

MATHEMATICAL MODEL FOR THE ANALYSIS OF EXPERT ASSESSMENTS IN EDUCATION

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ABSTRACT

In this paper we develop and study mathematical model for the analysis of the educational numerical expert evaluations that characterize both the educational and psychological levels of student training (scholars and/or students) required in order to continue their further studies successfully. To solve the constructed mathematical model an iterative algorithm is developed. Besides, it is proved that algorithm convergence as well as its convergence rate is determined. A numerical experiment illustrating how an iterative algorithm function is implemented was conducted. The obtained results show that by means of using the developed model as well as algorithm required for finding its solution there could be ranked both the true ratings of students based on the overall expert evaluations and the experts themselves in two ways – using the levels of "objectivity" and "coherence".

Keywords: Mathematical model, expert assessments, objectivity, coherence.

INTRODUCTION

The problem of expert evaluation analysis in educational activities is referred to as the global problem of optimum decision making or even the acceptable one in the presence of inaccurate and/or missing information relevant to the studied object, process or phenomenon. Among the expert methods used in teaching activities the following methods should be mentioned (for instance, see [1-4] and respective references given in these): a method of group experts evaluations; the method of individual expert evaluation; the method of paired comparisons and multidimensional scaling; methods based on multidimensional grouping; the method of independent characteristics synthesis; the sociometric method; the testing method; interviewing; opinion polling; the morphological expert method; the method of self estimation; hermeneutic methods.

Obviously, the expertise procedure involves passing through the following phases:

- preparatory phase, which consists of decision-making procedures for examination, the procedure for selection and creation of technical working group; of the objectives development procedure, preparing plan and schedule of examination; of the selection process and creation of the expert committee/group;
- phase of technical working group activity, which in its turn consists of a procedure for determining the rules of both own group and expert committee/group operation; of procedures ensuring the technical side of examination, including the presence of examined students, negotiating the time of the examination, preparation of the technical and material background for examination, etc.; and of the procedure allowing to develop complementary materials;
- phase of the expert committee/group operation;

- the final phase, which is devoted to the simultaneous solution of the following interrelated objectives:
 - 1) based on scientific analysis of the overall expert evaluation it is necessary to rank students in accordance with their true ratings;
 - 2) on the same basis of expert evaluation it is required to rank the experts themselves on the degree of their "consistency" and "objectivity";
 - 3) it is required to take into account the influence of experts' "coherence" and "objectivity" degree to the "truth" degree of student ratings in determining the final true ratings of students..

As the title of present work suggests, the study of the authors of this paper will be relevant only to the final phase of examination procedure, when from the side of DM (decision maker) it is required to make the best decisions on the above simultaneously solved three interrelated problems. At this final stage, there are traditionally used various mathematical models and algorithms for making the final decision by the DM. These models and algorithms may have different levels of complexity and adequacy, and could be described in terms of various areas of mathematics – probability theory, the theory of differential equations in partial derivatives, the theory of matrix games, differential games theory, operations research, etc. Mathematical models at the final phase of the examination procedure form the basis of the experts' survey planning, data collection and analysis of expert opinion, furthermore not only being expressed in numerical form (for instance, see [5, 6] and respective references given in these). Below in the next section of this paper it is offered as one of the mathematical models that uses only a numerical assessment of expert opinions.

FORMULATION OF THE PROBLEM AND CONSTRUCTION OF A MATHEMATICAL MODEL

Before we proceed directly to construction of a mathematical model, we should note that in problems of expert survey planning, data collection and analysis of expert opinion, as a rule, a feasible decision, including the optimal solution is made based on correlated experts' opinions (for instance, see [7 -9]), i.e. those experts whose opinions (as noted above, the expression of opinions may have non-numeric form, but in case if the opinions of experts are expressed as numbered values, these numbers can be also fractional) substantially differ from the opinions of most experts in the committee, they are excluded from the Commission of Experts or their opinions are discarded. Such a way of processing may happen, for example, in judging some kind of sports, as well as in the process of making compromises on economic issues in enterprises in which the DM is the Board of Directors, shareholders, etc. It is obvious that such an approach to the selection of an acceptable (even not optimal) solution, when they are not taken into account sharply contradicting expert opinions and evaluations there can be obtained a distorted final examination assessment, where the distorted measure remains unassessed and, moreover, there also remains unexplored potential impact of this measure on the final evaluation of examination. Hence, such approach does not reduce the influence of distorted expert evaluations on the final decision of DM. In addition, there is another major drawback concerning the experts themselves, regardless of whether the chosen approach for the examination evaluation is taken on the basis of experts' correlated opinions: shortcomings usually arise during the preparatory phase of examination procedure when a selection and further creation of the expert committee take place. Namely, some members of the expert group

