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On the Fatou Components of Semiconjugated Transcendental Entire functions

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Abstract. In this paper, we show that if f and h are two transcendental entire functions which are semiconjugated by an entire map g, where f has no Siegel disk and no Baker domain, then $g(F(f)) \subset F(h)$.

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1. Introduction

Let f be a non-constant entire function and let f^n, $n \in \mathbb{N}$, denote the nth iterate of f. The set of normality $F(f)$ is defined to be the set of points $z \in \mathbb{C}$, such that the sequence $\{f^n\}$ is normal in some neighbourhood of z, and $J = J(f) = \mathbb{C} \setminus F(f)$. $F(f)$ and $J(f)$ are called the Fatou set and the Julia set of f, respectively. These sets play a fundamental role in complex dynamics, (see [2, 5, 7, 9]) for an introduction to this theory. It is well known that $F(f)$ is open (which may be empty) and $J(f)$ is closed. Let f and h be two entire functions and let $g : \mathbb{C} \to \mathbb{C}$ be a non-constant continuous function such that

$$g \circ f = h \circ g. \tag{1}$$

Then we say that f and h are semiconjugated by g and we call g a semiconjugacy [4]. Further, if $f = h$, then (1) reduces to $g \circ f = f \circ g$ and in this case we say that f and g are permutable entire functions. If U is a component of $F(f)$, then $f(U)$ lies in some component V of $F(f)$. In fact, $V \setminus f(U)$ is either empty or a single point [6]. By a slight abuse of language, we write $V = f(U)$ even when $V \setminus f(U)$ is a singleton. If all $f^n(U)$ with $n \in \mathbb{N}$ are different components of $F(f)$, then U is called a wandering domain.

The behaviour of f^n in the periodic component is fairly well understood. In fact, if U is a periodic component of period p, then we have one of the following five possibilities [3].

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i) U **is an immediate attractive basin of** z_0**:** U contains an attracting periodic point z_0 of period p and $f^{np}(z) \to z_0$ as $n \to \infty$ for every $z \in U$.

ii) U **is a Leau domain:** ∂U contains a periodic point z_0 of period p and $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$. Further, $(f^p)'(z_0) = 1$, if $z_0 \in \mathbb{C}$ and if $z_0 = \infty$, then $(g^p)'(0) = 1$, where $g(z) = 1/f(\frac{1}{z})$.

iii) U **is a Siegel disk:** There exists an analytic homeomorphism $\varphi : U \to D$, where D is the unit disk such that $\varphi\{f^p[\varphi^{-1}(z)]\} = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus Q$.

iv) U **is a Herman ring:** There exists an analytic homeomorphism $\varphi : U \to A$, where A is the annulus $A = \{z : 1 < |z| < r\}, r > 1$ such that $\varphi\{f^p[\varphi^{-1}(z)]\} = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R} \setminus Q$.

v) U **is a Baker domain:** There exists $z_0 \in \partial U$, such that $f^{np}(z) \to z_0$ as $n \to \infty$ for all $z \in U$, but $f^p(z_0)$ is not defined. In 1984, [1] Baker proved that if f, g are two permutable entire functions with
$f = g + K$, where K is some constant, then $J(f) = J(g)$. In this paper, we prove that if f and h are two entire maps which are semiconjugated by an entire map g, where f has no Siegel disk and no Baker domain and f and g satisfy one of the following conditions

(a) there exist a non-constant polynomial p and an entire map k such that $p(f(z)) = k(g(z))$;

(b) $f = g + K$, where K is some constant, $K \neq 0$;

(c) $f = gK$, where K is some constant, $K \neq 1$, and $K > 1/e$,

then $g(F(f)) \subset F(h)$.

2. Main Results and Their Proofs

Theorem 1. *Let f and h be two transcendental entire maps semiconjugated by an entire map g, where f has no Siegel disk and no Baker domain. If there exist a non-constant polynomial p and an entire map k such that*

$$p(f(z)) = k(g(z)),$$

then

$$g(F(f)) \subset F(h).$$

Theorem 2. *Let f and h be two transcendental entire maps which are semiconjugated by an entire map g, where f has no Siegel disk and no Baker domain, and if $f = g + K$, where K is some constant, $K \neq 0$, then $g(F(f)) \subset F(h)$.*

Theorem 3. *Let f and h be two transcendental entire maps which are semiconjugated by an entire map g, where f has no Siegel disk and no Baker domain, and if $f = gK$, where K is some constant $\neq 1$, and $K > 1/e$, then $g(F(f)) \subset F(h)$.*

In order to prove the main results, we need the following lemmas.

