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ORTHOGONAL STABILITY OF A MIXED TYPE ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION

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Abstract

In this paper, the authors investigate the orthogonal stability of a mixed type additive and quadratic functional equation of the form

$$f(x+2y)+f(x-2y)+4f(x) = 3[f(x+y)+f(x-y)]+f(2y)-2f(y) \quad (0.1)$$

with $x \perp y$, where $\perp$ is orthogonality in the sense of Rätz.

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1 Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [21] in 1940, concerning the stability of group homomorphisms.

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by D.H. Hyers [9] under the assumption that G_1 and G_2 are Banach spaces. In 1951 and in 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by T. Aoki [1] and Th.M. Rassias [15]. The Hyers - Ulam - Aoki - Rassias stability originates from this historical backgrounds see [2, 3, 6, 16, 20] . In 1982, J.M. Rassias [13, 14] provided a generalizations of the Hyers stability theorem which allows the Cauchy difference to be bounded. The stability phenomenon that was proved by J.M. Rassias is called the Ulam - Gavruta - Rassias stability by [18]. Very recently J.M. Rassias [19] introduced a new concept on stability called JMRassias Mixed type product-sum of powers of norms stability.

There are several orthogonality notations on a real normed space are available. But here, we present the orthogonality concept introduced by J.Rätz[17].This is given in the following definition.

Definition 1.1. *[17] A vector space X is called an orthogonality vector space if there is a relation $x \perp y$ on X such that*
(i) $x \perp 0$, $0 \perp x$ *for all* $x \in X$;
(ii) if $x \perp y$ *and* $x, y \neq 0$, *then* x, y *are linearly independent;*
(iii) $x \perp y$, $ax \perp by$ *for all* $a, b \in \mathbb{R}$;
(iv) if P is a two-dimensional subspace of X ; then
(a) for every $x \in P$ there exists $0 \neq y \in P$ such that $x \perp y$;
(b) there exists vectors $x, y \neq 0$ such that $x \perp y$ and $x + y \perp x - y$.
Any vector space can be made into an orthogonality vector space if we define $x \perp 0, 0 \perp x$ for all x and for non zero vector x, y define $x \perp y$ iff x, y are linearly independent. The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all x, $y \in X$. The pair $(X, \perp)$ is called anorthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.

The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), x \perp y \tag{1.1}$$

in which $\perp$ is an abstract orthogonally was first investigated by S.Gudder and D. Strawther [8]. R. Ger and J. Sikorska discussed the orthogonal stability of the equation (1.1) in [7].

Definition 1.2. *Let X be an orthogonality space and Y be a real Banach space. A mapping $f : X \to Y$ is called orthogonally quadratic if it satisfies the so called orthogonally Euler-Lagrange quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in X$ with $x \perp y$. The orthogonality Hilbert space for the orthogonally quadratic functional equation (1.2) *was first investigated by F. Vajzovic [22].*

Several other functional equations and its stability in orthogonality spaces was discussed in [4, 10, 11, 12, 18, 19]. Fridoum Moradlou, Hamid Vaezi and G. Zamani Eskandani [5], obtained the general solution and the generalized Hyers-Ulam-Rassias stability of a functional equation deriving from quadratic and additive functions of the form

$$f(x+2y) + f(x-2y) + 4f(x) = 3[f(x+y) + f(x-y)] + f(2y) - 2f(y). \tag{1.3}$$

In this paper, the authors discussed the orthogonal stability of a mixed type additive and quadratic functional equation of the form

$$f(x+2y) + f(x-2y) + 4f(x) = 3[f(x+y) + f(x-y)] + f(2y) - 2f(y). \tag{1.4}$$

with $x \perp y$ and investigates its Hyers - Ulam - Aoki - Rassias stability of (1.4), where $\perp$ is orthogonality in the sense of Ratz. Note that the function $f(x) = ax + bx^2$ is the solution of the functional equation (1.4).

