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Topology and Social Choice

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Abstract

Recent events in the global economy have caused many writers to argue that the market is driven by animal spirits, by irrational exuberance or speculation. At the same time, the economic downturn has apparently caused many voters in the United States, and other countries, to change their opininion about the the proper role of government. Unfortunately, there does not exist a general equilibrium (GE) model of the political economy, combining a formal model of the existence, and convergence to a price equilibrium, as well as an equilibrium model of political choice. One impediment to such a theory is the so-called chaos theorem (Saari, 1997) which suggests that existence of a political equilibrium is non-generic. This paper surveys the results in GE based on the theory of dynamical systems, emphasizing the role of structural stability. In this context it is natural to consider a preference field H for the society, combining economic fields, associated with the preferred changes wrought by agents in the economic market place, together with fields of preferred changes in the polity. A condition called *half-openess* of H is sufficient to guarantee existence of a local direction gradient, d, for the society. The paper argues that instead of seeking equilibrium, it is natural to examine the structural stability of the flow induced by the social preference field. Instead of focusing on equilibrium analysis, based on some version of the Brouwer fixed point theoerem, and thus on the assumption that the social world is topologically contractible, political economy could consider the nature of the dynamical path of change, and utilize notions from the qualitative theory of dynamical systems.

Mathematics Subject Classification: 37C10, 37C20, 37 H20.

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1 Introduction

When Galileo Galilei turned his telescope to the heavens in August, 1609, he inaugurated the modern era in science. In his *Sidereal Messenger* (1610) he wrote of the myriad stars in the milky way, the moons of Jupiter, each at a different period and distance from Jupiter. Jupiter's moons suggested it was a planet just like the earth. Morevoer the phases of Venus also suggested that it was a planet orbiting the Sun. These observations, together with Kepler's empirical "laws" on planetary orbits made it clear that the Copernican heliocentric model of the solar system (Copernicus, (1543) was not just a mathematical theory but a truth. Galileo waited 22 years before publishing *Dialogue concerning the Two Chief World Systems, Ptolemaic and Copernican,* for fear that he would be accused of heresy by the Church. Indeed, in 1633, he was found guilty of "vehement suspicion of heresy" and spent the years until his death under house arrest, but writing *Two New Sciences* (1638). Within fifty years Newton published *Philosophiae Naturalis Principia Mathematica* (1687), giving a mathematical model of physical reality, including celestial mechanics that provided the theoretical foundations for Kepler's Laws.

Even with the Newtonian mathematical model, it was unclear whether the solar sytem was "structurally stable". Although it was possible to compute the orbit of a single planet round the sun, the calculation of the influence of many planets on each other seemed technically difficult. Could these joint influences cause a planet to slowly change its orbit, perhaps causing it to spiral in to the sun? Structural stability for the orbital system of the planets means that the perturbations caused by these interactions, do not change the overall dynamic system. The failure of structural stability means that a slight perturbation of the dynamical system induces a change in the qualitative characteristics of the system. We can use the term "chaos" to refer to this breakdown.

It is only in the last twenty years or so that the implications of "chaos" have begun to be realized. In a recent book Kauffman (1993) commented on the failure of structural stability in the following way.

> One implication of the occurrence or non-occurrence of structural stability is that, in structurally stable systems, smooth walks in parameter space must [result in] smooth changes in dynamical behavior. By contrast, chaotic systems, which are not structurally stable, adapt on uncorrelated landscapes. Very small changes in the parameters pass through many interlaced bifurcation surfaces and so change the behavior of the system dramatically.

It is worth mentioning that the idea of structural stability is not a new one, though the original discussion was not formalized in quite the way it is today. The laws of motion written down by Newton in *Principia Mathematica* could be solved precisely giving a dynamical system that for the case of a planet (a point mass) orbiting the sun. However, the attempt to compute the entire system of planetary orbits had to face the problem of perturbations. Would the perturbations induced in each orbit by the other planets cause the orbital computations to converge or diverge? With convergence, computing the orbit of Mars, say, can be done by approximating the effects of Jupiter, Saturn perhaps, on the Mars orbit. The calculations would give a prediction very close to the actual orbit. Using the approximations, the planetary orbits could be computed far into the future, giving predictions as precise as calculating ability permitted. Without convergence, it would be impossible to make predictions with any degree of certainty. Laplace in his work *Mécanique Céleste* (1799-1825) had argued that the solar system (viewed as a formal dynamical system) is structurally stable (in our terms). Consistent with his view was the use of successive approximations to predict the perihelion (a point nearest the sun) of Haley's comet, in 1759, and to infer the existence and location of Neptune in 1846.

Structural stability in the three-body problem (of two planets and a sun) was the obvious first step in attempting to prove Laplace's assertion. In 1885 a prize was announced to celebrate the King of Sweden' s birthday. Henri Poincaré submitted his entry "Sur le problème des trois corps et les Equations de la Dynamique." This attempted to prove structural stability in a restricted three body problem. The prize was won by Poincaré although it was later found to contain an error. His work on differential equations in the 1880s and his later work, *New Methods of Celestial Mechanics* in the 1890's, developed qualitative techniques (in what we now call differential topology).

The Poincaré conjecture, that "a compact manifold, with the homotopy characteristics of the three-dimensional sphere, is indeed a three sphere" was one of the great unproven theorems of the twentieth century.[1] The theorem has recently been proved by Grigory Perelman.

The earlier efforts to prove this result has led to new ideas in topological geometry, that have turned out, surprisingly, to have profound implications for a better understanding of general relativity and the large scale structure of the universe. (See the discussion in O'Shea , 2007). Our physical universe is a three dimensional manifold, probably bounded and thus compact. The Ricci flow on this manifold is given by the partial differential equation $\delta_t(g_{ij}) = 2R_{ij}$. This equation is a way of characterizing the curvature of geodesics on this manifold. The equation has a deep relationship with the topological structure of the universe. Perelman's proof depends on understanding the nature of singularities associated with this equation. .

In passing it is worth mentioning that since there is a natural periodicity

[1] The generalized Poincaré conjecture in higher dimension was proved by Smale (1961) for dimension five or more, while Freedman (1982) proved it in dimension four.

to any rotating celestial system, the state space in some sense can be viewed as products of circles (that is tori). The examples mentioned below, such as periodic (rational) or a-periodic (non-rational) flow on the torus come up naturally in celestial mechanics.

One of the notions important in understanding structural stability and chaos is that of *bifurcation*. Bifurcation refers to the situation where a particular dynamical system is on the boundary separating qualitatively different systems. At such a bifurcation, features of the system separate out in pairs. However Poincaré also discovered that the bifurcation could be associated with the appearance of a new solution with period double that of the original. This phenomenon is central to the existence of a period-doubling cascade as one of the characteristics of chaos. Near the end of his *Celestial Mechanics*, Poincaré writes of this phenomenon:

> Neither of the two curves must ever cut across itself, but it must bend back upon itself in a very complex manner ...an infinite number of times.... I shall not even try to draw it...nothing is more suitable for providing us with an idea of the complex nature of the three body problem.[2]

Although Poincaré was led to the possibility of chaos in his investigations into the solar system, he concluded that though there were an infinite number of such chaotic orbits, the probability that an asteroid would be in a chaotic orbit was infinitesimal. Arnold showed in 1963 that for a system with small planets, there is an open set of initial conditions leading to bounded orbits for all time. Computer simulations of the system far into time also suggests it is structurally stable.[3] Even so, there are events in the system that affect us and appear to be chaotic (perhaps catastrophic would be a more appropriate term). It is certainly the case that the "N-body system" can display exceedingly complex, or chaotic phenomena (Saari and Xia, 1989; Saari, 2005).[4]

Poincare (1908) was led to the realization that deterministic systems could be chaotic. As he wrote:

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If

[2] Galison (2003: 74).

[3] Peterson, 1993.

[4] Although space is three dimensional, the Einsteinian universe also involves time, and the behavior of geodesics near space-time singularities may also be very complex. See the discussion of space-time singularities, such as black holes, in Hawking and Ellis (1973).

> that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

The impact of large asteroids may have a dramatic effect on the biosphere of the earth, and these have been suggested as a possible cause of mass extinction. The onset and behavior of the ice ages over the last 100,000 years is very possibly chaotic, and it is likely that there is a relationship between these violent climatic variations and the recent rapid evolution of human intelligence (Calvin, 1991, 2006).

More generally, evolution itself is often perceived as a gradient dynamical process, leading to increasing complexity. However, Gould (1989,1996) has argued over a number of years that evolution is far from gradient-like: increasing complexity coexists with simple forms of life, and past life has exhibited an astonishing variety. Evolution itself appears to proceed at a very uneven rate.[5]

"Empirical" chaos was probably first discovered by Lorenz (1962, 1963) in his efforts to numerically solve a system of equations representative of the behavior of weather. A very simple version is the non-linear vector equation

$$\frac{dx}{dt} = \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} -a(x_1 - x_2) \\ -x_1x_3 \ + \ a_2x_1 - x_2 \\ x_1x_2 - a_3x_3 \end{bmatrix}$$

which is chaotic for certain ranges of the three constants, a_1, a_2, a_3.

The resulting "butterfly" portrait winds a number of times about the left hole (as in Figure 1),then about the right hole,then the left,*etc.* Thus the "phase prortrait" of this dynamical system can be described by a sequence of winding numbers $(w_l^1, w_k^1, w_l^2, w_k^2,\ etc.)$. Changing the constants a_1, a_2, a_3 slightly changes the winding numbers. Note that the picture in Figure 1 is in three dimensions, The butterfly wings on left and right consist of infinitely many, infinitesimally close loops. The whole thing is called the Lorentz "strange attractor."

Given that chaos can be found in such a simple meteorological system, it is worthwhile engaging in a thought experiment to see whether "climatic chaos" is a plausible phenomenon. Weather occurs on the surface of the earth, so the

[5] While we are all aware that evolution can be interpreted as a gene game (Dawkins, 1976), less emphasis is put on the fact that sex implies that the game between genes is coalitional. The choice theorem presented below suggests that evolution itself may be chaotic.

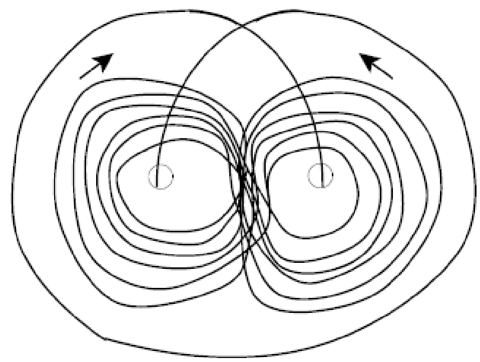

Figure 1: The butterfly

spatial context, or geo-sphere, is $S^2 \times I$, where I is an interval corresponding to the depth of the atmosphere and S^2 is the two-dimensional sphere, the surface of the earth. Below, we shall present a theoretical result to show that a certain kind of dynamical system on $S^2 \times I$ will exhibit a singularity, such as the eye of a hurricane.

Climate is affected by temporal periodicities, induced by the orbit of the earth round the sun and wobbles in the earth's rotation. In addition there are spatial periodicities or closed orbits in the geo-sphere. Chief among these must be the jet stream and the oceanic orbit of water from the southern hemisphere to the North Atlantic (the Gulf Stream) and back. The most interesting singularities are the hurricanes generated each year off the coast of Africa and channeled across the Atlantic to the Caribbean and the coast of the U.S.A. Hurricanes are self-sustaining heat machines that eventually dissipate if they cross land or cool water. It is fairly clear that their origin and trajectory is chaotic. While the topological structure of the geo-sphere allows us to infer the existence of a singularity, understanding weather, and more generally, climate itself, involves the analysis of an extremely complex dynamical system that depends on properties of the solar system.

Perhaps we can use this thought experiment to consider the global economy. First of all there must be local periodicities due to climatic variation. Since hurricanes and monsoons, *etc.* effect the economy, one would expect small chaotic occurrences. More importantly, however, some of the behavior of economic agents will be based on their future expectations about the nature of economic growth, *etc.* Thus one would expect long term expectations to affect large scale decisions on matters such as investment, fertility etc.

It is evident enough that the general equilibrium (GE) emphasis on the existence of price equilibria, while important, is probably an incomplete way to understand economic development. In particular, GE theory tends to downplay the formation of expectations by agents, and the possibility that this can lead to unsustainable "bubbles."

It is a key assumption of GE that agents' preferences are defined on the commodity space alone. If, on the contrary, these are defined on commodities *and* prices, then it is not obvious that the Arrow Debreu Theorem (1954) can be employed to show existence of a price equilibrium. More generally one can imagine energy engines (very like hurricanes) being generated in asset markets, and sustained by self-reinforcing beliefs about the trajectory of prices. It is true that modern decentralised economies are truly astonishing knowledge or data-processing mechanisms. From the perspective of today, the argument that a central planning authority can be as effective as the market in making "rational" investment decisions is very controversial. Hayek 's "calculation" argument used the fact that information is dispersed throughout the economy, and is, in any case, predominantly subjective. He argued essentially that only a market, based on individual choices, can possibly "aggregate this information (Hayek, 1954).

Recently, however, theorists have begun to probe the degree of consistency or convergence of beliefs in a market when it is viewed as a game. It would seem that when the agents "know enough about each other", then convergence in beliefs is a possibility (Arrow,1986; Aumann, 1976).

