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The analysis of an unfair contest model

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Abstract

The contests are usually "unfair" in the sense that outperforming all rivals may not be enough to be the winner, because some contestants are favored by the allocation rule, while others are handicapped. However, the roles of the contestants can have a transform. In other words, the contestant who is favored by the allocation rule at beginning of the contest is possibly handicapped with the passage of time. An unfair, two-player discriminatory contest (all pay auction) where the roles of the contestants have a transform, is analyzed. We characterize equilibrium strategies and provide closed form solutions to unfair contests.

Mathematics Subject Classification: 60G35, 60G20

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1 Introduction

It is well-known that unfair contests are strategically equivalent to all-pay auctions. In all-pay auctions, each bidder has to pay his bid regardless of whether he wins the auctions or not. In unfair contest, each contestant bears his effort costs no matter if he wins or not. Baye et al.[1] provided a comprehensive analysis of the all-pay auction under complete information. Lizzer and Persico [3] analyzed conditions that there existed a unique pure strategy equilibrium in general auction games, including the all-pay auction. Maskin and Riley [4] considered auctions in which contestants are asymmetric in the sense that the valuations of each bidder were drawn from different distributions. This also implies that the bidder with the highest valuation does no longer win the object with certainty. Feess, Muehlheusser, and Walzla [2] extended several known approaches for determining the equilibrium bidding strategies from a system of differential equations for analyzing contests with handicaps, and provided a closed form solution to the equilibrium strategies.

Our paper is related to Feess, Muehlheusser, and Walzla [2] as they analyze the unfair contests. We extend and modify their contest model for determining the equilibrium bidding strategies. Exactly, we consider a two-player discriminatory contest (all pay auction) where contestants have private information concerning the value of the prize to them and their roles can have a transform with the passage of time. We define the discriminatory level l is the function of time t while l is a constant in [2]. For any time t, we show that there exists a unique pure strategy Bayesian Nash Equilibrium (BNE) and provide a closed form solution for the equilibrium strategies.

The remainder of the paper is organized as follows. In Section 2, the basic model is presented. We analyze the equilibrium and derive our main theoretical results for the unfair contests in Section 3. A conclusion is given in Section 4.

2 The Model

We consider an unfair contest (all-pay auction) where two risk-neutral contestants indexed i=1,2 compete for a single prize to be awarded. Each contestant has valuation $v_i \in [0,1]$ for the prize which is drawn from a common distribution function $F(v) \in C^1$ satisfying $F(0) = 0$ where the density function $F'(v)$ is positively valued. The realization of v_i (contestant i's "type") is private information to contestant i. Each contestant can influence his chances of winning the prize by exerting effort which is denoted by b_i. In what follows, we analyze equilibria in which the effort strategy of contestant is a function of his type, i.e., $b_i : [0,1] \to R_0^+$.

The specific feature of this contest is the allocation rule. Denoting the identity of the winner by $W = 1$ or $W = 2$ and the discriminatory level be defined by $l(t)$ which is the function of time t, $l(t) \in (0, +\infty)$, we have

$$W = 1 \Leftrightarrow b_1 > l(t) \cdot b_2; W = 2 \Leftrightarrow b_2 > \frac{1}{l(t)} \cdot b_1, \tag{1}$$

where a coin is flipped in the case where $b_1 = l(t) \cdot b_2$ holds so that each contestant wins with probability $\frac{1}{2}$. Thus, contestant 1 wins the contest only if he exerts at least $l(t)$-times as much effort as contestant 2, while contestant 2 wins if he exerts at least $\frac{1}{l(t)}$-times as much effort as contestant 1. We define the value of $l(t)$ is

$$l(t) \begin{cases} \in (0,1), & t \in (0, \alpha), \\ = 1, & t = \alpha, \\ \in (1, +\infty), & t \in (\alpha, +\infty), \end{cases} \tag{2}$$