Lemma 1. *Suppose that W is a wandering domain of a transcendental entire function f. Then for any compact subset Ω of W, diam$[f^n(\Omega)] \to 0$ as $n \to \infty$.*

Proof. The proof follows on same lines as the proof of Lemma 8.2.2 in [2]. For completeness, we include it here. Suppose that the result is false. Then there is some compact set K of W, some $\delta > 0$, some increasing sequence n_j of positive integers such that for $j = 1, 2, \cdots$, we have that diam$[f^n(\Omega)] \geq \varepsilon$. As f^n is normal in W, there is a subsequence of f^{n_j} which converges locally uniformly on W to some analytic function g. For convenience, we relabel this subsequence and so assume that f^{n_j} itself has this property. If g is constant, with value α say, on W, then f^{n_j} converges uniformly to α on K and so for large j, f^{n_j} lies in an $\varepsilon/3$-neighbourhood of α. This contradicts the fact that diam$[f^n(\Omega)] \geq \varepsilon$. Thus we conclude that f^{n_j} converges to a non-constant g locally uniformly on W. Now take a point ζ in W such that $g'(\zeta) \neq 0$ and draw a small circle C with centre ζ such that its interior D lies in W, and which is such that $g(z) \neq g(\zeta)$, where z is on C. Then for $j \geq j_0$ say,

$$|f^{n_j} - g(z)| < \inf_{w \in C} |g(w) - g(\zeta)| < |g(z) - g(\zeta)|$$

on C, so by Rouche's Theorem, $f^{n_j}(D)$ contains a point $g(\zeta)$. A contradiction to the fact that W is a wandering domain. This completes the proof.

Lemma 2 ([1])**.** *If $\alpha \in J(f)$, if N is an open neighbourhood of α, and if Ω is a compact set which does not contain a Fatou-exceptional point of f, then there exists n_0 such that $f^n(N) \supset \Omega$ for all $n > n_0$.*

Lemma 3. *Let f and h be two transcendental entire functions, where f has no Siegel disk and no Baker domain and let g be a continuous and open map such that $g \circ f = h \circ g$, then if there is a subsequence f^{n_k}, with $n_k \to \infty$ which has a finite limit, say ξ in the component U of $F(f)$ which contains α, then $g(\alpha) \in F(h)$.*

Proof. Suppose $g(\alpha) \notin F(h)$. Then $g(\alpha) \in J(h)$. Let $\alpha \in F(f)$. Then there exists an open neighbourhood U of α such that $\overline{U} \subset F(f)$. Since f^{n_k} has a finite limit function, say ξ in U such that all $f^{n_k}(U)$ lie in a single compact set, say K_0 on which g is uniformly continuous. Now by Lemma 1, we have $g(f^{n_k}(U)) = h^{n_k}(g(U))$ has small diameter for all large n_k. Again $g(\alpha) \in J(h)$ and $g(K_0)$ is a compact set, which does not contains Fatou exceptional point of h, then by Lemma 2, there exists n_0 such that for $n > n_0$,

$$h^{n_k}(g(U)) \supset g(K_0).$$

Now

$$h^{n_k}(g(U)) = g(f^{n_k}(U)). \tag{2}$$

Choose a non-Fatou exceptional value $\eta \notin g(K_0)$. Then for any point $\delta \in U$, $g(\delta) \in g(U)$ and $h^{n_{k'}}(g(\delta)) = \eta$, where $n_{k'}$ is fixed. Thus $f^{n_{k'}}(\delta) \in f^{n_{k'}}(U) \subset K_0$, and so

$$g(f^{n_{k'}}(\delta)) \subset g(K_0),$$

which implies that $g(f^{n_{k'}}(\delta)) \neq \eta$, a contradiction to (2). Hence $g(\alpha) \in F(h)$.

Proof. [Theorem 1] Let $\alpha \in F(f)$. Then there exists a neighbourhood U of α such that $\overline{U} \subset F(f)$. By Lemma 3, we only need to consider the case $f^n \to \infty$ in U. Let $M = \max_{|w|=1} |k(w)|$. Since p is a non-constant polynomial, there exist a positive constant K such that $|p(z)| > M+1$ when $|z| > K$. Since $f^n \to \infty$ in U as $n \to \infty$, there exists n_0 such that for $n > n_0$ and $z \in U$, $|f^n(z)| > K$.
Thus $|f(z)| > K$ for every $z \in f^n(U)$ $(n > n_0)$. Now if $g(\alpha) \notin F(h)$, then for arbitrary large n, by expanding properties of Julia sets [see 8, p. 75], the sequence $\{h^n\}$ takes all values in $g(U)$ with at most one exception. Thus there exists $t = g(\xi), \xi \in U$, such that for some $m > n_0$,

$$1 > |h^m(t)| = |h^m(g(\xi))| = |g(f^m(\xi))|.$$

Thus $\delta = f^m(\xi) \in f^m(U)$, and so $|f(\delta)| > K$, and $|g(\delta)| < 1$. Hence
$M+1 < |p(f(\delta))| = |k(g(\delta))| \leq M$, which is a contradiction. Thus we have that $g(\alpha) \in F(h)$. Hence $g(F(f)) \subset F(h)$.