In fact the issue about the "truth-seeking capability" of human institutions is very old and dates back to the work of Condorcet (1785). Recent work suggests that there may be "belief cascades" or bubbles, which generate multiple paths of beliefs which diverge away from the truth. (Bikhchandani, Hirschleifer and Welsh,1992).

John Maynard Keynes work, *The General Theory of Employment, Interest and Money* (1936) was very probably the most influential economic book of the century. What is interesting about *The General Theory* is that it does appear to have grown out of work that Keynes did in the period 1906 to 1914 on the foundation of probability, and that was published eventually as the *Treatise on Probability* (1921). In the *Treatise*, Keynes viewed probability as a degree of belief.[6] He also wrote:

> The old assumptions, that all quantity is numerical and that all quantitative characteristics are additive, can no longer be sustained. Mathematical reasoning now appears as an aid in its symbolic rather than its numerical character. I, at any rate, have not the same lively hope as Condorcet, or even as Edgeworth, "Eclairer le Science morales et politiques par le flambeau de l'Algebre."

[6] The *Treatise* discusses earlier work by Condorcet (1795) and Laplace (1812, 1814) as well as Poincaré (1896), and comments on the conceptual link between Condorcet, Malthus and Darwin. See Schofield (2010) for comments on this link.

Macro-economics as it is practiced today tends to put a heavy emphasis on the empirical relationships between economic aggregates. Keynes' views,in *Treatise*, suggest that he was impressed neither by econometric relationships nor by algebraic manipulation. Moreover, his ideas on "speculative euphoria and crashes" would seem to be based on an understanding of the economy grounded neither in econometrics nor algebra but in the qualitative aspects of its dynamics. (See Minsky, 1975, 1986).

Obviously I have in mind a dynamical representation of the economy somewhere in between macro-economics and general equilibrium theory. The laws of motion of such an economy would be derived from modeling individuals' "rational" behavior as they process information, update beliefs and locally optimize.

As Akerlof and Shiller (2009) argue in their book,

> the business cycle is tied to feedback loops involving speculative price movements and other economic activity — and to the talk that these movements incite. A downward movement in stock prices, for example, generates chatter and media response, and reminds people of longstanding pessimistic stories and theories. These stories, newly prominent in their minds, incline them toward gloomy intuitive assessments. As a result, the downward spiral can continue: declining prices cause the stories to spread, causing still more price declines and further reinforcement of the stories.

At present it is not possible to construct such a micro-based macro-economy because the laws of motion are unknown. Nonetheless, just as simulation of global weather systems can be based on local physical laws, so may economic dynamics be built up from the local "rationality" of individual agents. However, the GE models discussed in this paper are based on the assumption that the political economic world is contractible, that is, it has the topological characteristic of a ball. This seems an unlikely assumption. (See Krugman, 2009, for a recent argument that the assumptions of economic theory are unrealistic.) Although the total set of resources may well be bounded, it does not appear to be the case that technological possibilies are similarly bounded. Indeed, the Enlightenment argument between Malthus ([1798], 1970) and Condorcet([1795], 1955) seems, at least in the developed world, to have been carried by Condorcet.

Although we might be optimistic about technological advance, recent economic events have caused concern about the validity of current economic theory. As a result, an extensive literature has developed over the last few years over the theory of efficient markets . This literature discusses the nature of herd instinct, the way markets respond to speculative behavior and the power law that characterizes market price movements.[7] Some of these analyses are based on a version of the market equilibrium theorem. In fact, much of the work on efficient markets is based on the Black-Scholes partial differential equation used to price options (Black and Scholes (1973; Merton, 1973). This equation is structurally similar to the Ricci flow equation mentioned above in relation to the Poincare conjecture. The recent collapse of the economy suggests that this equation is subject to chaotic singularities, whose qualitative nature is not understood.

Indeed Minsky's interpretation of Keynes's general theory focuses on the proposition that asset pricing is subject to an extreme degree of uncertainty.[8] The underlying idea here is that individuals do not know the true probability distribution on the various states of the world, but only have personal probability distributions, in the sense of Savage (1954), and make stochastic choices on the basis of this personal uncertainty.[9] Agents may also differ widely in how they treat "black swan" low probability events, as discussed in Hinich (2003), Taleb (2007) and Chichilinsky (2009). Since investement decisions are based on these uncertain evaluations, and these are the driving force of an advanced economy, the flow of the market can exhibit singularities, of the kind that recently nearly brought on a great depression. These singularities are time-dependent, and can be induced by endogenous belief-cascades, rather than by any change in economic or political fundmentals.[10]

More abstractly, the space in which economic and political behavior occurs may be thought of as a differentiable manifold of very high dimension. While GE asserts that there are "equilibria", these will depend on the dynamical domain in which they are defined. These domains are separated by singularities, where the qualitative nature of the system may be radically transformed. To illustrate, although topology asserts that there is a singularity in a flow on the geosphere, $S^2 \times I$, as described above, we need the complex mathematics of chaos theory to understand the creation of a hutrricane, and more generally to attempt to understand the qualitatice changes that can occur in weather and climate. Here I interpret a *singularity* not as a simple phenomenon such as the eye of the hurricane, but as a gate between different dynamical systems.

One of the concerns about climate is that it may exhibit complex singularities. For example, the spatially periodic, oceanic flow of water, including the Gulf stream, has switched off, and then on again, in the past. These switches can be interpreted as singularities that have caused catastrophic changes in

[7] See, for example, Barbera (2009), Cassidy (2009), Fox (2009), Mandelbrot and Hudson (2004), Taleb (2007) and Shiller (2003,2005).

[8] Minsky (1975).

[9] See also Kurz and Motolese (2001).

[10] Schofield (2006a, 2010).

climate, and have in turn been caused by subtle changes in the underlying periodicities of the system. Since the end of the last ice age, during the period of the holocene of the last twelve thousand years, humankind has benefited from a structurally stable and mild climate domain, conducive to agriculture. Were human activity to be sufficient to "force" the bio-sphere[11] through one of these singularities, then the dynamical system that we would face would be qualitatively completely different.[12]

It is increasingly understood that the dynamics of the geo-sphere and bio-sphere interact through multiple feedback mechanisms (Calvin, 2006). The melting of the icecaps resulting from a temperature change modifies their albedo, reflecting less heat energy, further raising global temperature, increasing oceanic volume, affecting forest evapotransportation as well as the oceanic algae populations. Methane can be liberated from deep ocean domains. Cloud formations may change as the weather system is transformed, and intense families of hurricanes spawned in the oceans. All these possible changes are deeply chaotic because they involve a fundamental change in the nature of the equilibrium between the oceans, the land and the atmosphere.

For this reason the future we face exhibits the kind of fundamental uncertainty that Keynes emphasized. It can be argued that the degree of uncertainty is so extreme that we should plan for the future with extreme risk aversion (Stern, 2007; Coyle, 2011).

While GE may assert the existence of a general full-employment equilibrium, recent events suggest that economic behavior in our sophisticated markets may also induce complex or chaotic singularities in the flow of the economy. Indeed, it has dawned on us that these lurches from one crisis to another make it even more difficult to see how to plan for the future.[13]

These suggestions are meant to indicate that there are deep connections between economic and social choice and mathematical models derived from differential topology and geometry. In my view, the qualitative theory of dynamical systems will have a major role in constructing a dynamical theory of the political economy and its effect on the bio-sphere. In what follows I shall review some aspects of the qualitative theory of dynamical systems in order to suggest how this theory may be constructed.

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[11] I use the term bio-sphere for the whole system of life *etc.* within the geo-sphere. Both systems are extremely complex and, I believe, resist purely quantitative analysis.

[12] Metaphorically speaking, it would be like passing through a black hole into a totally different universe.

[13] Recent economic events have led to severe disagreement about how to attempt to deal with climate change. It was only because of pressure from Obama that the Copenhagen Accord was agreed to, in December 2009, by the United States together with four key emerging economies - China, Brazil, India and South Africa. .

2 Genericity on the space of utilities

We first introduce the idea of a topology on the set $U(W)^N$ of smooth utility profiles. Details of this topology can be found in Golubitsky and Guillemin (1973), Smale (1973), Hirsch (1976), and Saari and Simon (1977).

Definition 2.1

Let W be a compact subset of the $w-$dimensional commodity space $\mathbb{R}^w$. A profile u: $W \twoheadrightarrow \mathbb{R}^n$ for a society N of size n belongs to $U(W)^N$ if the Jacobian function

$$J[u] : W \twoheadrightarrow Mat(w,n) : (x) \longrightarrow \left[\frac{\partial u_i}{\partial x_j}\right]_{j=1,\dots,w}^{i=1,\dots,n}$$

is everywhere defined and continuous wrt the topologies on .the space W and the set of w by n matrices, $Mat(w,n)$.

(i) A set $V \subset U(W)^N$ is open in the C^0-*topology*, T_0, on $U(W)^N$ iff for any $u \in V, \exists\delta > 0$ such that

$$\left\{u' \in U(W)^N : \ \|u_i'(x) - u_i(x)\| < \delta, \forall x \in W, \forall i \in N\right\} \subset V,$$

where $\|-\|$ is the Euclidean norm on W. Write $(U(W)^N, T_0)$ for this topological space.

(ii) Let $\|\|_{w.n}$ be the natural norm on the set of matrices $Mat(w,n)$.

(iii) A set $V \subset U(W)^N$ is open in the C^1-*topology*, T_1, on $U(W)^N$ iff for any $u \in V, \exists\ \delta_1, \delta_2 > 0$ such that

$$\left\{ \begin{array}{c} u' \in U(W)^N : \ \|u_i'(x) - u_i(x)\| < \delta_1, \\ \text{and } \ \|J[u_i'](x) - J[u_i](x)\|_{n,w} < \delta_2, \\ \forall i \in N, \forall x \in W \end{array} \right\} \subset V.$$

Write $(U(W)^N, T_1)$ for this topological space.

Comment

In general if T_1 and T_2 are two topologies on a space U, then say T_2 is *finer than* T_1 iff every open set in the T_1-topology is also an open set in the T_2-topology. T_2 is strictly finer than T_1 iff T_2 is finer than T_1 and there is a set V which is open in T_2 but which is not open in T_1. The C^1-topology on $U(W)^N$ is strictly finer than the C^0-topology on $U(W)^N$

Comment . A set $V \subset U(W)^N$ is called a *residual* set in the topology on $U(W)^N$ iff it is the countable intersection of open dense sets in the topology. It can be shown that $U(W)^N$ is a Baire space, so any residual subset of $U(W)^N$

in the C^1-topology is itself dense. Let K be a property which can be satisfied by a smooth profile, and let

$$U[K] = \{u \in U(W)^N : u \text{ satisfies } K\}.$$

Then K is called a *generic property* iff $U[K]$ contains a residual set in the C^1-topology.

The Debreu-Smale Theorem (Smale, 1973, 1974; Debreu,1976; Saari and Simon, 1977) has shown that the existence of isolated price equilibria is a generic phenomenon in the space, $(U(W)^N, T_1)$, of smooth utility profiles.

3 Structural Stability of a Vector Field

The adjustment process for an economy can be viewed a vector field on the $(w-1)$ dimensional price simplex, $\Delta :$ that is at every price vector $p(t)$, at time t, there exists a rule that changes $p(t)$ by the equation $\frac{dp(t)}{dt} = \xi(p)$, where $\xi(p)$ is the excess demand (the difference between the total demand at the price vector p, and the supply, given by the total endowments, $\mathbf{e} \in W^n$, of the society).

At an equilibrium price vector p^*, the excess demand $\xi(p^*) = 0$ so $\frac{dp(t)}{dt}\mid_{p^*} = 0$, and the price adjustment process has a stationary point. The flow on Δ can be obtained by integrating the differential equation. Now consider the excess demand function ξ as a map from $U(W)^N \times W^n$ to the metric space $\mathcal{V}^1(\Delta)$ of vector fields on Δ. Here $\mathcal{V}^1(\Delta)$ is enowed with the natural topology induced from the metric $\| - \|$on $\mathcal{V}^1(\Delta)$. Thus given a profile u and a vector $\mathbf{e} \in W^n$ of initial endowments, we let

$$\xi : U(W)^N \times W^n \longrightarrow \mathcal{V}^1(\Delta).$$

The genericity theorem given above implies that, in fact, there is an open dense set V in $(U(W)^N, T_1)$ such that ξ is indeed a C^1 vector field on Δ.

An obvious question to ask is how ξ changes as the parameters u and $\mathbf{x} \in W^n$ change. One way to do this is to consider small perturbations in a vector field ξ and determine how the *phase portrait* of ξ changes.

It should be clear that small perturbations in the utility profile or in $\mathbf{x}$ may be sufficient to change ξ so that the orbits change in a qualitative way. A *phase portrait* $\tau(\xi)$ for a vector field, ξ is the picture obtained by integrating the vector field.

If two vector fields, ξ_1, and ξ_2 have phase portraits that are homeomorphic, then $\tau(\xi_1)$ and $\tau(\xi_2)$ are qualitatively identical (or similar). Thus we say ξ_1 and ξ_2 are *similar* vector fields if there is a homeomorphism $h : \Delta \to \Delta$ such that each orbit in the phase portrait $\tau(\xi_1)$ of ξ_1 is mapped by h to an orbit in $\tau(\xi_2)$.