where $\alpha \in (0, +\infty)$, and the value of α may be different due to the different forms of function $l(t)$. For simplifying the following analysis, we assume $l(t)$ is a continuous and strictly increasing function of t. When $l(t) \in (1, +\infty), (t \in (\alpha, +\infty))$, we can find that contestants 1 and 2 will be referred to as the "handicapped" and the "favored" contestant, respectively. However, when $l(t) \in (0, 1), (t \in (0, \alpha))$, contestant 1 will be referred to as the "favored" contestant and the contestant 2 will be the "handicapped". We will refer to the case $l(t) = 1, (t = \alpha)$ as a "fair" contest and the roles of the contestants begin to have a transform at that time. In other words, from the time $t = \alpha$ where $l(t) = 1$, the role of the contestant 1 will be transferred from "handicapped" to "favored" with the passage of time, and the role of the contestant 2 will have a opposite transform.

We find that the payoffs at time t for given effort levels b_1 and b_2 are

$$\pi_1(b_1, b_2, v_1; l(t)) = \begin{cases} v_1 - b_1, & b_1 > l(t) \cdot b_2, \\ \dfrac{1}{2}v_1 - b_1, & b_1 = l(t) \cdot b_2, \\ -b_1, & b_1 < l(t) \cdot b_2, \end{cases} \tag{3}$$

and

$$\pi_2(b_1, b_2, v_2; l(t)) = \begin{cases} v_2 - b_2, & b_2 > \dfrac{1}{l(t)} \cdot b_1, \\ \dfrac{1}{2}v_2 - b_2, & b_2 = \dfrac{1}{l(t)} \cdot b_1, \\ -b_2, & b_2 < \dfrac{1}{l(t)} \cdot b_1. \end{cases} \tag{4}$$

Therefore, at time t, the expected payoffs are

$$\Pi_1(\cdot) = v_1 \cdot Pr(b_1 > l(t) \cdot b_2(v_2)) - b_1, \tag{5}$$

$$\Pi_2(\cdot) = v_2 \cdot Pr(b_2 > \frac{1}{l(t)} \cdot b_1(v_1)) - b_2. \tag{6}$$

3 Equilibrium analysis

Provided that $l(t)$ is a function of t, for any identified time $t = t_0$, $t_0 \in (0, +\infty)$, the unfair contest we analyze is a static game with incomplete information and the equilibrium concept is BNE. A vector of effort level $(b_1^*(v_1), b_2^*(v_2))$ is a BNE if the following set of conditions holds

$$\Pi_i(b_i^*(v_i), b_j^*(v_j); l(t)) \geq \Pi_i(b_i, b_j^*(v_j); l(t)) \tag{7}$$

for all $v_i \in [0, 1]$ and $b_i \in R_0^+$. In equilibrium, no contestant can increase his expected payoff by choosing an effort strategy other than b_i^*, given that the opponent adheres to his equilibrium strategy. We cite a definition given in [1].

Definition 3.1 *Consider a set* $A \subset \Re$ *and a function* z*:* $A \to \Re$*. Then define* $D_z := \{a \in A : z(a) \in \Re^+\}$.

Lemma 3.2 *(Equilibrium effort strategies)* $b_i^* : D_{b_i} \to (0, b_i(1))$ *where* $i = 1, 2$ *is an increasing bijection between non-empty subsets of [0,1] and differentiable almost everywhere.*

Proof: Firstly, we show that the structure of the payoff function induces non-decreasing strategies. For any $v_i', v_i \in [0,1]$ with $v_i' > v_i$, From Eq.(7), the incentive compatibility requires

$$\Pi_i(b_i(v_i), v_i, l(t)) \geq \Pi_i(b_i(v_i'), v_i, l(t)),$$

$$\Pi_i(b_i(v_i'), v_i', l(t)) \geq \Pi_i(b_i(v_i), v_i', l(t)).$$

Taking the sum of both conditions yields

$$\Pi_i(b_i(v_i'), v_i', l(t)) - \Pi_i(b_i(v_i'), v_i, l(t)) \geq \Pi_i(b_i(v_i), v_i', l(t)) - \Pi_i(b_i(v_i), v_i, l(t)).$$

From (5) and (6), we get

$$(v_1' - v_1)Pr(b_1(v_1') > l(t) \cdot b_2) \geq (v_1' - v_1)Pr(b_1(v_1) > l(t) \cdot b_2),$$

$$(v_2' - v_2)Pr(b_2(v_2') > \frac{1}{l(t)} \cdot b_1) \geq (v_2' - v_2)Pr(b_2(v_2) > \frac{1}{l(t)} \cdot b_1).$$

This only holds if $b_i(v_i') \geq b_i(v_i)$ which proves monotonicity.