Definition 1.3. *A mapping $f : A \to B$ is called orthogonal additive and quadratic respectively, if it satisfies the mixed type functional equation (1.4) for all $x, y \in A$, with $x \perp y$ where A be an orthogonality space and B be a real Banach space.*

Through out this paper, let $(A, \perp)$ denote an orthogonality normed space with norm $\| \cdot \|_A$ and $(B, \| \cdot \|_B)$ is a Banach space. We define

$$D\, f\,(x,y) = f(x+2y)+f(x-2y)+4f(x)-3[f(x+y)+f(x-y)]-f(2y)+2f(y).$$

for all $x, y \in A$, with $x \perp y$.

2 Hyers - Ulam - Aoki - Rassias Stability of (1.4)

In this section, we present the Hyers - Ulam - Aoki - Rassias stability of the orthogonal functional equation (1.4).

Theorem 2.1. *Let α and $s(s<1)$ be nonnegative real numbers. Let $f_a : A \to B$ be an odd mapping satisfying*

$$\|D\ f_a\,(x,y)\|_B \leq \alpha\left\{\|x\|_A^s + \|y\|_A^s\right\} \tag{2.1}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : A \to B$ such that

$$\|f_a(y) - L(y)\|_B \leq \frac{\alpha}{(2-2^s)}\,\|y\|_A^s \tag{2.2}$$

for all $y \in A$. The function $L(y)$ is defined by

$$L(y) = \lim_{n\to\infty} \frac{f_a(2^n y)}{2^n} \tag{2.3}$$

for all $y \in A$.

Proof. Replacing (x, y) by $(0, 0)$ in (2.1), we get $f_a(0) = 0$. Setting (x, y) by $(0, y)$ in (2.1), we obtain

$$\|f_a(2y) - 2f_a(y)\|_B \leq \alpha\,\|y\|_A^s \tag{2.4}$$

for all $y \in A$. Since $y \perp 0$, we have

$$\left\|\frac{f_a(2y)}{2} - f_a(y)\right\|_B \leq \frac{\alpha}{2}\,\|y\|_A^s \tag{2.5}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 2 in (2.5) and summing resulting inequality with (2.5), we arrive

$$\left\|\frac{f_a(2^2 y)}{2^2} - f_a(y)\right\|_B \leq \frac{\alpha}{2}\left\{1 + \frac{2^s}{2}\right\}\|y\|_A^s \tag{2.6}$$

for all $y \in A$. In general, using induction on a positive integer n, we obtain that

$$\begin{aligned}\left\|\frac{f_a(2^n y)}{2^n} - f_a(y)\right\|_B &\leq \frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{2^{s\,k}}{2^k}\,\|y\|_A^s \\ &\leq \frac{\alpha}{2}\sum_{k=0}^{\infty}\frac{2^{s\,k}}{2^k}\,\|y\|_A^s\end{aligned} \tag{2.7}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f_a(2^n y)/2^n\}$, replace y by $2^m y$ and divide by 2^m in (2.7), for any $n, m > 0$, we obtain

$$\begin{aligned}\left\|\frac{f_a\,(2^n 2^m y)}{2^{(n+m)}} - \frac{f_a(2^m y)}{2^m}\right\|_B &= \frac{1}{2^m}\left\|\frac{f_a\,(2^n 2^m y)}{2^n} - f_a\,(2^m y)\right\|_B \\ &\leq \frac{1}{2^m}\frac{\alpha}{2}\ \sum_{k=0}^{n-1}\frac{2^{s\,k}}{2^k}\,\|2^m y\|_A^s \\ &\leq \frac{\alpha}{2}\ \sum_{k=0}^{\infty}\frac{1}{2^{(1-s)(k+m)}}\,\|y\|_A^s\,.\end{aligned} \tag{2.8}$$

As $s < 1$, the right hand side of (2.8) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f_a(2^n y)/2^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $L : A \to B$ such that

$$L(y) = \lim_{n\to\infty} \frac{f_a(2^n y)}{2^n} \quad \forall \quad y \in A.$$

Letting $n \to \infty$ in (2.7), we arrive the formula (2.2) for all $y \in A$. To prove L satisfies (1.4), replace (x, y) by $(2^n x, 2^n y)$ in (2.1) and divide by 2^n, it follows that

$$\frac{1}{2^n}\Big\| f_a(2^n(x+2y)) + f_a(2^n(x-2y)) + 4f_a(2^n x) - 3[f_a(2^n(x+y)) + f_a(2^n(x-y))] - f_a(2^n\ 2y) + 2f_a(2^n y)\Big\|_B \le \frac{\alpha}{2^n}\left\{\|2^n x\|_A^s + \|2^n y\|_A^s\right\}.$$