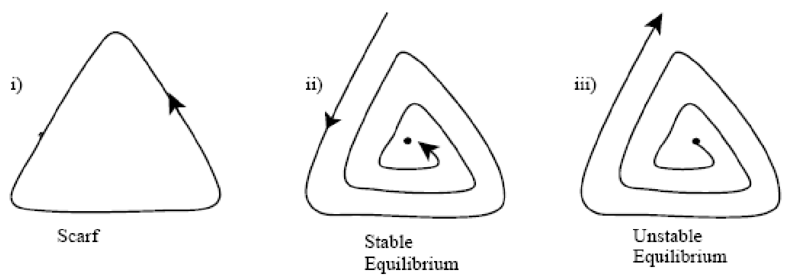

Figure 2: Scarf's example

To illustrate this, consider the Scarf (1960) example where each of the orbits of the excess demand function, ξ_1, say, comprises a closed orbit (homeomorphic to S^1) as in Figure 2(i).[14] Now consider the vector field ξ_2 whose orbits approach an equilibrium price vector p^*. The phase portraits of ξ_2 is given in Figure 2(ii).

The price equilibrium in Figure 2(ii) is *stable* since $lim_{t\to\infty}\ p(t) \to p^*$. Obviously each of the orbits of ξ_2 is homeomorphic to the half open interval $(-\infty, 0]$. Moreover $(-\infty, 0]$ and S^1 are not homeomorphic, so ξ_1 and ξ_2 are not similar.

It is intuitively obvious that the vector field, ξ_2 can be obtained from ξ_1 by a "small perturbation, in the sense that $\| \xi_1 - \xi_2 \| < \delta$, for some small $\delta > 0$. When there exists a small perturbation ξ_2 of ξ_1, such that ξ_1 and ξ_2 are dissimilar, then ξ_1 is called *structurally unstable*. On the other hand, it should be plausible that, for any small perturbation ξ_3 of ξ_2 then ξ_3 will have a phase portrait $\tau(\xi_3)$ homeomorphic to $\tau(\xi_2)$, so ξ_2 and ξ_3 will be similar. The vector field ξ_2 is called *structurally stable*. Notice that structural stability of ξ_2 is a much more general property than stability of the equilibrium point p^* (where $\xi_2(p^*) = 0$). In Fig 2(iii) is a vector field, ξ_4, say, with an unstable equilibrium point. Yet ξ_4 is a structurally stable vector field.

All that we have said on Δ can be generalised to the case of a "smooth manifold" Y, (that is a topological space that is smooth and is locally Euclidean). So let $\mathcal{V}^1(Y)$ be the topological space of smooth vector fields on Y and $\mathcal{P}(Y)$ the collection of phase portraits on Y. The topology on $\mathcal{V}^1(Y)$ is obtained by using a local metric on the vector field.

[14]It is interesting that this vector field is the same as the game dynamic obtained from the classic cycle game of rock beats scissors beats paper. See Hofbauer and Sigmund (1988).

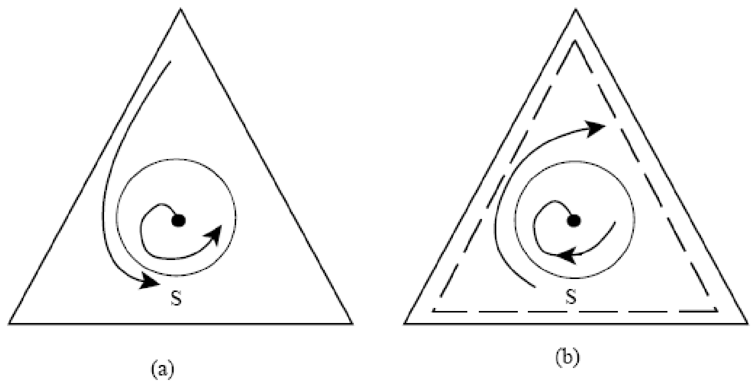

Figure 3: Attractors and Repellors

Definition 3.1

(i) Let $\xi_1, \xi_2 \in \mathcal{V}^1(Y)$. Then ξ_1 and ξ_2 are said to be *similar* (written $\xi_1 \sim \xi_2$) iff there is a homeomorphism $h : Y \to Y$ such that an orbit σ is in the phase portrait $\tau(\xi_1)$ of ξ_1 iff $h(\sigma)$ is in the phase portrait of $\tau(\xi_2)$.

(ii) The vector field ξ is *structurally stable* iff there exists an open neighborhood V of ξ in $\mathcal{V}^1(Y)$ such that $\xi' \sim \xi$ for all $\xi' \in V$.

(iii) A property K of vector fields in $\mathcal{V}^1(2)$ is *generic* iff the set $\{\xi \in \mathcal{V}^1(Y) : \xi$ satisfies $K\}$is residual in $\mathcal{V}^1(Y)$.

A residual set, V, is the countable intersection of open dense sets, and, when $\mathcal{V}^1(Y)$is a "Baire space," V will itself be dense.

Figure 3 gives two examples of structurally stable fields. In 3(i) there is a stable attractor, the circle, S^1, while in 3(ii) the circle is a repellor.

It was conjectured that structural stability is a generic property. This is true if the dimension of Y is 2, but is false otherwise (Smale 1966, Peixoto 1962).

Before discussing the Peixoto-Smale Theorems, it will be useful to explore further how we can qualitatively "classify" the set of phase portraits on a manifold Y. The essential feature of this classification concerns the nature of the critical or *singularity* points of the vector field on W and how these are constrained by the topological nature of W. We now introduce the Euler characteristic, $\chi(W)$ of W.

Example 1.

To illustrate the Euler characteristic, let $W = S^1 \times S^1$ be the *torus* (the skin of a donut) and let $f : W \to \mathbb{R}$ be the height function, as in Figure 4.

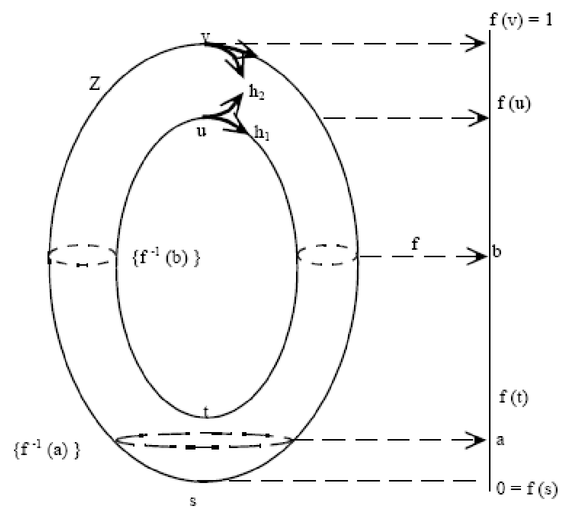

Figure 4: The height function on the torus

At the bottom point,s, of the torus, we let $f(s) \ = \ 0$, so that near s we can represent f as

$$f : (h_1, h_2) \to 0 \ + \ {h_1}^2 \ + \ {h_2}^2.$$

Note that the Hessian, Hf, of f at s is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,which is positive definite. The index of s is the number of negative eigenvalues of the Hessian of f at s, which is 0. At the next critical point, t, we can write $f \ : \ (h_1, h_2) \ \to$ $f(t) + {h_1}^2 - {h_2}^2$.Clearly $Hf(t) \ = \ \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, so t is a saddle, with index 1. The next critical point is also a saddle, u, near which f can be represented as $(h_1, h_2) \to f(u) - {h_1}^2 + {h_2}^2$ Finally the top critical point v is a local maximum and f is represented near v by $(h_1, h_2) \to f(u) \ - \ {h_1}^2 \ - \ {h_2}^2$.Thus the index of v is 2.

This example allows us to introduce the idea of the Euler characteristic $\chi(W)$of a manifold W. If W has dimension, w, let $c_i(W, f)$ be the number of critical points of index i, of the function $f : W \to \mathbb{R}$ and define the index of f on W to be

$$\chi(W, f) \ = \ \sum_{i=0}^{w} (-1)^i \ c_i(W, f).$$

For example the height function, $f : W \to \mathbb{R}$ on the torus W has

(i) $c_0(W, f) \ = \ 1$, since s has index 0,

(ii) $c_1(W, f) \ = \ 2$, since both t and u have index 1,

(iii) $c_2(W, f) = 1$, since $v$has index 2.

Thus $\chi(W, f) = 1 - 2 + 1 = 0$. In fact, it can be shown that $\chi(W, f)$is independent of f, when W is compact. It is an *invariant* of the smooth manifold W, and is written $\chi(W)$) and is known as the Euler characteristic of W.

We have shown $\chi(Torus) = 0$.

Example 2.

The sphere S^2 has an index 0 critical point at the bottom and an index 2 critical point at the top, so $\chi(S^2) = c_0 + c_2 = 1 + 1 = 2$. Clearly the circle, S^1, has $\chi(S^1) = c_0 + c_1 = 1 - 1 = 0$.

More generally, $\chi(S^k) = 0$ if k is odd and $= 2$ if k is even.

To compute $\chi(B^n)$for the closed $n-$ball, take the sphere S^n and delete the top hemisphere. The remaining bottom hemisphere is diffeomorphic to B^n. By this method we have removed the index n critical point at the top of S^n.

Now,

$$\begin{aligned}
\chi(S^{2k+1}) &= \sum_{i=0}^{2k} (-1)^i \; c_i \; (S^{2k+1}) - c_n(S^{2k+1}) = 0, \\
\text{so } \chi(B^{2k+1}) &= \sum_{i=0}^{2k} (-1)^i \; c_i \; (S^{2k+1}) = 1. \\
\chi(S^{2k}) &= \sum_{i=0}^{2k-1} (-1) c_i(S^{2k}) + c_n(S^{2k}) = 2, \\
\text{so } \chi(B^{2k+1}) &= \sum_{i=0}^{2k-1} (-1) c_i(S^{2k}) = 1
\end{aligned}$$

Example 3.

Let us return to the example of the torus $W = S^1 \times S^1$examined above. We defined a height function $f : W \to \mathbb{R}$ and considered the four critical points $\{s, t, u, v\}$of f, where v was an index 2 critical point (a local maximum of f). Near v, f could be represented as

$$f(h_1, h_2) = f(v) - {h_1}^2 - {h_2}^2.$$

Now f defines a gradient vector field ξ where

$$\xi(h_1, h_2) = -df(h_1, h_2)$$

The field ξ may be interpreted as the law of motion under a potential energy field, f, so that the system flows from the "source", v, towards the "sink", s, at the bottom of the torus.

The Euler characteristic can be interpreted as an *obstruction* to the non-existence of equilibria, or of fixed points. For example suppose that $\chi(W) = 0$. Then it is possible to construct a vector field ξ on W without zeros. Then there will exist a function $f : W \to W$ which follows the trajectories of ξ a small distance, ϵ, say. But then this function f is homotopic to the identity.

That is to say, from each point x construct a path $c_x : [0,1] \to Y$ with $c_x(0) = x$ and $c_x(1) = f(x)$ whose gradient $\frac{dc}{dt}(t)$ at time t is given by the vector field ξ at the point $x' = c_x(t)$. Say that f is *induced* by the vector field, ξ. The homotopy $F : [0,1] \times Y \to Y$ is then given by $F(0,x) = x$ and $F(t,x) = c_x(t)$. Since c_x is continuous, so is F. Thus F is a homotopy between f and the identity on W. A function $f : W \to W$ which is homotopic to the identity is called a *deformation* of Y.

If $\chi(W) = 0$ then it is possible to find a vector field ξ on W without singularities and then construct a deformation f of Y induced by ξ. Since $\xi(x) = 0$ for no x, f will not have a fixed point: that is, there exists no $x \in W$ such that $f(x) = x$. Conversely if f is a deformation on W and $\chi(W) \neq 0$, then the homotopy between f and the identity generates a vector field ξ. Were f to have no fixed point, then ξ would have no singularity. If f and thus ξ have the right behavior on the boundary, then ξ must have at least one singularity. This contradicts the fixed point free property of f.

The Lefshetz Fixed Point Theorem.

If W is a manifold with $\chi(W) = 0$ then there exists a fixed point free deformation of W. If $\chi(W) \neq 0$ then any deformation of W has a fixed point.

The "Hairy Ball" Theorem.

Any vector field ξ on S^{2n} (of even dimension) must have a singularity. However there exists a vector field ξ on S^{2n+1}(of odd dimension) such that $\xi(p) = 0$ for no $p \in S^{2n+1}$.

To illustrate Figure 5 shows a vector field on S^2 where the flow is circular on each of the circles of latitude, but both north and south poles are singularities. The flow is evidently non-gradient, since no potential function, f, can increase around a circular orbit.

Example 4.

(1) For a more interesting deformation of the torus $W = S^1 \times S^1$, consider Figure 6(i).

The closed orbit at the top of the torus is a repellor, R, say. Any flow starting near to R winds towards the bottom closed orbit, A, an attractor. There are no singularities, and the induced deformation is fixed point free.