Now we prove continuity by contradiction. We firstly consider $t \in (\alpha, +\infty)$ where $l(t) \in (1, +\infty)$, assuming that b_1 is not continuous at $v_1 \in (0,1)$. Stated differently $b_1(x) > \lim_{\varepsilon \to 0} b_1(v_1 - \varepsilon) \equiv b_{-1}(v_1)$, which means that contestant 2 will not choose any effort level that $b_2 \in (b_{-1}(v_1)/l(t), b_1(v_1)/l(t))$ as he can always reduce costs while the probability of winning the contest remains unchanged. Anticipating this, there is no reason for contestant 1 to increase effort from $b_{-1}(v_1)$ to $b_1(v_1)$. Hence, we end up with a contradiction. Furthermore, as $F'(v) \neq 0 \ \forall v \neq 0$, this result holds for all $v_1 \in (0,1]$.

Considering $t \in (\alpha, +\infty)$ where $l(t) \in (1, +\infty)$, we assume that $b_i(v_i)$ is not strictly increasing on D_{b_i}. Therefore, we can find an interval $I \subseteq (0,1]$ of finite length with $b_i(v_i) \equiv \underline{b} > 0, \forall v_i \in I$. Given such a strategy profile of contestant i, contestant j maximizes his expected payoff as given in(4) or (5). We set $i = 1$ and $j = 2$. Now assuming that contestant 2 chooses $(\underline{b} - \varepsilon)/l(t)$ for some valuation v_2, his pay-off is

$$v_2 Pr(\underline{b} - \varepsilon > b_1) - (\underline{b} - \varepsilon)/l(t).$$

If contestant 2 chooses $(\underline{b} + \varepsilon)/l(t)$, his expected pay-off is

$$v_2 Pr(\underline{b} + \varepsilon > b_1) - (\underline{b} + \varepsilon)/l(t).$$

Contestant 2 profits from such a deviation as can be seen when $\varepsilon \to 0$

$$\begin{aligned}&\lim_{\varepsilon\to 0}(v_2 Pr(b_1 > \underline{b}+\varepsilon) - \frac{\underline{b}+\varepsilon}{l(t)} - (v_2 Pr(b_1 > \underline{b}-\varepsilon) - \frac{\underline{b}-\varepsilon}{l(t)}))\\ &= \lim_{\varepsilon\to 0}(v_2 \cdot Pr(b_1 \in [\underline{b}-\varepsilon, \underline{b}+\varepsilon]) - 2\frac{\varepsilon}{l(t)})\\ &= v_2 Pr(b_1 = \underline{b}) > 0.\end{aligned}$$

Therefore contestant 2 will always exert effort slightly above $\underline{b}/l(t)$ instead of slightly below. This contradicts continuity. Analogously, a gap in effort strategies of contestant 1 can be deduced from a plateau in contestant 2's equilibrium strategy. This proves strict monotonicity on D_{b_i}. Therefore effort strategies are differentiable almost everywhere and a bijection from D_{b_i} onto $(0, b_i(1)]$.

If $t \in (0,\alpha)$ where $l(t) \in (0,1)$, we identify $g(t) = \frac{1}{l(t)}$ and can find that $W = 2 \Leftrightarrow b_2 > g(t) \cdot b_1; W = 1 \Leftrightarrow b_1 > \frac{1}{g(t)} \cdot b_2$, whose form is identical to the form of Eq.(1). Thus the Lemma 1 is proved when $t \in (0,\alpha)$

If $l(t) \in (1,+\infty)$, the contestant 1 is "handicapped" and the contestant 2 is "favored". $W = 1 \Leftrightarrow b_1 > l(t) \cdot b_2; W = 2 \Leftrightarrow b_2 > \frac{1}{l(t)} \cdot b_1$, where the W means the winner. When $l(t) \in (0,1)$, the contestant 1 is "favored" and the contestant 2 is "handicapped". At this time, we define $g(t) = \frac{1}{l(t)}$. Then we find that $W = 2 \Leftrightarrow b_2 > g(t) \cdot b_1; W = 1 \Leftrightarrow b_1 > \frac{1}{g(t)} \cdot b_2$, whose form is identical to the form of Eq.(1). By the same approach used before, we conclude that this Lemma is correct if $l(t) \in (1,+\infty)$. It completes the proof.