Proof. [Theorem 2] Let $\alpha \in F(f)$, and a neighbourhood U of α such that $\overline{U} \subset F(f)$. Then by Lemma 3, we only need to consider the case $f^n \to \infty$ in U. Take a constant A such that $A > |K| + 1$. There exists n_0 such that $|f^n| > A$ in U for $n > n_0$, and hence $|f^n| > A$ for $z \in f^n(U), n > n_0$. To complete the proof, let $g(\alpha) \notin F(h)$. Then for arbitrary large n, the sequence $\{h^n\}$ takes all values in $g(U)$ with atmost one exception [see 8, p. 75]. Therefore, there exists $t = g(\xi), ; \xi \in U$, such that for some $m > n_0$,

$$1 > |h^m(t)| = |h^m(g(\xi))| = |g(f^m(\xi))|,$$

which implies that $|g(f^m(\xi))| < 1$. Thus $\eta = f^m(\xi) \in f^m(U)$ and
$|g(\eta)| = |g(f^m(\xi))| < 1$. Since $|f^n(\xi)| > A$ for all $\xi \in U$ and for all $n > n_0$, and so $|f(\eta)| = |f(f^m(\xi))| = |f^{m+1}(\xi)| > A$ for $\xi \in U$. Also

$$|K| = |f(\eta) - g(\eta)| > |f(\eta)| - |g(\eta)| > A - 1,$$

a contradiction for $A > |K| + 1$.
Thus we have that $g(\alpha) \in F(h)$. Hence $g(F(f)) \subset F(h)$. Next example illustrates that there exist transcendental entire maps f, g and h such that $f \neq h$ and satisfying the conditions in Theorem 2.

Example 1. *Let $f(z) = e^z + K$, $K > 0$, $g(z) = e^z$ and $h(z) = e^{z+K}$. Then $f(z) = g(z) + K$ and $(g \circ f)(z) = (h \circ g)(z)$. Also $f(z) = e^z + K \neq h(z)$.*

Proof. [Theorem 3] Take $\alpha \in F(f)$, and a neighbourhood U of α such that $\overline{U} \subset F(f)$. Then by Lemma 3, we shall consider only the case when $f^n \to \infty$ in U. Take a constant A such that $A > |K|$. Then there exist n_0 such that for $n > n_0, |f^n| > A$ holds in U, and hence $|f(z)| > A$ for $z \in f^n(U), (n > n_0)$. To complete the proof, let $g(\alpha) \notin F(h)$. Then by expanding properties of

Julia sets [see 8, p. 75] for arbitrary large n, the sequence $\{h^n\}$ takes all values in $g(U)$, with atmost one exception. Therefore, there exist $t = g(\xi), \xi \in U$, such that for some $m > n_0$,

$$1 > |h^m(t)| = |h^m(g(\xi))| = |g(f^m(\xi))|,$$

and so $|g(f^m(\xi))| < 1$ Thus $\eta = f^m(\eta) \in f^m(U)$ and $|g(\eta)| = |g(f^m(\xi))| < 1$. Also, since $|f^n| > A$ in U for $n > n_0$, and so

$$|f(\eta)| = |f(f^m(\xi))| = |f^{m+1}(\xi)| > A \text{ for } \xi \in U.$$

Now $|f(\eta)| > A$, and so $|Kg(\eta)| > A$ for $\eta \in f^m(U)$, which implies that $|g(\eta)| > A/K$. Thus $A/K < |g(\eta)| < 1$, a contradiction for $A > |K|$. Thus we have that $g(\alpha) \in F(h)$. Hence $g(F(f)) \subset F(h)$. Again we provide an example which illustrates that there exist transcendental entire maps f, g and h such that $f \neq h$ and satisfying the conditions in Theorem 3.

Example 2. *Let $f(z) = Ke^z, g(z) = e^z$ and $h(z) = e^{Kz}$ $(K \neq 1)$, and $K > 1/e$ be three transcendental entire functions. Clearly, $f(z) \neq h(z)$ and $g \circ f = h \circ g$.*

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