(2) Not all flows on the torus W need have closed orbits. Consider the flow

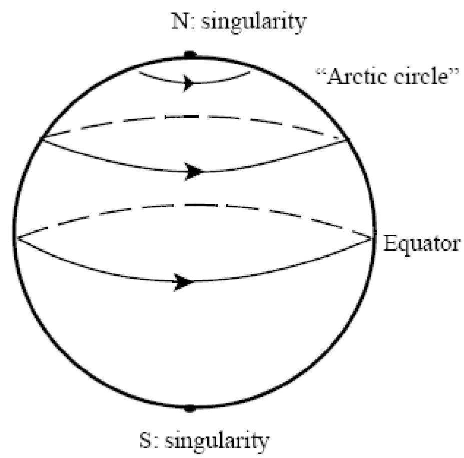

Figure 5: A non gradient flow on the sphere

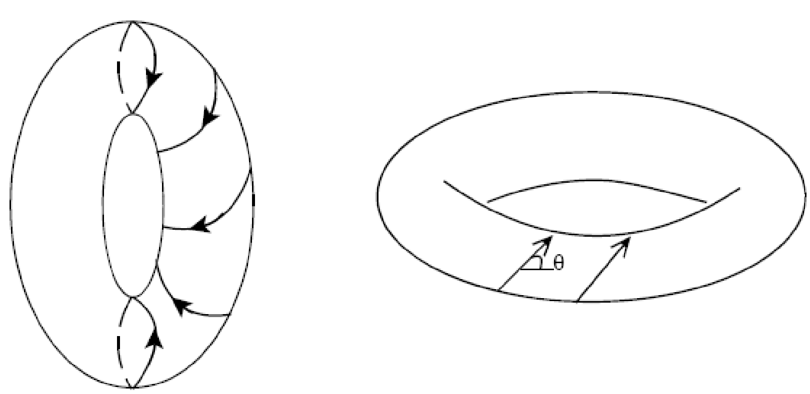

Figure 6: Flows on the torus

on $S^1 \times S^1$,given in Figure 6 (ii). If the tangent of the angle, θ, is rational, then the orbit through a point is closed, and will consist of a specific number of turns round W. However suppose this flow is perturbed. There will be, in any neighborhood of θ, an irrational angle. The orbits of an irrational flow will not be closed. To relate this to the Peixoto Theorem which follows, with rational flow there will be an infinite number of closed orbits. However the phase portrait for rational flow cannot be homeomorphic to the portrait for irrational flow. Thus any perturbations of rational flow gives a non-homeomorphic irrational flow. Clearly any vector field on the torus which gives rational flow is structurally unstable.

Peixoto-Smale Theorem

(i) If dim $W = 2$ and W is compact, then structural stability of vector fields on W is generic.

(ii) If dim $W \geq 3$, then structural stability is non-generic.

Peixoto (1962) proved part (i) by showing that structurally stable vector fields on compact W (of dimension 2) must satisfy the following properties:

(1) there are a finite number of non-degenerate isolated singularities (that is, critical points which can be sources, sinks, or saddles)

(2) there are a finite number of attracting or repelling closed orbits

(3) every orbit (other than closed orbits) starts at a source or saddle, or winds away from a repellor and finishes at a saddle or sink, or winds towards an attractor

(4) no orbit connects saddle points.

Peixoto showed that for any vector field ξ on W and any neighborhood V of ξ in $\mathcal{V}^1(W)$ there was a vector field ξ' in V that satisfied the above four conditions and thus was structurally stable. Systems satisfying the above four conditions are called Morse-Smale systems and are structurally stable in two dimensions.

Although we have not carefully defined the terms used above, they should be intuitively clear. To illustrate, Figure 7(i) shows a structurally unstable flow, where an orbit connects two saddles. Figure 7(ii) shows that after perturbation a qualitatively different phase portrait is obtained.

In Figure 7(i), A and B are connected saddles, C is a repellor (orbits starting near to C leave it) and D is a closed orbit. A small perturbation disconnects A and B as shown in Figure 7(ii), and orbits starting near to D (either inside or outside) approach D, so it is an attractor.

Smale's (1966) proof that structural stability was non-generic in three or more dimensions was obtained by constructing a diffeomorphism $f : Y^3 \rightarrow Y^3$(with $Y = S^1 \times S^1 \times S^1$). This induced a vector field $\xi \in \mathcal{V}^1(Y)$ that had the property that for a neighborhood V of ξ in $\mathcal{V}^1(Y)$, no ξ' in V was structurally stable. In other words, every ξ' when perturbed led to a qualita-

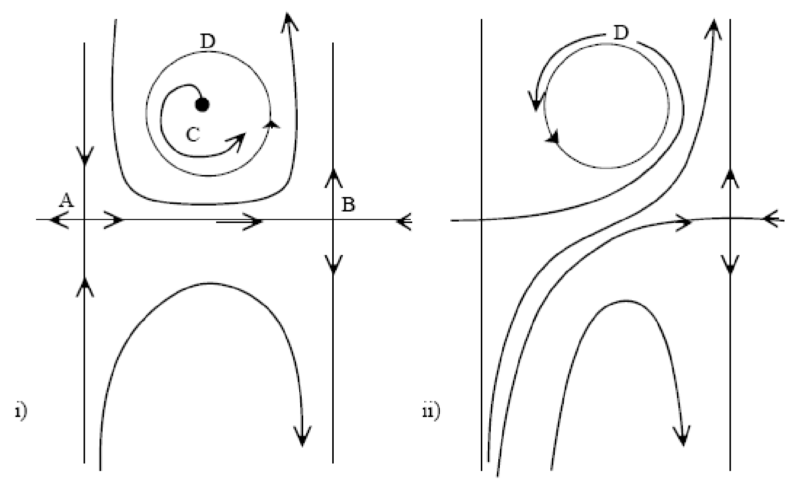

Figure 7: structurally unstable and stable flows

tively different phase portrait. We could say that ξ was *chaotic*. Any attempt to model ξ by an approximation ξ', say, results in an essentially different vector field. See Wiggins (1988).

SMD theorem.

Suppose that there are at least as many economic agents (n) as commodities. Then it is possible to construct a well-behaved economy $(u, \mathbf{x})$ with monotonic, strictly convex preferences induced from smooth utilities, u, and an endowment vector $\mathbf{x} \in W^n$, such that *any* vector field in $\mathcal{V}^1(\Delta)$ is generated by the excess demand function for the economy $(u, \mathbf{x})$.

Variants of this SMD theorem are given in Sonnenschein (1972), Mantel (1974), and Debreu (1974). A generalization can be found in Saari and Simon (1978) with further discussion in Saari (1995). As we have discussed in this section, because the simplex Δ has $\chi(\Delta) = 1$, then the excess demand vector field, ξ, will always have at least one singularity. In fact, from the Debreu-Smale theorem, we expect ξ to generically exhibit only a finite number of singularities. Aside from these restrictions, ξ, is essentially unconstrained.

As we saw above, the vector field ξ of the Scarf example was structurally unstable, but any perturbation of ξ led to a structurally stable field ξ', say, either with an attracting or repelling singularity. It is possible to find $(u, \mathbf{x})$ such that the induced vector field ξ on Δ is chaotic—in some neighborhood V of ξ there is no structurally stable field. Any attempt to model ξ by ξ', say, must necessarily incorporate some errors, and these errors will multiply in some fashion as the phase portrait is mapped. In particular the flow generated by ξ through some price vector, $p \in \ \Delta$ can be very different from the flow generated by ξ' through p' This phenomenon has been called "sensitive dependence on initial conditions."

Example 5.

As an application, we may consider a generalized flow, by defining at each point $x \ \in \ W$ a *set* $H(x)$of vectors in the tangent space at x. This flow generated by H could be induced by a utility profile $\{u : W \to \mathbb{R}^n\}$, of the form

$$H_N(u)(x) \ = \ \{v \ \in \ T_xW : (du_i(x) \centerdot v) > \ 0, \ \forall \ i \ \in \ N\}.$$

Here $(du_i(x) \centerdot v)$ is the scalar product of $du_i(x)$, regarded as a vector in $\mathbb{R}^w$, and the vector $v \in T_xW$, the tangent space at x. That is to say $v \ \in \ H_N(u)(x)$ iff each utility function increases in the direction v. Then $H_N(u)(x) \ = \ \Phi$, the empty set, whenever $x \ \in \ \overset{0}{\Theta}(u_1, \ldots, u_n)$, the "critical Pareto set" of the profile.$\{u : W \to \mathbb{R}^n\}$. We use this idea in the next section to consider social choice.

4 Existence of a Choice

Arguments for the existence of an equilibrium or choice are based on some version of Brouwer's fixed point theorem, which we can regard as a variant of the Lefshetz fixed point theorem. Brouwer's theorem asserts that any continuous function $f : B \to B$ between the finite dimensional ball, or indeed any compact convex set in $\mathbb{R}^w$, has the fixed point property. Figure 8 suggests the proof. If f has no fixed point, then there is a continuous retraction $h : B \to S$, to the sphere. Since the ball is contractible, and the sphere is not, h cannot be continuous. By contradiction, f has the fixed point property.[15]

This section will consider the use of variants of the Brouwer theorem, to prove existence of an equilibrium of a general social choice mechanism. We shall argue that the condition for existence of an equilibrium will be violated if there are cycles in the underlying mechanism.

Let W be the set of alternatives and, as before, let X be the set of all subsets of W. A *preference correspondence*, P, on W assigns to each point $x \in W$, its *preferred set* $P(x)$. Write $P\colon W \to X$ or $P\colon W \twoheadrightarrow W$ to denote that the image of x under P is a set (possibly empty) in W. For any subset V of W, the restriction of P to V gives a correspondence $P_V\colon V \twoheadrightarrow V$. Define

[15] This can be generalized to the infinite dimensional case if B is a closed convex subset of a complete normed vector space, and the image of f is compact. We use a version of this fixed point result below.

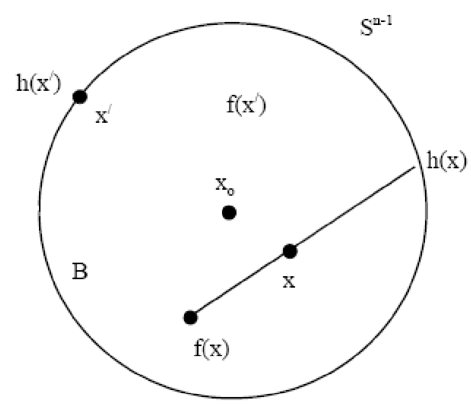

Figure 8: A retraction of the ball onto the sphere

$P_V^{-1}\colon V \twoheadrightarrow V$ such that for each $x \in V$,

$$P_V^{-1}(x) = \{y : x \in P(y\} \cap V.$$

$P_V^{-1}(x) = \{y : x \in P(y\} \cap V$. The sets $P_V(x), P_V^{-1}(x)$ are sometimes called the *upper* and *lower* preference sets of P on V. When there is no ambiguity we delete the suffix V. The *choice* of P from W is the set

$$C(W, P) = \{x \in W\colon P(x) = \Phi\}.$$

Here Φ is the empty set. The choice of P from a subset, V, of W is the set

$$C(V, P) = \{x \in V\colon P_V(x) = \Phi\}.$$

Call C_P a *choice function* on W if $C_P(V) = C(V, P) \neq \Phi$ for every subset V of W. We now seek general conditions on W and P which are sufficient for C_P to be a choice function on W. Continuity properties of the preference correspondence are important and so we require the set of alternatives to be a topological space.

Definition 4.1

Let W, Y be two topological spaces. A correspondence $P\colon W \twoheadrightarrow Y$ is

(i) *Lower hemi-continuous* (*lhc*) iff, for all $x \in W$, and any open set $U \subset Y$ such that $P(x) \cap U \neq \Phi$ there exists an open neighborhood V of x in W, such that $P(x') \cap U \neq \Phi$ for all $x' \in V$.

(ii) *Upper hemi-continuous* (*uhc*) iff, for all $x \in W$ and any open set $U \subset Y$ such that $P(x) \subset U$, there exists an open neighborhood V of x in W such that $P(x') \subset U$ for all $x' \in V$.

(iii) *Lower demi-continuous* (*ldc*) iff, for all $x \in Y$, the set

$$P^{-1}(x) = \{y \in W : x \in P(y)\}$$

is open (or empty) in W.

(iv) *Upper demi-continuous* (*udc*) iff, for all $x \in W$, the set $P(x)$ is open (or empty) in Y

(v) *Continuous* iff P is both *ldc* and *udc*.

We use acyclicity, where a correspondence $P : W \twoheadrightarrow W$ is *acyclic* if it is impossible to find a cycle $x_t \in P(x_{t-1}), x_{t-1} \in P(x_{t-2}), .., x_1 \in P(x_t)$.

We shall use lower demi-continuity of a preference correspondence to prove existence of a choice. In some cases, however, it is possible to make use of lower hemi-continuity. Note that if P is ldc then it is lhc.

We shall now show that if W is compact, and P is an acyclic and ldc preference correspondence $P : W \twoheadrightarrow W$, then $C(W, P) \neq \Phi$. First of all, say a preference correspondence $P : W \twoheadrightarrow W$ satisfies the *finite maximality property* (FMP) on W iff for every finite set V in W, there exists $x \in V$ such that $P(x) \cap V = \Phi$.

Lemma 4.1 (Walker 1977)

If W is a compact, topological space and P is an ldc preference correspondence that satisfies FMP on W, then $C(W, P) \neq \Phi$.

This follows readily, using compactness to find a finite subcover, and then using FMP.

Corollary 4.1.

If W is a compact topological space and P is an acyclic, ldc preference correspondence on W, then $C(W, P) \neq \Phi$.

As Walker (1977) noted, when W is compact and P is ldc, then P is acyclic iff P satisfies FMP on W, and so either property can be used to show existence of a choice. A second method of proof to show that C_P is a choice function is to substitute a convexity property for P rather than acyclicity.