We will show that an equilibrium is unique when it exists. Lemma 3.2 ensures existence of the inverse mapping $\rho_i : (0, b_i^*(1)) \to D_{b_i}$, and $\rho_i(b) = b_i^{-1}(b)$. The maximization problem for contestant 1 when contestant 2 is playing some strategy $b_2(v_2)$ is given by

$$\max_{b_1} v_1 \cdot Pr(b_1 > l(t) \cdot b_2(v_2)) - b_1 = \max_{b_1} v_1 \cdot F(\rho_2(\frac{b_1}{l(t)})) - b_1, \tag{8}$$

while for contestant 2, when contestant 1 is playing strategy $b_1(v_1)$, we have

$$\max_{b_2} v_2 \cdot Pr(b_2 > \frac{1}{l(t)} \cdot b_1(v_1)) - b_2 = \max_{b_2} v_2 \cdot F(\rho_1(l(t) \cdot b_2)) - b_2. \tag{9}$$

The following system of first order differential equations gives the first order conditions of these maximization problems

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{l(t)})) \cdot \rho_2'(\frac{b_1(v_1)}{l(t)}) \cdot \frac{1}{l(t)} = 1, \tag{10}$$

$$v_2 \cdot F'(\rho_1(l(t) \cdot b_2(v_2))) \cdot \rho_1'(l(t) \cdot b_2(v_2)) \cdot \ l(t) = 1. \tag{11}$$

For a given set of initial conditions, Eqs(10) and (11) determine a unique trajectory of effort strategies [3]. Namely, there is a single pair of initial conditions, such that a solution to Eqs.(10) and (11) is unique. Additionally, the initial conditions follow from the subsequent results concerning the properties of the equilibrium effort distributions $G_{i=1,2} = F(\rho_i(b_i^*))$ which is the mapping from D_{G_i} to (0,1].

Lemma 3.3 *(Equilibrium effort distribution) In any BNE, the effort distributions G_1 and G_2 have the following properties*
(i) $D_{G_1} = (0, b_1^*(1)]$ and $D_{G_2} = (0, b_2^*(1)]$ where $b_1^*(1) = l(t) \cdot b_2^*(1)$.
(ii) G_i is continuous and strictly monotone increasing for any time t, $i = 1, 2$.
(iii) If $G_i(0) > 0$, then $G_{j\neq i}(0) = 0$.
(iv) There is a single set of admissible initial conditions.

Proof: Part(i): $b_i(0) = 0$ determines the lower bound of D_{G_i}. Contestant 1 can never be better off by exerting effort excessively than $l(t) \cdot b_2^{max}$. Thus $b_1 \leq l(t) \cdot b_2^{max}$ holds. Analogously, neither will contestant 2 exert more effort than necessary to win the contest with probability 1. Thus $b_2 \leq \frac{1}{l(t)} \cdot b_1^{max}$ holds. Of course, this must also be true for b_1^{max} and b_2^{max} where $b_1^{max} \leq l(t) \cdot b_2^{max}$ and $b_2^{max} \leq \frac{1}{l(t)} \cdot b_1^{max}$. Rearranging yields