Taking limit as $n \to \infty$ in the above inequality, we get

$$L(x+2y) + L(x-2y) + 4f(x) = 3[L(x+y) + L(x-y)] + L(2y) - 2L(y)$$

for all $x, y \in A$ with $x \perp y$. Therefore $L : A \to B$ is an orthogonally additive mapping which satisfies (1.4). To prove the uniqueness of L, let L' be another orthogonally additive mapping satisfying (1.4) and the inequality (2.2). Then

$$\begin{aligned}\|L(y) - L'(y)\|_B &= \frac{1}{2^n}\|L(2^n y) - L'(2^n y)\|_B \\ &\le \frac{1}{2^n}\left(\|L(2^n y) - f_a(2^n y)\|_B + \|f_a(2^n y) - L'(2^n y)\|_B\right) \\ &\le \frac{2\ \alpha}{[2-2^s]}\frac{1}{2^{n(1-s)}}\|y\|_A^s \\ &\to\ 0 \quad \text{as } n \to \infty\end{aligned}$$

for all $y \in A$. Therefore L is unique. This completes the proof of the theorem. □

Now we will provide an example to illustrate that the functional equation (1.4) is not stable for $s = 1$ in Theorem 2.1.

Example 2.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\begin{aligned} |f(x+2y)+f(x-2y)+4f(x)-3f(x+y)-3f(x-y) \\ -f(2y)+2f(y)| \leq 30\mu\,(|x|+|y|) \end{aligned} \tag{2.9}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ such that

$$|f(x) - A(x)| \leq \gamma|x| \qquad for\ all \quad x \in \mathbb{R}. \tag{2.10}$$

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (2.9).

If $x = y = 0$ then (2.9) is trivial. If $|x| + |y| \geq 1$ then the left hand side of (2.9) is less than 30μ. Now suppose that $0 < |x| + |y| < 1$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \leq |x| + |y| < \frac{1}{2^k}, \tag{2.11}$$

so that $2^{k-1}x < 1,\ 2^{k-1}y < 1$, and consequently

$$\begin{aligned} 2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(x+y), 2^{k-1}(x-y), , \\ 2^{k-1}(x+2y), 2^{k-1}(x-2y), 2^{k-1}(2y) \in (-1,1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} 2^n(x), 2^n(y), 2^n(x+y), 2^n(x-y) \\ 2^n(x+2y), 2^n(x-2y), 2^n(2y) \in (-1,1) \end{aligned}$$

and

$$\begin{aligned} \phi(2^n(x+2y) + \phi(2^n(x-2y)) + 4\phi(2^n(x)) - 3\phi(2^n(x+y)) - 3\phi(2^n(x-y)) \\ - \phi(2^n(2y)) + \phi(2^n y) = 0 \end{aligned}$$

for $n = 0, 1, \dots, k-1$. From the definition of f and (2.11), we obtain that

$$\begin{aligned}
&|f(x+2y) + f(x-2y) + 4f(x) - 3f(x+y) - 3f(x-y) - f(2y) + 2f(y)| \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n} \Big| \phi(2^n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(x)) - 3\phi(2^n(x+y)) \\
&\qquad\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} \Big| \phi(2n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(4x)) - 3\phi(2^n(x+y)) \\
&\qquad\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n} 15\mu = 2\mu \times \frac{15}{2^k} = 30\mu\left(|x| + |y|\right).
\end{aligned}$$

Thus f satisfies (2.9) for all $x, y, z \in \mathbb{R}$ with $0 < |x| + |y| < 1$.

We claim that an additive functional equation (1.4) is not stable for $s = 1$ in Theorem 2.1. Suppose on the contrary that there exist an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ satisfying (2.10). Since f is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f(x)| \le (\gamma + |c|)\,|x|. \tag{2.12}$$

But we can choose a positive integer m with $m\mu > \gamma + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \dots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{\mu 2^n x}{2^n} = m\mu x > (\gamma + |c|)\, x$$

which contradicts (2.12). Therefore the additive functional equation (1.4) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality (2.1).