Definition 4.2.

(i) If W is a subset of a vector space, then the *convex hull* of W is the set, Con$[W]$, defined by taking all convex combinations of points in W.

(ii) W is *convex* iff $W = $ Con$[W]$. (The empty set is also convex.)

(iii) W is *admissible* iff W is a compact, convex subset of a topological vector space.

(iv) A preference correspondence $P : W \twoheadrightarrow W$ on a convex set W is *convex* iff, for all $x \in W$, $P(x)$ is convex.

(v) A preference correspondence $P\colon W \twoheadrightarrow W$ is *semi-convex* iff, for all $x \in W$, it is the case that $x \notin \text{Con}(P(x))$.

Fan (1961) has shown that if W is admissible and P is ldc and semi-convex, then $C(W,P)$ is non-empty.

Fan Theorem (Fan, 1961, Bergstrom, 1975).

If W is an admissible subset of a Hausdorff topological vector space, and $P\colon W \twoheadrightarrow W$ a preference correspondence on W which is ldc and semi-convex then $C(W,P) \neq \Phi$.

The proof uses the KKM lemma due to Knaster, Kuratowski and Mazurkiewicz (1929). Yannelis and Prabhakar (1983) have used the KKM lemma to prove Browder's fixed point theorem, that a correspondence $Q\colon W \twoheadrightarrow W$ has a fixed point, $x \in Q(x)$, whenever Q is convex valued and everywhere non-empty. There is a useful corollary to the Fan theorem. Say a preference correspondence on an admissible space W satisfies the *convex maximality property* (CMP) iff for any finite set V in W, there exists $x \in \text{Con}(V)$ such that $P(x) \cap \text{Con}(V) = \Phi$.

Corollary 4.2.

Let W be admissible and $P\colon W \twoheadrightarrow W$ be ldc and semi-convex. Then P satisfies the convex maximality property.

The original form of the Theorem by Fan made the assumption that $P\colon W \twoheadrightarrow W$ was *irreflexive* (in the sense that $x \notin P(x)$ for all $x \in W$) and *convex*. Together these two assumptions imply that P is semi-convex. Bergstrom (1975) extended Fan's original result to give the version presented above.

Note that the Fan Theorem is valid without restriction on the dimension of W. Indeed, Aliprantis and Brown (1983) have used this theorem in an economic context with an infinite number of commodities to show existence of a price equilibrium. Bergstrom (1992) also showed that when W is finite dimensional then the Fan Theorem is valid when the continuity property on P is weakened to lhc. Bergtrom (1992) also used this theorem to show existence of a Nash equilibrium of a game $G = \{(P_1, W_1), .P_i, W_i), ..(P_n, W_n) : i \in N\}$. Here the i^{th} stategy space is finite dimensional W_i and each individual has a preference P_i on the joint strategy space $P_i : W^N = W_1 \times W_2 ... \times W_n \twoheadrightarrow W_i$. The Fan Theorem can be used, in principle to show existence of an equilibrium in complex economies with externalities. Define the Nash improvement correspondence by $P_i^* : W^N \twoheadrightarrow W^N$ by $y \in P_i^*(x)$ whenever $y = (x_1, ..x_{i-1}, x_i^*, ..., x_n)$, $x = (x_1, .., x_{i-1}, x_i, .., x_n)$, and $x_i^* \in P_i(x)$ The joint Nash improvement correspondence is $P_N^* = \cup P_i^* : W^N \twoheadrightarrow W^N$. The Nash equilibrium of a game G is a vector $\mathbf{z} \in W^N$ such that $P_N^*(\mathbf{z}) = \Phi$.

Bergstrom Theorem (Bergstrom, 1992).

For a game, G, on finite dimensional admissible W^N, if all $P_i^* : W^N \twoheadrightarrow W^N$ are lhc and semi-convex, then there exists a Nash equilibrium.

This theorem can be used to show existence of an equilibrium for an abstract economy with lhc and semiconvex preferences. Define an exchange economy to be a game

$$G(P, \mathbf{e}) = \{\mathbf{e},(P_0, \Delta), (P_1, W_1), .P_i, W_i), ..(P_n, W_n) : i \in N\}$$

where Δ is the price simplex, and $\mathbf{e} = (e_1, \;\; ,e_n) \in W^N \subset (\mathbb{R}^w)^n$ is a vector of initial endowments. A competitive equilibrium $(\mathbf{x}$,$p)$ for $G(P, \mathbf{e})$ is the choice set $C(P_N, B_N(p))$ By this notation we mean $(B_N)(p) = (B_1 \times ..B_n)(p)$ with $(B_i)(p) = \{\mathbf{x}_i : (p \centerdot \mathbf{x}_i) \leq (p \centerdot \mathbf{e}_i)\}$ being the budget set for i at the price vector p. We require each P_i to be null on $(B_i)(p)$ and $\Sigma \mathbf{x}_i < \Sigma \mathbf{e}_i$.

For $i \in N$, extend $P_i^* : W^N \twoheadrightarrow W^N$ to

$$P_i^{**} : \Delta \times W^N \twoheadrightarrow \Delta \times W^N \text{ by } (p', y) \in P_i^{**}((p, x) \text{ if } p' = p, \text{ and } y \in P_i^*(x).$$

Define the *price adjustment mechanism* $P_0^* : \Delta \times W^N \twoheadrightarrow \Delta$ by

$$(p', y) \in P_0^*((p, x) \text{ if } (p' - p) \centerdot (\sum_{i \in N}(x_i - e_i)) > 0.$$

The term $\sum_{i \in N}(x_i - e_i)$ is a measure of excess demand, so this adjustment process is designed so that the prices change to reduce aggregate excess demand. Finally define

$$P_0^{**} : \Delta \times W^N \twoheadrightarrow \Delta \times W^N \text{ by } (p', y) \in P_0^{**}((p, x) \text{ if } p' \in P_0^*((p, x) \text{ and } y = x.$$

By restricting the domain of the individual preferences correspondences to the budget sets defined by the price vector p, it is possible to interpret a Nash equilibrium of this game as a free-disposal economic equilibrium, under which total demand may be less than total supply. Additional conditions, such as no-satiation can then be used to gurantee equality of supply and demand. The Nash equilibrium gives a competitive equilibrium $(\mathbf{x}^*$,$p)$ with $\mathbf{x}_i^* \in (B_i)(p)$ for each i. No satiation means that if $\mathbf{x} \in P_i(\mathbf{x}^*)$ then $\mathbf{x}_i \notin (B_i)(p)$.

Numerous applications of the procedure have been made by Shafer and Sonnenschein (1975) and Borglin and Keiding (1976), etc. to show existence of such an economic equilibrium. Note however, that these results all depend on semi-convexity of the preference correspondences.

5 Dynamical Choice Functions

The discussion above of a vector field $\xi : U(W)^N \times \;\; W^n \longrightarrow \mathcal{V}^1(\Delta)$ is a differential analogue of the price adjustment process just discussed.

We now use the Lefshetz Fixed Point Theorem to suggest that equilibria may exist, even when W is not convex. In the spirit of the above discussion of vector fields, we consider a *generalized preference field* $H : W \twoheadrightarrow TW$, on a manifold W.

We use this notation to mean that at any $x \in W$, $H(x)$ is a cone in the tangent space T_xW above x. That is, if a vector $v \in H(x)$, then $\lambda v \in H(x)$ for any $\lambda > 0$. If there is a smooth curve, $c : [-1, 1] \to W$, such that the differential $\frac{dc(t)}{dt} \in H(x)$, whenever $c(t) = x$, then c is called an integral curve of H. An integral curve of H from x=$c(o)$ to y =$\lim_{t\to 1}c(t)$ is called an H -*preference curve* from x to y. The preference field is called *S-continuous* if $H(x)$ is open in T_xW and whenever $v \in H(x)$ then there is an integral curve, c, in a neighborhood of x with $\frac{dc(0)}{dt} = v$. The choice $C(W, H)$ of H on W is defined by $C(W, H) = \{x \in W : H(x) = \Phi\}$. Say H is *half open* if at every $x \in W$, either $H(x) = \Phi$ or there exists a vector $v' \in T_xW$ such that $(v' \centerdot v) > 0$ for all $v \in H(x)$. We can say in this case that there is, at x, a *direction gradient* d in the cotangent space T^*_xW of linear maps from T_xW to $\mathbb{R}$ such that $d(v) > 0$ for all $v \in H(x)$. If H is S-continuous and half-open, then there will exist such a continuous direction gradient $d\ V \to T^*V$ on a neighborhood V of x.

Choice Theorem.

If H is an S-continuous half open preference field, on a finite dimensional compact manifold, W, with $\chi(W) \neq 0$, then $C(W, H) \neq \Phi$. If H is not half open then there exists an H -*preference cycle* through $\{x_1, x_2, x_3, .x_r.x_1\}$. For each arc (x_s, x_{s+1}) there is an H -*preference curve* from x_s to x_{s+1}, with a final H -*preference curve* from x_r to x_1.

This Theorem was proved in Schofield (1984a), using the Fan-Bergstrom and Lefshetz Theorems. For the result we do not need convexity, but we do need finite dimensionality.The result is illustrated in Figure 9. As the left hand figure in Figure 9 suggests, the field is half open on S^1, but $\chi(S^1) = 0$, and there is no choice. On the other hand $\chi(S^2) \neq 0$, and $C(W, S^2) \neq \Phi$, in the right hand figure of Figure 9.

Such a field on S^2 could be used to model plate techtonics. The Choice Theorem implies the existence of a *singularity* of the field, H. Note that there may be global cycles, around the equator, for example, but no localized cycles. Another example is given by a model of cosmology on S^4. Such a singularity may correspond to the "big bang" at the beginning of time.[16] There seem to be deep connections between the nature of singularies in our universe and the possibility of cyclic geodesics in space time.

Example 6.

To illustrate this, consider the example due to Kramer (1973), with $N = \{1, 2, 3\}$.Let the preference relation $P_{\mathbb{D}}\colon W \twoheadrightarrow W$ be generated by a set of

[16] This is how I interpret the singularity theorems of Hawking and Ellis (1973).See also Penrose (2010).

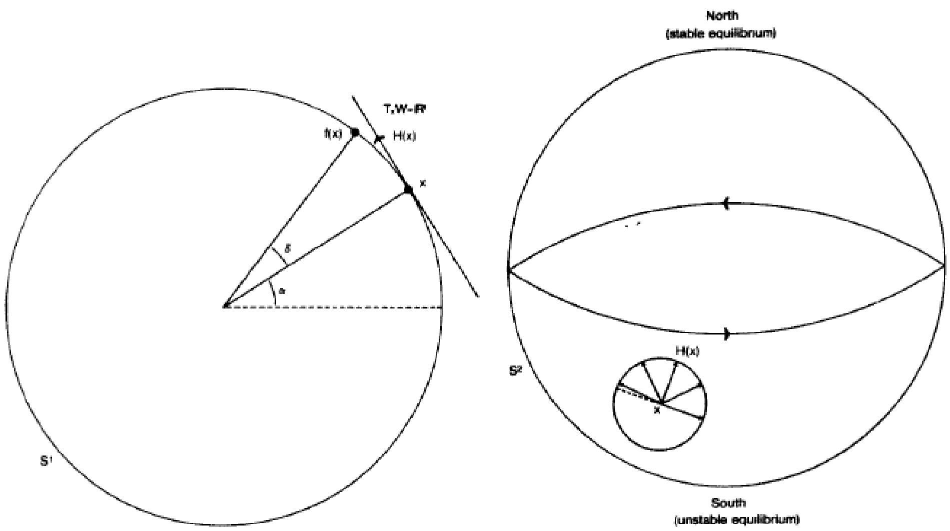

Figure 9: Flows on the 1 and 2-sphere

decisive coalitions, $\mathbb{D} = \{\{1,2\},\{1,3\},\{2,3\}$, so that $y \in P_{\mathbb{D}}(x)$ whenever two voters prefer y to x.Suppose further that the preferences of the voters are characterized by the direction gradients

$$\{du_i(x)\colon i = 1,2,3\}$$

as in Figure 10.

As the figure makes evident, it is possible to find three points $\{a,b,c\}$ in W such that

$$\begin{aligned} u_1(a) &> u_1(b) = u_1(x) > u_1(c) \\ u_2(b) &> u_2(c) = u_2(x) > u_2(a) \\ u_3(c) &> u_3(a) = u_3(x) > u_3(b). \end{aligned}$$

That is to say, preferences on $\{a,b,c\}$ give rise to a *Condorcet cycle.* Note also that the set of points $P_{\mathbb{D}}(x)$, preferred to x under the voting rule, are the shaded "win sets" in the figure. Clearly $x \in \operatorname{Con} P_{\mathbb{D}}(x)$, so $P_{\mathbb{D}}(x)$ is not semi-convex. Indeed it should be clear that in *any* neighborhood V of x it is possible to find three points $\{a',b',c'\}$ such that there is *local* voting cycle, with $a' \in P_{\mathbb{D}}(b'), b' \in P_{\mathbb{D}}(c'), c' \in P_{\mathbb{D}}(a')$. We can write this as

$$a' \to c' \to b' \to a'.$$

Not only is there a voting cycle, but the Fan theorem fails, and we have no reason to believe that $C(W,P_{\mathbb{D}}) \neq \Phi$.