$$b_2^{max} \leq \frac{1}{l(t)} \cdot b_1^{max} \leq b_2^{max},$$

from which we can deduce that $b_2^{max} = \frac{1}{l(t)} \cdot b_1^{max}$ or $b_1^{max} = l(t) \cdot b_2^{max}$.
Part(ii): It follows from our assumptions on $F(v)$ and Lemma 3.2.
Part(iii): Suppose $G_j(0) = g > 0$, for all $v_i \in [0, 1]$, there is some positive effort level $x > 0$ for contestant i such that he is strictly better off than with choosing $b_i = 0$. If $b_i = 0$, contestant i loses when $b_j > 0$ (which happens with probability $1-g$, and wins with probability $\frac{1}{2}$ when $b_j = 0$ (which happens with probability g) so that his expected payoff is $\Pi_i = 0 \cdot (1-g) + \frac{v_i}{2} \cdot g = \frac{v_i \cdot g}{2}$. When choosing a positive effort level $x > 0$, he wins with certainty when $b_j = 0$. We have

$$\Pi_i(x, l(t)) = v_i \cdot G_j(l(t) \cdot x) - x \geq v_i \cdot g - x > v_i \cdot \frac{g}{2} = \Pi_i(0, l(t))$$

or

$$\Pi_i(x, l(t)) = v_i \cdot G_j(\frac{x}{l(t)}) - x \geq v_i \cdot g - x > v_i \cdot \frac{g}{2} = \Pi_i(0, l(t)),$$

which holds if $x < \frac{g}{2} \cdot v_i$. Thus, we arrive at the desired result.
Part(iv): As the first order conditions consist of two first order ordinary differential equations which are Lipschitz continuous for $v_i > 0$. Any set of initial

conditions ($b_i(v_i) = c_i, i = 1, 2$) determines unique trajectories $b_i(v_i)$. Using parts(i) and (iii) together with the so-called no-crossing property of equilibrium effort levels [3] implies that there is only one admissible set of initial conditions[3]. The proof is completed.

For given time t, the following analyzed method is appropriate whether $0 < l(t) < 1$ or $1 \leq l(t) < +\infty$.

Considering a bijection $k(v_1, t)$ which maps every type of contestant 1 onto the type of contestant 2 whose equilibrium effort level is $\frac{1}{l(t)}$-times as much as contestant 1's, we get

$$k(v_1, l(t)) = \rho_2(\frac{b_1^*(v_1)}{l(t)}) \tag{12}$$

Analogously, $k^{-1}(v_2, l(t)) = \rho_1(l(t) \cdot b_2^*(v_2))$ gives that type of contestant 1 who will choose $l(t)$-times as much effort as contestant 2 when his type is v_2. Note that *Definition 1*, Eq.(12) defines a bijection between the domains D_{b_1} and D_{b_2} of the two equilibrium strategies. We can rewrite the first-order conditions with $k(v_1, l(t))$ and separate the dependent and independent variables to draw a closed form solution for $k(\cdot)$.

Lemma 3.4 *Define* $H(x) = \int_x^1 \frac{F'(y)}{y} dy$ *such that* $\frac{d}{dx} H(\cdot) = -\frac{F'(x)}{x} < 0$. *Then we have*

(i). $k(v_1, l(t)) = H^{-1}(l(t) \cdot H(v_1))$ *satisfying*

$$\frac{d}{d(l(t))} k(v_1, l(t)) \begin{cases} > 0, & l(t) \in (0, 1), \\ < 0, & l(t) \in (1, +\infty), \end{cases}$$

and $k(v_1, 1) = v_1$.

(ii). $k^{-1}(v_2, t) = H^{-1}(\frac{1}{l(t)} \cdot H(v_2))$ *satisfying*

$$\frac{d}{d(l(t))} k^{-1}(v_2, l(t)) \begin{cases} < 0, & l(t) \in (0, 1), \\ > 0, & l(t) \in (1, +\infty), \end{cases}$$

and $k^{-1}(v_2, 1) = v_2$.

(iii). $\begin{cases} \lim\limits_{t \to 0} k(v_1, l(t)) = 1 \\ \lim\limits_{t \to 0} k^{-1}(v_2, l(t)) = 0 \end{cases}$ *implies that contestant 1's (contestant 2's) probability of winning tends to 1 (0) as* $t \to 0$.

(iv). $\begin{cases} \lim\limits_{t\to+\infty} k(v_1, l(t)) = 0 \\ \lim\limits_{t\to+\infty} k^{-1}(v_2, l(t)) = 1 \end{cases}$ *implies that contestant 1's (contestant 2's) probability of winning tends to 0 (1) as* $t \to +\infty$.