Theorem 2.2. *Let β and $s (s < 2)$ be nonnegative real numbers. Let $f_q : A \to B$ be an even mapping satisfying*

$$\|D\ f_q\,(x,y)\|_B \le \beta\left\{\|x\|_A^s + \|y\|_A^s\right\} \tag{2.13}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $M : A \to B$ such that

$$\|f_q(y) - M(y)\|_B \le \frac{\beta}{4 - 2^s}\,\|y\|_A^s \tag{2.14}$$

for all $y \in A$. The function $M(y)$ is defined by

$$M(y) = \lim_{n\to\infty} \frac{f_q(2^n y)}{4^n} \tag{2.15}$$

for all $y \in A$.

Proof. Replacing (x, y) by $(0, 0)$ in (2.13), we get $f_q(0) = 0$. Setting (x, y) by $(0, y)$ in (2.13), we obtain

$$\|f_q(2y) - 4f_q(y)\|_B \leq \beta \|y\|_A^s \tag{2.16}$$

for all $y \in A$. Since $y \perp 0$, we have

$$\left\|\frac{f_q(2y)}{4} - f_q(y)\right\|_B \leq \frac{\beta}{4} \|y\|_A^s \tag{2.17}$$

for all $y \in A$. Now replacing y by $2y$ and dividing by 4 in (2.17) and summing resulting inequality with (2.17), we arrive

$$\left\|\frac{f_q(2^2 y)}{4^2} - f_q(y)\right\|_B \leq \frac{\beta}{4}\left\{1 + \frac{2^s}{4}\right\} \|y\|_A^s \tag{2.18}$$

for all $y \in A$. In general, using induction on a positive integer n, we obtain that

$$\begin{aligned}\left\|\frac{f_q(2^n y)}{4^n} - f_q(y)\right\|_B &\leq \frac{\beta}{4}\sum_{k=0}^{n-1} \frac{2^{sk}}{4^k} \|y\|_A^s \\ &\leq \frac{\beta}{4}\sum_{k=0}^{\infty} \frac{2^{sk}}{4^k} \|y\|_A^s\end{aligned} \tag{2.19}$$

for all $y \in A$. In order to prove the convergence of the sequence $\{f_q(2^n y)/4^n\}$, replace y by $2^m y$ and divide by 4^m in (2.19), for any $n, m > 0$, we obtain

$$\begin{aligned}\left\|\frac{f_q(2^n 2^m y)}{4^{(n+m)}} - \frac{f_q(2^m y)}{4^m}\right\|_B &= \frac{1}{4^m}\left\|\frac{f_q(2^n 2^m y)}{4^n} - f_q(2^m y)\right\|_B \\ &\leq \frac{1}{4^m}\frac{\beta}{4}\sum_{k=0}^{n-1}\frac{2^{sk}}{4^k}\|2^m y\|_A^s \\ &\leq \frac{\beta}{4}\sum_{k=0}^{\infty}\frac{1}{2^{(2-s)(k+m)}}\|y\|_A^s .\end{aligned} \tag{2.20}$$

As $s < 2$, the right hand side of (2.20) tends to 0 as $m \to \infty$ for all $y \in A$. Thus $\{f_q(2^n y)/4^n\}$ is a Cauchy sequence. Since B is complete, there exists a mapping $M : A \to B$ such that

$$M(y) = \lim_{n\to\infty} \frac{f_q(2^n y)}{4^n} \quad \forall \quad y \in A.$$

Letting $n \to \infty$ in (2.19), we arrive the formula (2.14) for all $y \in A$. To prove M satisfies (1.4) and it is unique the proof is similar to that of Theorem 2.1 $\square$

Now we will provide an example to illustrate that the functional equation (1.4) is not stable for $s = 2$ in Theorem 2.2.