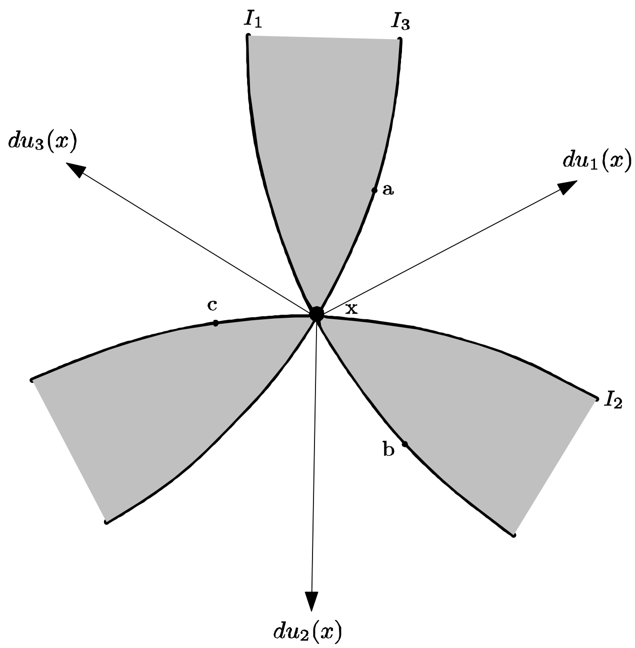

Figure 10: A chaotic general flow

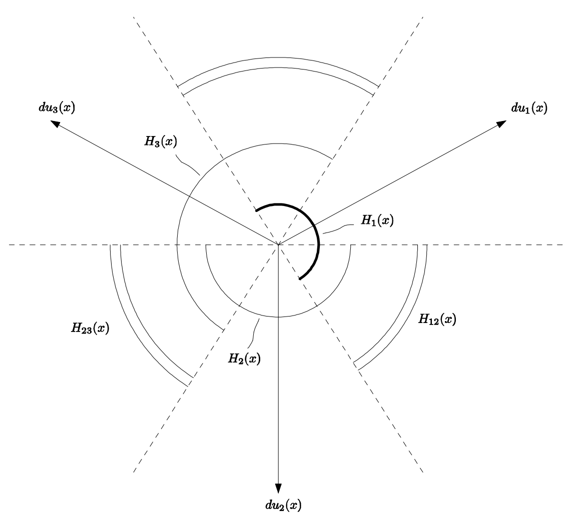

Figure 11: The failure of half-openess of a preference field

We can translate this example into one on preference fields by writing

$$H_{\mathbb{D}}(u) = \cup H_M(u) : W \twoheadrightarrow TW$$

where each $M \in \mathbb{D}$ and

$$H_M(u)(x) \ = \ \{v \ \in \ T_xW : (du_i(x) \centerdot v) > \ 0, \ \forall \ i \ \in \ M\}.$$

Figure 11 shows the three difference preference fields $\{H_i : i = 1, 2, 3)$ as well as the intersections H_M, for $M = \{1, 2\}$ etc.
Obviously the joint preference field $H_{\mathbb{D}} : W \twoheadrightarrow TW$ fails the half open property at x. Although $H_{\mathbb{D}}$ is S-continuous, we cannot infer that $C(W, H) \neq \Phi$. If we define

$$Cycle(W, H) \ = \ \{x \ \in W : H(x) \text{ is not half open}\}.$$

then at any point in $Cycle(W, H)$ it is possible to construct local cycles in the manner just described.

The choice theorem can then be interpreted to mean that for any S-continuous field on W, if $\chi(W) \ \neq \ 0$ then

$$Cycle(W, H) \cup C(W, H) \neq \Phi.$$

For a voting rule, $\mathbb{D}$ it is possible to guarantee that $Cycle(W, H) = \Phi$ and thus that $C(W, H) \neq \Phi$, by restricting the dimension of W.

Definition 5.1

(i) Let $\mathbb{D}$ be a family of subsets of N. If the collegium, $K(\mathbb{D}) = \cap\{M \in \mathbb{D}\}$ is non-empty then $\mathbb{D}$ is called *collegial* and the *Nakamura number* $\kappa(\mathbb{D})$ is defined to be ∞.

(ii) If the collegium $K(\mathbb{D})$ is empty then $\mathbb{D}$ is called *non-collegial*. Define the *Nakamura number* in this case to be $\kappa(\mathbb{D}) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D}$ and $K(\mathbb{D}') = \Phi\}$.

Nakamura Theorem.

If $u \in U(W)^N$ and $\mathbb{D}$ has Nakamura number $\kappa(\mathbb{D})$ with $dim(W) < \kappa(\mathbb{D}) - 2$ then $Cycle(W, H_{\mathbb{D}}(u)) = \Phi$. If in addition, $\chi(W) \ \neq \ 0$ then $C(W, H_{\mathbb{D}}(u)) \neq \Phi$.

This result is proved in Schofield (1984b) using results of Nakamura (1979). See also Strnad (1985).

Unfortunately, the Nakamura number for majority voting rules is either 3 or 4, depending on whether n is odd or even, so this result can generally only be used to prove a median voter like theorem in one dimension for a society of size odd. However, the result can be combined with the Fan Theorem to prove existence of equilibrium for a political economy with voting rule $\mathbb{D}$, when the dimension of the public good space is bounded by $\kappa(\mathbb{D})-2$ (Konishi, 1996). Recent work in political economy generally only considers a public good space of one dimension (Acemoglu and Robinson,2006). Note however, that if $\mathbb{D}$ is collegial, then $Cycle(W, H_{\mathbb{D}}(u)) = \Phi$. Such a rule can be called oligarchic, and this inference provides a theoretical basis for comparing democracy and oligarchy (Acemoglu, 2008)

Extending this equilibrium result to higher dimension faces a difficulty caused by the following theorem.

Saari Theorem.

For any non-collegial $\mathbb{D}$, there exists an integer $w(\mathbb{D}) > \kappa(\mathbb{D})$ such that $dim(W) > w(\mathbb{D})$ implies that $C(W, H_{\mathbb{D}}(u)) = \Phi$ for all u in a residual subspace of $(U(W)^N, T_1)$. If. in addition,$\chi(W) \neq 0$, then $Cycle(W, H_{\mathbb{D}}(u)) \neq \Phi$ generically.

This result was essentially proved by Saari (1997), building on earlier results by McKelvey (1976, 1979), Schofield (1978, 1983) and McKelvey and Schofield (1987). See Saari (1985a,b, 2001a,b, 2008) for related analyses.[17]

Recent work has attempted to avoid negative conclusions of this result by using the Brouwer fixed point theorem to seek existence of a belief equilibrium for a society $N_{\tau+1}$ of size $n_{\tau+1.}$ time $\tau+1$. In this context we let

$W_E = W_1 \times W_2 ... \times W_{n_{\tau+1}} \times \Delta$ be the economic product space, where W_i is the commodity space for citizen i and Δ is a price simplex.. Let $W_{\mathbb{D}}$ be a space of political goods, governed by a rule $\mathbb{D}$. At time $\tau+1$, $W_{\tau+1} = W_E \times W_{\mathbb{D}}$ is the political economic space.

At τ, each individual, i, is described by a utility function $u_i : W_\tau \to \mathbb{R}$, so the population profile is given by $u : W_\tau \to \mathbb{R}^{n_\tau}$. Beliefs at τ about the future $\tau+1$ are given by a stochastic rule, $\mathbb{Q}_\tau$, that transforms the agents' utilities from those at time τ to those at time $\tau+1$. Thus $\mathbb{Q}_\tau$generates a new profile for $N_{\tau+1}$at $\tau+1$ given by $\mathbb{Q}_\tau(u) = u' : W_{\tau+1} \to \mathbb{R}^{\mathbf{n\tau+1}}$. The utility and beliefs of i will depend on the various sociodemographic subgroups in the society $N_{\tau.}$ that i belongs to, as well as information about about the current price vector in Δ.

Thus we obtain a transformation on the function space $[W_{\boldsymbol{\tau}} \to \mathbb{R}^{\mathbf{n\tau}}]$ given by

$$[W_{\boldsymbol{\tau}} \to \mathbb{R}^{\mathbf{n\tau}}] \to {}^{\mathbb{Q}_\tau} \to [W_{\boldsymbol{\tau}} \to \mathbb{R}^{\mathbf{n\tau+1}}] \to [W_{\boldsymbol{\tau}} \to \mathbb{R}^{\mathbf{n\tau}}]$$

[17] Although this result formally applies to voting rules, Schofield (2009b) argues that it is applicable to any non-collegial social mechanism, and as a result can be interpreted to imply that chaos is a generic phenomenon in coalitional systems. Recent work by Collier (2009) suggests that chaos is indeed endemic in parts of Africa, particularly in the Congo.

Much more speculatively, evolution is driven by random associations of genes, and it seems entirely possible that the amazing fertility shown by evolution (Dawkins, 2009) is due precisely to the phenomenon of local chaos.

The second transformation here is projection onto the subspace $[W_{\boldsymbol{\tau}} \rightarrow \mathbb{R}^{n\tau}]$ obtained by restrictiong to changes to the original population $N_{\tau.}$ and space.

A *dynamic belief equilibrium* at τ for $N_{\tau.}$ is fixed point of this transformation. Although the space $[W_{\boldsymbol{\tau}} \rightarrow\mathbb{R}^{n\tau}]$ is infinite dimensional, if the domain and range of this transformation are restricted to *equicontinous* functions (Pugh, 2002), then the domain and range will be compact. Penn (2009) shows that if the domain and range are convex then a generalized version of Brouwer's fixed point theorem can be applied to show existence of such a dynamic belief equilibrium. This notion of equilibrium was first suggested by Hahn (1973) who argued that equilibrium is located in the mind, not in behavior.

However, the Saari theorem suggests that the validity of Penn's theorem will depend on how the the model of political choice is constructed.

An alternative idea is to consider a generalized preference field $H : W_\tau \twoheadrightarrow TW_\tau$ for the society on W_τ, and to model the flow generated by H . By extending the flow over some time interval $\Upsilon = [\tau, \tau + 1]$, we could then examine whether it appears to be structurally stable.

In the above, we used the term *structural stability* for the property that the qualitative features of the flow are not changed by small perturbations in the underlying parameters of the flow. The term *chreod* was used by Rene Thom ([1966], 1994) in the context of evolutionary or biological processes to describe such a dynamical system that returns to a steady trajectory. This term is derived from the Greek word for "necessary" and the word for "pathway". A *social chreod* is therefore a structurally stable path through time, where the state space W_Υ now involves not only characteristics, such as factor endowments and prices, p, in W_E, but also the beliefs and thus the preferences of individuals, particularly as regards the risk postures that are embedded in their preferences.

One advantage of such a modeling exercise is that we do not need not need to use Penn's equilibrium theorem to determine the transformation $\mathbb{Q}_\tau(u) =u'$. Instead, the society N_τ is characterised at τ by a family $\{du_i : W_\tau{\rightarrow}T^*W_\tau : i \in N_\tau\}$ of normalized direction gradients. Here $du_i(x,p) : TW_\tau{\rightarrow}\mathbb{R}$ specifies the local change in utility for i at a point x, and price p,when composed with a vector $v \in TW_\tau$. Just as in the model for the choice theorem, we can then define, for any coalition $M \subset N_\tau$, the preference field $H_M^*(x) \subset T_xW_\tau$ at $x \in W_\tau$, consisting of vectors in T_xW_τ that are both preferred and feasible for the coalition at x. Taking $H = \cup H_M^* \cup H_\Delta$, where H_Δ is a field on the price simplex, gives the generalised field $H : W_\Upsilon \twoheadrightarrow TW_\Upsilon$ on the tangent bundle TW_Υ above W_Υ.

Because of the SMD theorem it is possible that the flow on the component Δ of W_τ at time τ could be exotic. Such a flow could induce changes in beliefs about the future sufficient to cause discontinuities or bifurcations in the preference field, H. Since Smale's Theorem indicates that structural stability is non generic, it is unlikely that the flow generated by H would be structurally stable. One the other hand, there may be conditions under which the field H is half open. If indeed, H is half open on TW_Υ, then there will exist a *local social direction gradient*, $d : W_\Upsilon \rightarrow T^*W_\Upsilon$ with the property that $d((x)(v) > 0$ for every $v \in H(x)$, at every $x \in W_\Upsilon$. We now consider a model such that H may permit a local social direction gradient.

6 Stochastic Choice

To construct such a social preference field, we first consider political choice on a compact set of proposals. This model is an extension of the standard multiparty stochastic model, modified by inducing asymmetries in terms of the preferences of voters.