Proof: Using $k(v_1, l(t))$, the conditions (10) and (11) can be transformed into a set of differential equations expressed in a single variable v_1. Substituting $k(v_1)$ for v_2 in Eq.(11) yields

$$v_1 \cdot F'(\rho_2(\frac{b_1(v_1)}{l(t)})) \cdot \rho_2'(\frac{b_1(v_1)}{l(t)}) = 1, \tag{13}$$

$$k(v_1) \cdot F'(\rho_1(l(t)b_2(k(v_1)))) \cdot \rho_1'(l(t) \cdot b_2(k(v_1))) \cdot l(t) = 1. \tag{14}$$

From (13) and (14), we get

$$v_1 \cdot F'(\rho_2(\frac{b_1}{l(t)})) \cdot \rho_2'(\frac{b_1}{l(t)}) \cdot \frac{1}{l(t)} = k(v_1) \cdot F'(\rho_1(l(t) \cdot b_2(k(v_1)))) \cdot \rho_1'(l(t) \cdot b_2(k(v_1))) \cdot l(t) \tag{15}$$

Moreover, it follows from the definition of $k(\cdot)$ that

$$\frac{dk(v_1, l(t))}{dv_1} = \rho_2'(\frac{b_1(v_1)}{l(t)}) \cdot \frac{db_1(v_1)}{dv_1} \cdot \frac{1}{l(t)}. \tag{16}$$

Thus, we can rewrite Eq.(15) as

$$\begin{aligned} & v_1 \cdot F'(k(v_1, l(t))) \cdot \frac{dk(v_1 l(t))}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} \\ & = k(v_1) \cdot F'(\rho_1(l(t)b_2(\rho_2(\frac{b_1(v_1)}{l(t)})))) \cdot \rho_1'(l(t)b_2(\rho_2(\frac{b_1(v_1)}{l(t)}))) \cdot l(t) \\ & \Leftrightarrow v_1 \cdot F'(k(v_1, l(t))) \cdot \frac{dk(v_1, l(t))}{dv_1} \cdot \frac{1}{\frac{db_1(v_1)}{dv_1}} = k(v_1) \cdot F'(\rho_1(b_1)) \cdot \rho_1'(b_1) \cdot l(t) \\ & \Leftrightarrow v_1 \cdot F'(k(v_1, l(t)) \cdot \frac{dk(v_1, l(t))}{dv_1} = k(v_1) \cdot F'(v_1) \cdot \rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} \cdot l(t). \end{aligned} \tag{17}$$

Finally, as $\rho_1(b_1(v_1)) = v_1$, it follows that $\rho_1'(b_1) = \frac{dv_1}{db_1}$ which implies that $\rho_1'(b_1) \cdot \frac{db_1(v_1)}{dv_1} = 1$. Hence, we end up with a single differential equation

$$\frac{dk(v_1, l(t))}{dv_1} = \frac{l(t) \cdot k(v_1, l(t)) \cdot F'(v_1)}{v_1 \cdot F'(k(v_1, l(t)))}, \tag{18}$$

where the boundary condition $k(1, l(t)) \equiv 1$ and our assumptions on $F(v)$ guarantee a unique solution for $k(\cdot)$. Analogously, we get

$$\frac{dk^{-1}(v_2, l(t))}{dv_2} = \frac{k^{-1}(v_2, l(t)) \cdot F'(v_2)}{l(t) \cdot v_2 \cdot F'(k^{-1}(v_2, l(t)))}. \tag{19}$$

To derive a solution in closed form, we separate dependent and independent variables of differential equations (18) and (19) to yield

$$\frac{dk}{k}F'(k) = l(t)\frac{dv_1}{v_1}F'(v_1), \tag{20}$$

$$\frac{dk^{-1}}{k^{-1}}F'(k^{-1}) = \frac{dv_2}{l(t)\cdot v_2}F'(v_2). \tag{21}$$