- cannot objectively (in the undistorted way) evaluate the object of expertise due to the lack of qualification. In this case the assessments of experts are usually independent from each other and, therefore, are inconsistent (*due to the lack of "experts" qualification Giordano Bruno – "unrepentant, stubborn and inflexible heretic" – was deprived of priest rank, excommunicated and sentenced by the court to "the most merciful punishment, without shedding of blood"; other well-known example is the conclusion of the three experts from Inquisition on the book of Galileo Galilei "Dialogue Concerning the Two Major World Systems – Ptolemaic and Copernican"; other well-known example of pedagogy can serve as expert opinions of the professors from the ÉcolePolytechnique, who rejected twice to study in the college genius Évariste Galois; as well as many other examples*);

- may deliberately distort the evaluation, pursuing different goals, not associated with the expertise itself. In this case the evaluation of such experts tend to agree (*with a loud fresh example is the intentional expert opinion about whether Iraq possessed "an unprecedented stock of mass destruction weapons – chemical and biological weapons, as well as the "presence of mobile stations for the production of biological weapons, based on trucks", etc.*).

Thus, in problems of expert survey planning, data collection and analysis of expert opinion, distortions are quite possible (for both reasons – due to lack of qualifications of the experts or intentionally) in expert assessments. Therefore, there arises a need to construct a mathematical model that would allow minimize the consequences of the above mentioned shortcomings of traditional models taking into account only correlated expert opinions.

Verbal formulation of the problem studied in this paper is the following: there are n experts (e.g. teachers or professors), where each of them evaluates each of m subjects (such as pupils or students) on the basis of the overall of such k educational characteristics like performance, ability to work hard, inclination towards exact sciences, social activity, etc. It is assumed that the i -th ($i = \overline{1, n}$) expert evaluates the j -th ($j = \overline{1, m}$) student with a single a_{ij} ($i = \overline{1, n}; j = \overline{1, m}$), number that gives an integral characteristic of k education

parameters/characteristics, for example, the mean value $a_{ij} = \frac{\sum_{l=1}^k a_{ij}^{\{l\}}}{k}$, where $a_{ij}^{\{k\}}$ is a numerical score (may be not a natural number) of i -th ($i = \overline{1, n}$) expert on l -th ($l = \overline{1, k}$) educational characteristic on the pedagogical characteristics of the j -th ($j = \overline{1, m}$) student. It is required to

- rank students in accordance with the overall final grades;
- determine the degree of "objectivity" of each examiner, considering the grades being put to each student;
- rank the experts themselves by both degrees of "objectivity" and "coherence";
- identify the influence of "coherence" degree of experts evaluations on the true ratings of students.

Remark 1. In general, experts can evaluate students on each of the k educational parameters/characteristics. Then, obviously, instead of one matrix of expert assessments, which is available in the considered problem, we shall get exactly n matrices $A_i = \{a_{ij}^{\{l\}}\}_{j=1, m}^{l=1, k}$ ($i = \overline{1, n}$): where each i -th ($i = \overline{1, n}$) expert has its own grades matrix, which elements consist of the grades, put by this expert to all students by all the educational parameters/characteristics. It is obvious that the problem considered in the present paper is a particular case of this general problem. However, the mathematical model developed below as well as the subsequent mathematical calculations could be generalized also for this common problem using the same approach and same ideas, which are outlined below.

In order to construct a mathematical model of the above mentioned problem, let us introduce the following designation:

– a column vector $x = (x_1, \dots, x_m)^T$ of dimension $m \times 1$ denotes the required final grades of students, where the x_j coordinate of this vector shows the true rating of the j -th ($j = \overline{1, m}$) student; Hence, the vector x means the required ranking of students based on experts evaluation results;

–as the w_i ($i = \overline{1, n}$) there is designated the required degree of "objectivity" of the i -th ($i = \overline{1, n}$) expert. Obviously, that w_i is inversely proportional (with proportionality coefficient p) to the grades divergence of i -the expert, put to all m students, in comparison to the grades of the other commission experts, put to all m students: the lower is the w_i value the greater is the difference between grades of the i -the expert from the rest of the ratings inside the expert group;

–a column vector $w = (w_1, \dots, w_n)^T$ of dimension $n \times 1$ denotes the required degree of "coherence influence" of experts;

–the number $w_{\max} \stackrel{\text{def}}{=} \max_{i=\overline{1, n}} \{w_i\}$ means the highest possible degree of "objectivity" of expert evaluation;

– a parameter ss (sensitivity switch) denotes the sensitivity coefficient of the model to the "coherence" of experts: at $ss = 0$ there should be obtained using which the DM makes a decision on the ratings of students by reducing the assessments of all the experts together and not taking into account the correlation of experts opinions; increasing the value of sensitivity coefficient ss of the model there should be increased the extent to which correlated opinions of experts in making final decisions regarding the true ratings of students is taken into account.