Example 2.2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{4^n} \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\begin{aligned} &|f(x+2y) + f(x-2y) + 4f(x) - 3f(x+y) - 3f(x-y) \\ &\quad - f(2y) + 2f(y)| \leq 5 \times 4^2 \left(|x|^2 + |y|^2\right) \end{aligned} \tag{2.21}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ such that

$$|f(x) - Q(x)| \leq \gamma |x|^2 \qquad \textit{for all} \quad x \in \mathbb{R}. \tag{2.22}$$

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(2^n x)|}{|4^n|} = \sum_{n=0}^{\infty} \frac{\mu}{4^n} = \frac{4\mu}{3}.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (2.21).

If $x = y = 0$ then (2.21) is trivial. If $|x|^2 + |y|^2 \geq \frac{1}{4}$ then the left hand side of (2.21) is less than 20μ. Now suppose that $0 < |x|^2 + |y|^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$\frac{1}{4^{k+1}} \leq |x|^2 + |y|^2 < \frac{1}{4^k}, \tag{2.23}$$

so that $4^{k-1}x^2 < \frac{1}{4}$, $4^{k-1}y^2 < \frac{1}{4}$, and consequently

$$\begin{aligned} &2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(x+y), 2^{k-1}(x-y), , \\ &\qquad 2^{k-1}(x+2y), 2^{k-1}(x-2y), 2^{k-1}(2y) \in (-1, 1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(x), 2^n(y), 2^n(x+y), 2^n(x-y) \\ 2^n(x+2y), 2^n(x-2y), 2^n(2y) \in (-1, 1)$$

and

$$\phi(2^n(x+2y) + \phi(2^n(x-2y)) + 4\phi(2^n(x)) - 3\phi(2^n(x+y)) - 3\phi(2^n(x-y)) \\ - \phi(2^n(2y)) + \phi(2^n y) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (2.23), we obtain that

$$\begin{aligned} &|f(x+2y) + f(x-2y) + 4f(x) - 3f(x+y) - 3f(x-y) - f(2y) + 2f(y)| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{4^n} \Big| \phi(2^n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(x)) - 3\phi(2^n(x+y)) \\ &\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{4^n} \Big| \phi(2n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(4x)) - 3\phi(2^n(x+y)) \\ &\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{4^n} 15\mu = \frac{4\mu}{3} \times \frac{15}{4^k} = 5 \times 4^2 \mu \left(|x|^2 + |y|^2 \right). \end{aligned}$$

Thus f satisfies (2.21) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^2 + |y|^2 < \frac{1}{4}$.

We claim that the quadratic functional equation (1.4) is not stable for $s = 2$ in Theorem 2.2. Suppose on the contrary that there exist a quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ satisfying (2.22). Since f is bounded and continuous for all $x \in \mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.2, Q must have the form $Q(x) = cx^2$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f(x)| \leq (\gamma + |c|)\, |x|^2. \tag{2.24}$$

But we can choose a positive integer m with $m\mu > \gamma + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{4^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)^2}{4^n} = m\mu x^2 > (\gamma + |c|)\, x^2$$

which contradicts (2.24). Therefore the quadratic functional equation (1.4) is not stable in sense of Ulam, Hyers and Rassias if $s = 2$, assumed in the inequality (2.13).

Now we are ready to prove our main theorem.

Theorem 2.3. *Let θ and $s(s < 1)$ be nonnegative real numbers. Let $f : A \to B$ be a mapping satisfying*

$$\|D\ f\,(x, y)\|_B \le \theta\,\{\|x\|_A^s + \|y\|_A^s\} \tag{2.25}$$

for all $x, y \in A$, with $x \perp y$. Then there exists a unique orthogonally additive mapping $L : A \to B$ and a unique orthogonally quadratic mapping $M : A \to B$ such that

$$\|f(y) - L(y) - M(y)\|_B \le \left[\frac{\theta}{2 - 2^s} + \frac{\theta}{4 - 2^s}\right] \|y\|_A^s \tag{2.26}$$

for all $y \in A$. The function $L(y)$ and $M(y)$ are defined in (2.3) and (2.15) respectively for all $y \in A$.

