) > \frac{(p+3)\sigma + p-1}{(p+1)\sigma} \left[ - \left( \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_1, \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_0 \right) - (u_1, u_0)
\]

\[
- (u_{1x}, u_{0x}) - \frac{1}{2} \left\| \left( -\partial_x^2 \right)^{-\frac{1}{2}} u_0 \right\|^2 - \frac{1}{2} \left\| u_0 \right\|^2 - \frac{1}{2} \left\| u_{1x} \right\|^2 \right]
\]

which contradicts (3.1). Thus, the theorem is proved.  

**Theorem 3.2** Assume that \(u_0, u_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})\). Suppose that \(E(0) > 0\), \(H(u, 0) > 0\) and (3.1) holds for some \(\sigma > \sigma_m\). Then the weak solution of (1.1), (1.2) is globally defined for every \(t \in [0, \infty)\).

**Proof.** The proof of this theorem follows from adding some arguments to the local existence result of Theorem 2.3 of [8]. If \(\left( -\partial_x^2 \right)^{-\frac{1}{2}} u_1 \in L^2(\mathbb{R})\), then \(\left( -\partial_x^2 \right)^{-\frac{1}{2}} u_t \in L^2(\mathbb{R})\). By the assumption \(H(u, 0) > 0\) and the sign preserving property of \(H(u, t)\), we have \(H(u, t) > 0\).
Thus, $I_0(u) > 0$ for every $t > 0$. From the energy identity it follows that

$$E(0) = \frac{1}{2} \left( \left\| (-\partial_x^2)^{-\frac{1}{2}} u_t \right\|^2 + \|u_t\|^2 + \|u_{xt}\|^2 \right) + \frac{p-1}{2(p+1)} \|u\|^2_{H^1} + \frac{1}{p+1} I(u)$$

$$\geq \frac{1}{2} \left( \left\| (-\partial_x^2)^{-\frac{1}{2}} u_t \right\|^2 + \|u_t\|^2 + \|u_{xt}\|^2 \right) + \frac{p-1}{2(p+1)} \|u\|^2_{H^1}.$$

Therefore, $\|u\|_{H^1}$ and $\|u_t\|_{H^1}$ are bounded for every $t > 0$. The previously mentioned local existence theory completes the proof. ■

**Remark 3.3** In Section 2, we introduce the first new functional $I_\sigma(u)$, which is more general than the ones introduced before in some papers on fourth order Boussinesq equation [14]. However, in the case of $E(0) > 0$ we are not able to prove the sign invariance of $I_\sigma(u)$, because we see that for $E(0) > 0$, $I_\sigma(u)$ is always non-positive. Eventually, a satisfactory result comes from $H(u,t)$ which includes not only the initial displacement $u_0$, but also the initial velocity $u_1$.

**References**