Now, using the introduced designations, we can start constructing the required mathematical model. First of all, let us note that some items/grades a_{ij} ($i = \overline{1, n}; j = \overline{1, m}$) of the grades matrix A can be equal to zero or even negative. Just like in the theory of zero-sum matrix game elements of the payoff array are overridden by the positive elements, in the considered problem, without loss of generality, we will also require that the elements of the grades matrix A were positive. This can always be achieved by increasing each element of this

matrix by the same number, for example, the number of $\left| \min_{\substack{i=\overline{1, n} \\ j=\overline{1, m}}} \{a_{ij}\} \right| + 1$. It is obvious that from the

mathematical point of view resulting matrix is equivalent to the original grades matrix. Therefore, further we will initially assume that $a_{ij} > 0 \quad \forall i = \overline{1, n}; j = \overline{1, m}$. Thus, let us determine the requested true rating of the j -th ($j = \overline{1, m}$) student in proportion to the aspect ratio p to the weighted sum of expert assessments a_{ij} ($i = \overline{1, n}$) with the degree of "objectivity" w_i ($i = \overline{1, n}$):

$$x_j = p \cdot \sum_{i=1}^n w_i \cdot a_{ij} \quad (\forall j = \overline{1, m}). \quad (1)$$

Further, from the meaning of introduced w_i ($i = \overline{1, n}$), w_{\max} and ss it follows that the difference $w_{\max} - w_i$, which characterizes the deviation of the degree of "objectivity" of the i -the expert from the largest possible (i.e. ideal objectivity from the available) degree of "objectivity" of experts evaluation, and the sum $\sum_{j=1}^m \left| a_{ij} - p \cdot \sum_{l=1}^n w_l \cdot a_{lj} \right|$, which characterizes the amount of accumulated discrepancy between the true ratings of students and corresponding grades of the i -th expert, should be proportional to the coefficient of proportionality ss , which is, as it was prior mentioned, the sensitivity coefficient of the model to the "coherence" degree of experts:

$$w_{\max} - w_i = ss \cdot \sum_{j=1}^m \left| a_{ij} - p \cdot \sum_{l=1}^n w_l \cdot a_{lj} \right|. \quad (2)$$

Combining (1) and (2) gives us the required mathematical model that is continuously dependent on the parameter ss :

$$\begin{cases} x_j = p \cdot \sum_{i=1}^n w_i \cdot a_{ij} \quad \forall j = \overline{1, m}; \\ w_i = w_{\max} - ss \cdot \sum_{j=1}^m |a_{ij} - x_j| \quad \forall i = \overline{1, n}. \end{cases} \quad (3)$$

DEVELOPMENT OF AN ITERATIVE ALGORITHM FOR SOLVING MATHEMATICAL MODEL (3)

In order to solve the constructed model (3), first let us rewrite it in a compact matrix form. For this purpose we will introduce the matrix of residuals X_{Residual} of dimension $n \times m$, defined as $X_{\text{Residual}} \stackrel{\text{def}}{=} \left\{ |a_{ij} - x_j| \right\}_{i=1, n}^{j=1, m}$.

Furthermore, we will introduce a constant column vector $W_{\max} \stackrel{\text{def}}{=} \left(\underbrace{w_{\max}, \dots, w_{\max}}_n \right)^T$ of dimension $n \times 1$. Then the

model (3) tolerates the following matrix form with parameter ss :

$$\begin{cases} x = p \cdot A^T w; \\ w = W_{\max} - ss \cdot X_{\text{Residual}} I, \end{cases} \quad (4)$$

where I denotes the unit column vector of dimension $m \times 1$.

In the model (4) the unknown values are the vectors x and w , to find which we offer the following iterative process:

$$\begin{cases} x^0 = I; \quad w^0 = W_{\max}; \\ x^{l+1} = p \cdot \frac{A^T w^l}{\sum_{i=1}^m \left| \sum_{j=1}^n a_{ji} \cdot w_j^l \right|} \quad \forall l = 0, 1, \dots; \\ w^{l+1} = W_{\max} - ss \cdot X_{\text{Residual}}^{l+1} I \quad \forall l = 0, 1, \dots; \end{cases} \quad (5)$$

Convergence of the iterative process (5) at $ss = 0$ is obvious. Due to the fact that all the discrete functions that are involved in the iterative process, are continuous functions on the parameter ss , it is easy to see that the convergence of the iterative process (5) for values of parameter $ss \in [0, 1]$ is provided unconditionally. Questions concerning the stability and convergence rate of iterative process (5), fall beyond the scope of this paper.