We define a stochastic electoral model, $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$,which utilizes sociodemographic variables and voter perceptions of character traits. For this model we assume that voter i utility is given by the expression

$$\begin{aligned} u_{ij}(x_i, z_j) &= \lambda_j + \mu_j(z_j) + (\theta_j \cdot \eta_i) + (\alpha_j \cdot \tau_i) - \beta\|x_i - z_j\|^2 + \varepsilon_j. \quad (1) \\ &= [u^*_{ij}(x_i, z_j)] + \varepsilon_j \quad (2) \end{aligned}$$

The points $\{x_i \in W_\mathbb{D} : i \in N = \{1, ...n\}\}$ are the preferred policies of the voters in the political or policy space $W_\mathbb{D}$, and $\mathbf{z} = \{z_j \in W_\mathbb{D} : j \in Q = \{1, ...q\}\}$ are the positions of the agents/ candidates. The term $\|x_i - z_j\|$ is simply the Euclidean distance between x_i and z_j. The error vector $(\varepsilon_{1.}, .., \varepsilon_j, .., \varepsilon_q)$ is distributed by the iid Type I extreme value distribution, as assumed in empirical multinomial logit estimation (MNL). The symbol $\boldsymbol{\theta}$ denotes a set of k -vectors $\{\theta_j : j\epsilon Q\}$ representing the effect of the k different sociodemographic parameters (class, domicile, education, income, religious orientation, etc.) on voting for agent j while η_i is a k-vector denoting the i^{th} individual's relevant "sociodemographic" characteristics. The compositions $\{(\theta_j \cdot \eta_i)\}$ are scalar products, called the *sociodemographic valences* for j. These scalar terms characterize the various types of the voters.

The terms $\{(\alpha_j \cdot \tau_i)\}$ are scalars giving voter i's perceptions and beliefs. These can include perceptions of the character traits of agent j, or beliefs about the state of the economy, etc. We let $\boldsymbol{\alpha} = (\alpha_q, ...\alpha_1)$. A *trait score* can be obtained by factor analysis from a set of survey questions asking respondents about the traits of the agent, including 'moral', 'caring', 'knowledgable', 'strong', 'honest', 'intelligent', etc. The perception of traits can be augmented with voter perception of the state of the economy, etc. in order to examine how anticipated changes in the economy affect each agent's electoral support.

The *intrinsic* or *exogenous* valence vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ gives the general perception of the quality of the various candidates, $\{1, ..., q\}$. This vector satisfies $\lambda_q \geq \lambda_{q-1} \geq \cdots \geq \lambda_2 \geq \lambda_1$, where $(1, ..., q)$ label the candidates, and λ_j is the intrinsic valence of agent or candidate j. In empirical multinomial logit models, the valence vector $\boldsymbol{\lambda}$ is given by the intercept terms for each agent. Finally $\{\mu_j(z_j)\}$ represent the *endogenous* valences of the candidates. These valences depend on the positions $\{z_j \in W_{\mathbb{D}} : j \in Q\}$ of the agents.

Partial models are (i) $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\theta})$, called *sociodemographic*, with only intrinsic valence and sociodemographic variables (ii) $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, called *pure spatial,* (iii) $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \beta)$, called *joint* (iv) $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$, called *joint with traits.*

In all models, the probability that voter i chooses agent j, when party positions are given by $\mathbf{z}$ is:

$$\rho_{ij}(\mathbf{z}) = \Pr[[u_{ij}(x_i, z_j) > u_{il}(x_i, z_l)], \text{ for all } l \neq j].$$

A *local Political Nash equilibrium* (LNE) is a vector, $\mathbf{z}$, such that each agent, j, has chosen z_j to locally maximize the expectation $\frac{1}{n}\Sigma_i \rho_{ij}(\mathbf{z})$.

The type I extreme value distribution,Ψ, has a cumulative distribution with the closed form

$$\Psi(h) = \exp[-\exp[-h]],$$

while its pdf has variance $\frac{1}{6}\pi^2$.

With this distribution it follows, for each voter i, and agent, j, that

$$\rho_{ij}(\mathbf{z}) = \frac{\exp[u^*_{ij}(x_i, z_j)]}{\sum_{k=1}^{q} \exp u^*_{ik}(x_i, z_k)}. \tag{3}$$

This game is an example of what is known as a *Quantal response game* (McKelvey and Palfrey, 1995; McKelvey and Patty, 2006; Levine and Palfrey, 2007). Note that the utility expressions $\{u^*_{ik}(x_i, z_k)\}$ can be estimated from surveys that include vote intentions.

In more complex forms of this game it is possible to model the choice of each voter whether or not to vote. This will depend on the voter estimates of the probability of being *pivotal* (casting the deciding vote for a candidate).

Once the voter probabilities over a given set $\mathbf{z} = \{z_j : j \epsilon Q\}$ are computed , then estimation procedures allow these probabilities to be computed for each possible finite set. This allows the determination and proof of existence of *local Nash equilibrium* (LNE), namely a vector, $\mathbf{z}^*$, such that each agent, j, chooses z^*_j to locally maximize its expected vote share, given by the expectation $V_j((\mathbf{z}^*) = \frac{1}{n}\Sigma_i \rho_{ij}(\mathbf{z}^*)$.

It can be shown that the first order condition for a LNE is that the *marginal electoral pull* at $\mathbf{z}^* = (z^*_1, .., z^*_j, ..z^*_q)$ is zero. For candidate j, this is defined to be

$$\frac{d\mathcal{E}_j^*}{dz_j}(z_j^*) = \left[z_j^{el} - z_j^*\right]$$
$$\text{where } z_j^{el} \equiv \sum_{i=1}^{n} \varpi_{ij} x_i$$

is the *weighted electoral mean* of candidate j.

Here the weights $\{\varpi_{ij}\}$ are individual specific, and defined at the vector $\mathbf{z}^*$ by:

$$[\varpi_{ij}] = \left[\frac{[\rho_{ij}(\mathbf{z}^*) - \rho_{ij}(\mathbf{z}^*)^2]}{\sum_{k\in N}[\rho_{kj}(\mathbf{z}^*) - \rho_{kj}(\mathbf{z}^*)^2]}\right] \qquad (4)$$

Because the candidate utility functions $\{V_j : W_{\mathbb{D}} \rightarrow \mathbb{R}\}$ are differentiable, the second order condition on the Hessian of each V_j at $\mathbf{z}^*$ can then be used to determine whether $\mathbf{z}^*$ is indeed an LNE. Proof of existence of such an LNE will then follow from some version of the Choice Theorem. For example, in the model $\mathbb{M}(\boldsymbol{\lambda}, \beta)$, all weights are equal to $\frac{1}{n}$, so the electoral mean $\frac{1}{n}\sum x_i$ satisfies the first order condition, as suggested by Hinich (1977). For example, Table 1 gives the estimates for the conditional logit model for the 2008 US presidential election, based on the estimated positions given in Figure 12.

Obviously, the full model $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$ given in column 4 has superior loglikelihoods to the other partial models. Given the estimated parameters for the model $\mathbb{M}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \boldsymbol{\alpha}, \beta)$, the theory implies convergence of both candidates to the electoral mean, contradicting Figure 12.

To put together the political and economic models, we now consider, for each individual i at time τ and state $x \in W_\tau$ a stochastic preference cone $H_i^*(x) \subset T_x W_\tau$. That is, let $\{du_i : W_\tau \rightarrow T^* W_\tau : i \in N_\tau\}$ define the set of utility direction gradients for the society, N_τ, at time τ. Let $H_i^*(x)$ be a probability distribution over the set of vectors $\{v \in T_x W_\tau : du_i(x)(v) > 0\}$. As in a QRE, each individual chooses an action in $H_i^*(x)$, and this generates a curve $c_{i,t} : [-1. + 1] \twoheadrightarrow W_{i,\tau}$ where $W_{i,\tau}$ is the i^{th} individual's a commodity space at time τ.

Aggregating these curves gives a path $c_{N,t} : [-1. + 1] \twoheadrightarrow W_\Upsilon$ for the society in the interval $\Upsilon = [\tau, \tau + 1]$. The aggregate social path in W_Υ will of course involve changes in the price vector, along the lines discussed above. Because the individual actions are stochastic,, there is no guarantee that the the path of price changes is a Gaussian random walk. However, the path may permit a social direction gradient, $d : W_\Upsilon \rightarrow T^* W_\Upsilon$, as suggested above, but defined over an interval, $\Upsilon = [\tau, \tau + 1]$ from one election at time τ to the next election at time $\tau + 1$. The jumps occasioned by by a policy switch at election time, τ, may very well induce changes in the social preference cone $H_{\tau+1}^*$.

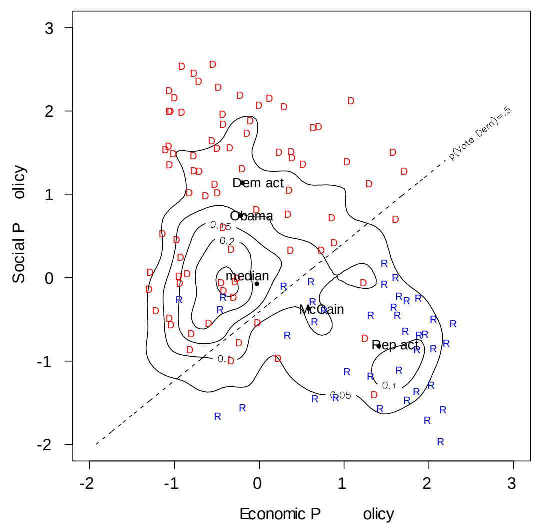

Figure 12: Estimated voter distribution, candidate positions and activist positions in 2008. (Democrat activists are red and Republican activists are blue.)

7 Conclusion.

As this review has been at pains to point out, all the equilibrium theorems are based on assumptions of convexity and/or acyclicity, and the stability of equilibria, rather than singularity. It has often been alleged that the basis for political irrationality can be found in some form of the chaos theorem applied to political choice. However, QRE models of political choice give statistically significant estimates of such choices. The existence of LNE in such models suggest that elections are intelligible and non-chaotic. It thus would appear that irrationality, in the form of bubble collapse and the like, may be due to the consequence of the SMD theorem or the existence of unstable singularities.

This paper suggests that the models that have been deployed in political economy are mathematically unsophisticated. Instead of focusing on equilibrium analysis, based on some version of the Brouwer fixed point theorem, and thus on the assumption that the social world is simply a ball, political economy could consider the nature of the dynamical path of change, and utilize notions from the qualitative theory of dynamical systems.

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Table 1. $\beta-$Spatial Conditional Logit Models for USA 2008

	$\mathbb{M}(\boldsymbol{\lambda},\beta)$	$\mathbb{M}(\boldsymbol{\lambda},\boldsymbol{\alpha},\beta)$	$\mathbb{M}(\boldsymbol{\lambda},\boldsymbol{\theta},\beta)$	Full:$\mathbb{M}(\boldsymbol{\lambda},\boldsymbol{\theta},\boldsymbol{\alpha},\beta)$
McCain valence λ	-0.84^{***}	-1.08^{***}	-2.60^{**}	-3.58^{***}
	(0.11)	(0.13)	(0.93)	(1.05)
Distance β	0.85^{***}	0.78^{***}	0.86^{***}	0.83^{***}
	(0.06)	(0.07)	(0.07)	(0.08)
McCain traits		1.30^{***}		1.36^{***}
		(0.17)		(0.19)
Obama traits		-1.02^{***}		-1.16^{***}
		(0.15)		(0.18)
Age			-0.01	-0.01
			(0.01)	(0.01)
Female			0.29	0.44
			(0.23)	(0.26)
Black			-4.16^{***}	-3.79^{***}
			(1.10)	(1.23)
Hispanic			-0.55	-0.23
			(0.41)	(0.45)
Education			0.15^{*}	0.22^{***}
			(0.06)	(0.06)
Income			0.03	0.01
			(0.02)	(0.02)
Working Class			-0.54^{*}	-0.70^{**}
			(0.24)	(0.27)
South			0.36	-0.02
			(0.24)	(0.27)
Observations	781	781	781	781
log Like	-298.63	-243.14	-250.25	-206.88
AIC	601.27	494.28	520.50	437.77
BIC	610.59	512.92	567.11	493.69

Standard errors in parentheses. $^{*}pob < .05$; $^{**}prob < .01$; $^{***}prob < .001$

Vote for Obama is the baseline outcome

References

[1] D. Acemoglu, J. Robinson, Economic Origins of Dictatorship and Democracy, Cambridge University Press, (Cambridge, 2006).

[2] ———, Oligarchic versus democratic societies. J Eur Econ Assoc, 6 (2008), 1–44.

[3] G. A. Akerlof, R. J. Shiller, Animal spirits, Princeton University Press, (Princeton, 2009).

[4] C. Aliprantis, D. Brown, Equilibria in markets with a Riesz space of commodities. J Math Econ, 11 (1983), 189–207.

[5] K. Arrow, Rationality of self and others in an economic system. J Bus 59 (1986) S385–S399.

[6] K. Arrow, G. Debreu, Existence of an equilibrium for a competitive economy. Econometrica, 22 (1954) 265–90.

[7] R. J. Aumann, Agreeing to disagree, Annals Stat, 4 (1976) 1236–1239.

[8] R. Barbera, The Cost of Capitalism: Understanding Market Mayhem, New York, McGraw Hill (New York, 2009).

[9] T. Bergstrom, The existence of maximal elements and equilibria in the absence of transitivity. Typescript: University of Michigan, (1976).

[10] T. Bergstrom, in W. Neuefeind, R. Riezman (Eds.), Economic Theory and International Trade, Springer (Berlin, 1992).

[11] S. Bikhchandani, D. Hirschleifer, I. Welsh, A theory of fads, fashion, custom, and cultural change as information cascades. J Polit Econ 100 (1992) 992–1026.

[12] F. Black, M. Scholes, The pricing of options and corporate liabilities. J Polit Econ 81 (1973), 637–654.

[13] A. Borglin, H.Keiding, Existence of equilibrium actions and of equilibrium: A note on the new existence theorems. J Math Econ 3 (1976), 313–316.