Using $H(x) = \int_x^1 \frac{F'(y)}{y}dy$ gives rise to

$$H(k) = l(t)\cdot H(v_1), \tag{22}$$

$$H(k^{-1}) = \frac{1}{l(t)}H(v_2), \tag{23}$$

which are equivalent to

$$k(v_1, l(t)) = H^{-1}(l(t)H(v_1))$$

and

$$k^{-1}(v_2, l(t)) = H^{-1}(\frac{1}{l(t)}H(v_2)).$$

As stated in the Lemma. The contestant 2 wins when $v_1 \leq k^{-1}(v_2, l(t)$ which occurs with probability $F(k^{-1}(v_2, l(t)))$ and contestant 1 wins when $v_2 \leq k(v_1, l(t))$ which occurs with probability $F(k(v_1, l(t)))$. Therefore, the results (i),(ii),(iii) and (iv) are valid by using the definition of $k(v_1, l(t))$ and $k^{-1}(v_2, l(t))$. It completes the proof.

With the closed form solution for $k(\cdot)$ and its inverse, we obtain the main result.

Theorem 3.5 *There exists a unique pure-strategy BNE in which contestant 1 chooses*

$$b_1^*(v_1) = \int_{max\lfloor 0,k^{-1}(0,l(t))\rfloor}^{v_1} l(t)\cdot k(V, l(t))F'(V)dV \tag{24}$$

and in which contestant 2 chooses

$$b_2^*(v_2) = \frac{b_1^*(k^{-1}(v_2, l(t))}{l(t)}. \tag{25}$$

Proof: The derivative of the equilibrium effort strategy with respect to v_1 must satisfy $\frac{db_1(v_1)}{dv_1} = \frac{1}{\rho_1'(b_1)}$. Moreover, using the definition of $k(\cdot)$, we have

$\rho_1'(b_1) = \rho_1'(l(t) \cdot b_2(k(v_1, l(t))))$ such that $\frac{db_1(v_1)}{dv_1} = \frac{1}{\rho_1'(l(t) \cdot b_2(k(v_1, l(t))))}$ holds. From Eq.(24) it has

$$\frac{1}{\rho_1'(l(t) \cdot b_2(k(v_1, l(t))))} = l(t) \cdot k(v_1, l(t)) \cdot F'(v_1). \tag{26}$$

Thus, we have

$$\frac{db_1(v_1)}{dv_1} = l(t) \cdot k(v_1, l(t)) \cdot F'(v_1). \tag{27}$$

Together with $b_1(max[0, k^{-1}(0, l(t))]) = 0$ and the definition of $k(v_1, l(t))$, closed form solution for the equilibrium effort strategies are given by

$$b_1^*(v_1) = \int_{max[0,k^{-1}(0)]}^{v_1} l(t) \cdot k(V, l(t)) \cdot dF(V), \tag{28}$$

$$b_2^* = \frac{b_1 * (k^{-1}(v_2))}{l(t)}. \tag{29}$$

This completes the proof.

From Theorem 3,5, we find that there exists a unique pure-strategy BNE whether $0 < l(t) < 1$ or $1 \leq l(t) < +\infty$. Considering an extreme situation where the auction is taken continuously in the context of the unfair contest, in the period $t \in (0, T)$, the total efforts $B_1(v_1)$ and $B_2(v_2)$ exerted by contestants 1 and 2, respectively, are expressed by

$$B_1(v_1) = \int_0^T \int_{max\lfloor 0,k^{-1}(0,l(t))\rfloor}^{v_1} l(t) \cdot k(V, l(t))F'(V)dVdt, \tag{30}$$

and

$$B_2(v_2) = \int_0^T \frac{b_1^*(k^{-1}(v_2, l(t))}{l(t)} dt. \tag{31}$$

4 Conclusion

We have analyzed a two-player unfair contest where the roles of the contestants can have a transform. In the discriminatory contest we discussed, the allocation rule is asymmetric. The discriminatory level $l(t)$ we defined is the function of time t, whose change can make the roles of the contestants have a transform with the passage of time. We show that for a given time t, there exists a unique pure strategy equilibrium. As a result, inefficiencies based on inefficient allocation arise only from the possibility that the favored player wins the contest although his value is lower. We conclude that the rational and reasonable expectation to the discriminatory level $l(t)$ is the most important factor to decision-making process.

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