Remark 2. As it can be seen from the model (4) and the algorithm (5), components of vector x , which are the requested real ratings of students are determined up to the constant factor $p > 0$, and this factor can be chosen arbitrarily, for example it can be equal to the number of students, i.e. $p = m$. From the model (4) and the iterative algorithm (5) it also can be noticed that the components of vector w , which are the required experts "objectivity" level, reflecting the degree of each expert ratings consistency with the ratings of other

$(n-1)$ experts, dependent on the constant parameter $ss \in [0,1]$, and this dependence is more complicated, than the dependence of the vector x on the factor p . The parameter $ss \in [0,1]$ can also be chosen arbitrarily, but it should ensure the satisfaction of the following condition $w_i^{l+1} > 0 \quad \forall i = \overline{1, n}$ at $\forall l = 0, 1, 2, \dots$.

NUMERICAL EXPERIMENT

As a numerical calculation, we consider the following computational experiment: an expert committee consisting of 5 professors acting as experts should assess the level of training of 14 last year undergraduate students in a 10-point scale. Students are wishing to start their Masters studies next year with a partial or full exemption from payment. Below is a matrix of the expert evaluation:

Table 1: Students' grades put by experts

EXPERTS	STUDENTS													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	3	6	8	2	6	0	0	4	3	2	8	8	4	5
2	9	10	4	4	1	3	4	6	7	7	7	3	1	8
3	5	3	7	8	3	5	6	4	1	5	3	6	8	4
4	4	12	3	5	5	5	8	10	7	7	11	8	9	9
5	5	6	7	8	9	10	11	12	7	6	5	4	3	1

Application of the mathematical models (4) and algorithm (5) to this computational experiment for the parameters values $p = 5$, $ss \in \{0.5; 1\}$ provides the following results (stop of the iterative process is carried out

by the condition $\|x^{l+1} - x^l\| \stackrel{def}{=} \sqrt{\sum_{j=1}^m (x_j^{l+1} - x_j^l)^2} \leq \varepsilon$, where $\varepsilon > 0$ is a given accuracy):

- 1) when the sensitivity coefficient $ss = 0.5$ we find that
 - "true" student ratings are as follows:

Table 2: "True" student grades for the sensitivity coefficient 0.5

Students	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ratings of Students	6.5	9.3	7.3	6.8	6	5.8	7.3	9	6.3	6.8	8.5	7.3	6.3	6.8

- "objectivity" ratings of experts are:

Table 3: "Objectivity" ratings of experts.

Experts	Objectivity Degrees of Experts
1	978
2	982
3	981
4	988
5	981

2) when the sensitivity coefficient $s_s = 1$ we find that

- "true" student ratings are as follows:

Table 4: "True" student grades for the sensitivity coefficient 1

Students	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ratings of Students	6.4	9.4	6.9	6.7	5.9	5.8	7.4	9.1	6.4	6.9	8.6	7.2	6.4	6.7

- "objectivity" ratings of experts are:

Table 5: "Objectivity" ratings of experts.

Experts	Objectivity Degrees of Experts
1	545
2	639
3	620
4	775
5	624

The program code of the algorithm (5), written using applied calculations package MathCAD 14.0, has the following form:

```
t := 1
f(t1,t) := 1
counter0 := 0
```

```

EE(ε , p , ss , w_max, A) :=
  E1 ← matrix(rows(A), 1, f)
  E2 ← matrix(cols(A), 1, f)
  x ← E2
  w ← w_maxE1
  error ← ∞
  while error > ε
    xprev ← x
    x ←  $\frac{p}{\sum_{i=1}^{\text{cols}(A)} \left| \sum_{j=1}^{\text{rows}(A)} (A_{j,i} \cdot w_j) \right|} \cdot A^T \cdot w$ 
    for i ∈ 1..rows(A)
      for j ∈ 1..cols(A)
        Xresidi,j ← |Ai,j - xj|
    w ← w_maxE1 - ss · Xresid · E2
    error ←  $\sqrt{\sum_{j=1}^{\text{cols}(A)} (x_j - xprev_j)^2}$ 
  (x)
  (w)
  
```

```

EE2 := [
  for sss ∈ 0, 0.01.. 1
    flag ← 0
    ALLRES ← EE(0.01, 100, sss , 100, A)
    WRES ← ALLRES2
    for z ∈ 1..rows(A)
      flag ← 1 if WRESz < 0
    if flag = 0
      counter0 ← counter0 + 1
      Rcounter0 ←  $\begin{pmatrix} \text{counter0} \\ \text{sss} \\ \text{ALLRES} \end{pmatrix}$ 
  ]
  R
  
```

Value of Counter: COUNTER=
Sensitivity Coefficient of Modelss, ss=
Desired Objectivity Degrees of Experts, w=
Desired Solution, X=

Counter := length (EE2) k := 1..Counter

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