[14] R. Brown, The Lefschetz Fixed Point Theorem. Scott and Foresman (Glenview, Illinois, 1971).

[15] W. H. Calvin, The Ascent of Mind, Bantam (New York, 1991).

[16] ________, A brain for all seasons: human evolution and abrupt climate change, Chicago University Press, (Chicago, Illinois, 2006).

[17] J. Cassidy, How markets fail: The logic of economic calamities, Farrar, Strauss and Giroux (New York, 2009)

[18] G. Chichilinsky, The foundations of statistics with black swans.Math Soc Sci, 59 (2010), 184-192.

[19] D. R. J. Chillingsworth, Differential topology with a view to applications, Pitman, (London, 1976).

[20] P. Collier, Wars, guns and votes. Harper, (New York, 2009).

[21] N. Condorcet, Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. Paris: Imprimerie Royale, translated in part in Condorcet: Foundations of social choice and political theory, I. McLean, F. Hewitt, Edward Elgar, Aldershot, (UK, [1785], 1994).

[22] N. Condorcet, Esquisse d'un tableau historique des progrès de l'esprit humain: Sketch for an historical picture of the progress of the human mind, Weidenfeld, (London, [1795], 1955).

[23] N. Copernicus, De revolutionibus orbium coelestium (On the revolutions of the celestial spheres), trans A. Duncan. Davis and Charles, (Newton Abbot UK, [1543], 1976).

[24] D. Coyle, The Economics of Enough, Princeton University Press, (Princeton, 2011).

[25] R. Dawkins, The selfish gene. Oxford University Press, (Oxford, 1976).

[26] ________, The greatest show on earth, Random House, (London, 2009).

[27] G. Debreu, Excess demand functions. J Math Econ 1, (1974), 15–21.

[28] ________, The application to economics of differential topology and global analysis: regular differentiable economies. Am Econ Rev 66, (1976), 280–287.

[29] N. Eldredge, S. J. Gould, Punctuated equilibria: An alternative to phyletic gradualism, in T. J. M. (Ed.), Models in paleobiology, Norton, (New York, 1972).

[30] K. Fan, A generalization of Tychonoff's fixed point theorem. Math Annalen, 42 (1961), 305–310.

[31] J. Fox, The myth of the rational market, Harper, (New York, 2009).

[32] M. Freedman, The topology of four manifolds. J Differ Geom 17, (1982), 357–454.

[33] G. Galilei, The sidereal messenger, trans. and ed., A.Van Helden. University of Chicago Press, (Chicago, [1610], 1992).

[34] ________, History and demonstrations concerning sunspots, in The essential Galileo, trans. and ed., M. Finocchiaro. Hackett, (Indianapolis, [1613], 2008).

[35] ________, Dialogue concerning the two chief world systems, Ptolemaic and Copernican, trans. and ed., S. Drake, University of California Press, (Berkeley, [1632], 1967).

[36] ________, Two new sciences. Trans. and ed. S. Drake, University of Wisconsin Press, (Madison, WI. 1974)

[37] P. Galison, Einstein's clocks, Poincaré's maps, New York, (Norton, 2003).

[38] A. Gamble, Hayek: The iron cage of liberty, Westview, (Boulder, CO., [1946],1996).

[39] J. Gleick, Chaos: Making a new science, Viking, (New York, 1987).

[40] M. Golubitsky, V. Guillemin, Stable mappings and their singularities, Springer, (Berlin, 1973).

[41] S. J. Gould, Wonderful life, Norton, (New York, 1989).

[42] ________, Full house, Harmony Books, (New York, 1996).

[43] F. Hahn, On the notion of equilibrium in economics, Cambridge University Press, (Cambridge, 1973).

[44] S. W. Hawking, G. F. R., Ellis, The large scale structure of spacetime, Cambridge University Press, (Cambridge, 1973).

[45] F. A. Hayek, The use of knowledge in society. Am Econ Rev, 55, (1945), 519–530.

[46] M. J. Hinich, Equilibrium in spatial voting: The median voter theorem is an artifact. J Econ Theory, 16, (1977), 208–219.

[47] ________, Risk when some states are low-probability events. Macroecon Dyn, 7, (2003), 636–643.

[48] M. Hirsch, Differential topology, Springer, (Berlin, 1976).

[49] J. Hofbauer, K. Sigmund, The theory of evolution and dynamical systems, Cambridge University Press, (Cambridge, 1988).

[50] J. H. Hubbard, B. H. West, Differential equations: A dynamical systems approach, Springer, (Berlin, 1995).

[51] S. Kauffman, The origins of order, Oxford University Press, (Oxford, 1993).

[52] J. M. Keynes, Treatise on probability. Macmillan, (London, 1921).

[53] ———, The general theory of employment, interest and money, Macmillan, (London, 1936).

[54] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein beweis des fixpunktsatzes fur n-dimensionale simplexe. Fund Math, 14 (1929) 132–137.

[55] H. Konishi, Equilibrium in abstract political economies: with an application to a public good economy with voting. Soc Choice Welfare, 13 (1996) 43–50.

[56] P. Krugman, How did economists get it so wrong? New York Times, September 6, 2009.

[57] M. Kurz, M. Motolese, Endogenous uncertainty and market volatility. Econ Theor, 17 (2001) 97–544.

[58] P. S. Laplace, Théorie analytique des probabilités, Gauthiers-Villars, (Paris, 1812).

[59] ———, Essai philosophique sur les probabilités, Gauthiers-Villars, (Paris. 1814). A philosophical essay on probabilities, trans by F. Truscott and F. Emory, intro, by E. Bell. Dover, (New York, 1951)

[60] D. Levine, T. R. Palfrey, The paradox of voter participation. Am Polit Sci Rev, 101 (2007) 143–158.

[61] E. N. Lorenz, The statistical prediction of solutions of dynamical equations, Proc Internat Sympos Numerical Weather Prediction, Tokyo, (1962) 629–635..

[62] ———, Deterministic non periodic flow. J Atmos Sci, (1963) 130–141.

[63] ———, The essence of chaos, University of Washington Press, (Seattle, 1993).

[64] T. Malthus, An essay on the principle of population and a summary view of the principle of population, edited and with an introduction by A. Flew, Penguin, (Harmondsworth, UK, [1798], [1830] 1970).

[65] B. Mandelbrot, R. Hudson, The (mis)behavior of market,. Perseus, (New York, 2004).

[66] R. Mantel, On the characterization of aggregate excess demand. J Econ Theory, 7 (1972) 348–353.

[67] H. Margolis, It started with Copernicus: How turning the world inside out led to the scientific revolution, McGraw Hill, (New York, 2002).

[68] A. Mas-Colell, The theory of general economic equilibrium, Cambridge University Press, (Cambridge, 1985).

[69] R. D. McKelvey, Intransitivities in multidimensional voting models and some implications for agenda control. J Econ Theory, 12 (1976) 472–482.

[70] ________, General conditions for global intransitivities in formal voting models. Econometrica, 47 (1979) 1085–1112.

[71] R. D. McKelvey, J. W. Patty, A theory of voting in large elections. Games Econ Behav, 57 (2006)155–180.

[72] R. D. McKelvey, T. R. Palfrey, Quantal response equilibria in normal form games. Games Econ Behav, 10 (1995) 6–38.

[73] ________, Quantal response equilibria in extensive form games. Exp Econ, 1 (1998) 9–41.

[74] R. D. McKelvey, N. Schofield, Generalized symmetry conditions at a core point. Econometrica, 55 (1987) 923–933.

[75] R. C. Merton, Theory of rational option pricing. Bell J Econ Manag Sci, 4 (1973) 141–183.

[76] H. Minsky, John Maynard Keynes, Columbia University Press, (New York, 1975).

[77] ________, Stabilizing an unstable economy. Yale University Press, (New Haven, 1986).

[78] K. Nakamura, The vetoers in a simple game with ordinal preference.Int J Game Theory, 8 (1979) 55–61.

[79] I. Newton, Philosophiae naturalis principia mathematica, trans. and ed. , with an introduction by Motte, A. Prometheus Books, (Amherst, NY, [1687],1995).

[80] D. O'Shea, The Poincaré conjecture: In search of the shape of the universe, Walker, (New York, 2007).

[81] M. Peixoto, Structural stability on two-dimensional manifolds. Topology, 1 (1962) 101-120.

[82] E. Penn, A model of far-sighted voting. Am J Polit Sci, 53 (2009) 36-54.

[83] R. Penrose, Cycles of Time, Bodney Head, (London, 2010).

[84] I. Peterson, Newton's clock: Chaos in the solar system, Freeman, (New York, 1993).

[85] H. Poincaré, Calcul des probabilités, Gauthiers-Villars, (Paris, 1896).

[86] ________, New methods of celestial mechanics, Trans. and ed. by D. Goroff, D. American Institute of Physics, (New York, 1993).

[87] ________, Science and method. Cosimo Classics, (London, [1908], 2007).

[88] C. C. Pugh, Real mathematical analysis, Springer, (Berlin, 2002).

[89] T. Rader, Theory of general economic equilibrium. Academic Press, (New York, 1972).

[90] D. Saari, Price dynamics, social choice, voting methods, probability and chaos, in D. Aliprantis, O. Burkenshaw, and N. J. Rothman (Eds.), Lecture notes in economics and mathematical systems, No. 244. Springer, (Berlin, 1985).

[91] ________, A chaotic exploration of aggregation paradoxes. SIAM Rev 37, (1985) 37–52.

[92] ________, Mathematical complexity of simple economics. Notes Am Math Soc 42, (1995) 222-230.

[93] ________, The Generic Existence of a Core for q-Rules. Econ Theor 9, (1997) 219–260.

[94] ________, Decisions and elections: Explaining the unexpected. Cambridge University Press, (Cambridge, 2001).

[95] ________, Chaotic Elections, American Mathematical Society, (Providence, RI, 2001).

[96] ———, Collisions, rings and other Newtonian N-body problems. American Mathematical Society, (Providence, RI., 2005).

[97] ———, Disposing dictators, demystifying voting paradoxes, Cambridge University Press, (Cambridge, 2008).

[98] D. Saari, C. P. Simon, Singularity theory of utility mappings. J Math Econ 4, (1977) 217–251.

[99] ———, Effective price mechanisms. Econometrica 46, (1978) 1097–1125.

[100] D. Saari, Z. Xia, The existence of oscilatory and superhyperbolic motion in Newtonian systems. J Differ Equations 82, (1985) 342–355.

[101] L. J. Savage, The foundations of statistics, Wiley, (New York, 1954).

[102] H. Scarf, Some examples of global instability of the competitive equilibrium. Int Econ Rev 1, (1960) 157–172.

[103] N. Schofield, Instability of simple dynamic games. Rev Econ Stud 45, (1978) 575–594.

[104] ———, Existence of equilibrium on a manifold. Math Oper Res 9, (1984) 545–557.

[105] ———, Social Equilibrium and Cycles on Compact Sets. J Econ Theory 33, (1984) 59–71.

[106] ———, 2006. Architects of political change:Constitutional quandaries and social choice theory. Cambridge University Press, (Cambridge, 2006).

[107] ———, Equilibria in the spatial stochastic model of voting with party activists. Rev Econ Design 10, (2006) 183-203.

[108] ———, Social orders. Soc Choice Welf 34, (2010) 503-536.

[109] ——— and I. Sened, Multiparty democracy. Cambridge University Press, (Cambridge, 2006).

[110] ——— *et al.* Application of a Theorem in Stochastic Models of Elections.I J Math Math Sci, Article ID 562813, 30 pages, (2010) doi:10.1155/2010/562813.

[111] R. Shiller, The new financial order, Princeton University Press, (Princeton, 2003).

[112] ———, Irrational exuberance, Princeton University Press, (Princeton, 2005).

[113] R. Skidelsky, John Maynard Keynes: Hopes betrayed 1883-1920, Macmillan, (London, 1983).

[114] ———, John Maynard Keynes: The economist as saviour, 1920-1937. Macmillan, (London, 1992).

[115] ———, John Maynard Keynes: Fighting for Britain, 1937-1946. Macmillan, (London, 2000).

[116] S. Smale, Structurally stable systems are not dense. Am J Math 88, (1966) 491–496.

[117] ———, Global analysis and economics I: Pareto optimum and a generalization of Morse theory, in M. Peixoto, (Ed.) Dynamical systems.Academic Press, (New York, 1973).

[118] ———, Global analysis and economics IIA: Extension of a theorem of Debreu. J Math Econ 1, (1974) 1–14.

[119] ———, Dynamics in general equilibrium theory. Am Econ Rev 66, (1976) 288–294.

[120] H. Sonnenschein, Market excess demand functions. Econometrica 40, (1972) 649–663.

[121] N. Stern The economics of climate change, Cambridge University Press, (Cambridge, 2007).

[122] J. Strnad, The structure of continuous-valued neutral monotonic social functions. Soc Choice Welfare 2, (1985) 181–195.

[123] N. Taleb, The black swan. Random, (New York, 2007).

[124] R. Thom, Structural stability and morphogenesis. Benjamin, (Reading, MA., 1975).

[125] M. Walker, On the existence of maximal elements. J Econ Theory 16, (1977) 470–474.

[126] S. Wiggins, Global bifurcations and chaos, Springer, (Berlin, 1988).

[127] N. Yannelis, N. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces. J Math Econ 12, (1983) 233–245.

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