Proof. Let $f_e(y) = \dfrac{f_q(y) + f_q(-y)}{2}$ for all $y \in A$, then $f_e(0) = 0$. Hence

$$\begin{aligned}\|Df_e(x, y)\| &\le \frac{\theta}{2}\,\{(||x||_A^s + ||y||_A^s) + (|| - x||_A^s + || - y||_A^s)\}\\ &\le \theta(||x||_A^s + ||y||_A^s).\end{aligned} \tag{2.27}$$

By Theorem 2.2, we have

$$\|f_e(y) - M(y)\|_B \le \frac{\theta}{4 - 2^s}\,\|y\|_A^s \tag{2.28}$$

for all $y \in A$. Also, let $f_o(y) = \dfrac{f_a(y) - f_a(-y)}{2}$ for all $y \in A$, then $f_o(0) = 0$. Hence

$$\begin{aligned}\|Df_o(x, y)\| &\le \frac{\theta}{2}\,\{(||x||_A^s + ||y||_A^s) + (|| - x||_A^s + || - y||_A^s)\}\\ &\le \theta(||x||_A^s + ||y||_A^s).\end{aligned} \tag{2.29}$$

By Theorem 2.1, we have

$$\|f_o(y) - L(y)\|_B \le \frac{\theta}{2 - 2^s}\,\|y\|_A^s \tag{2.30}$$

for all $y \in A$. Define

$$f(y) = f_e(y) + f_o(y) \tag{2.31}$$

for all $y \in A$. From (2.28),(2.30) and (2.31), we arrive

$$\begin{aligned}\|f(y) - L(y) - M(y)\|_B &= \|f_e(y) + f_o(y) - L(y) - M(y)\|_B\\ &\le \|f_e(y) - M(y)\|_B + \|f_o(y) - L(y)\|_B\\ &\le \left[\frac{\theta}{2 - 2^s} + \frac{\theta}{4 - 2^s}\right] \|y\|_A^s\end{aligned}$$

for all $y \in A$. Hence the theorem is proved. □

Example 2.3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} \mu(x+x^2), & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2^n+1)}{(2^n)^2} \phi(2^n x) \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$\begin{aligned} &|f(x+2y) + f(x-2y) + 4f(x) - 3f(x+y) - 3f(x-y) \\ &\quad - f(2y) + 2f(y)| \leq 50\mu\,(|x| + |y|) \end{aligned} \tag{2.32}$$

for all $x, y \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ such that

$$|f(x) - A(x) - Q(x)| \leq \gamma |x + x^2| \qquad \textit{for all} \quad x \in \mathbb{R}. \tag{2.33}$$

Proof. Now

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{|2^n+1|}{|2^n|^2} \phi(2^n x) = \frac{10}{3}\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (2.32).

If $x = y = 0$ then (2.32) is trivial. If $|x| + |y| \geq 1$ then the left hand side of (2.32) is less than 50μ. Now suppose that $0 < |x| + |y| < 1$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \leq |x| + |y| < \frac{1}{2^k}, \tag{2.34}$$

so that $2^{k-1}x < 1,\ 2^{k-1}y < 1$, and consequently

$$\begin{aligned} &2^{k-1}(x), 2^{k-1}(y), 2^{k-1}(x+y), 2^{k-1}(x-y),, \\ &\qquad 2^{k-1}(x+2y), 2^{k-1}(x-2y), 2^{k-1}(2y) \in (-1,1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &2^n(x), 2^n(y), 2^n(x+y), 2^n(x-y) \\ &\qquad 2^n(x+2y), 2^n(x-2y), 2^n(2y) \in (-1,1) \end{aligned}$$

and

$$\begin{aligned} &\phi(2^n(x+2y) + \phi(2^n(x-2y)) + 4\phi(2^n(x)) - 3\phi(2^n(x+y)) - 3\phi(2^n(x-y)) \\ &\qquad - \phi(2^n(2y)) + \phi(2^n y) = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f and (2.34), we obtain that

$$\begin{aligned}
&|f(x+2y) + f(x-2y) + 4f(x) - 3f(x+y) - 3f(x-y) - f(2y) + 2f(y)| \\
&\leq \sum_{n=0}^{\infty} \frac{2^n+1}{4^n} \Big| \phi(2^n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(x)) - 3\phi(2^n(x+y)) \\
&\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\
&\leq \sum_{n=k}^{\infty} \frac{2^n+1}{4^n} \Big| \phi(2n(x+2y) + \phi(2^n(x-2y)) + \phi(2^n(4x)) - 3\phi(2^n(x+y)) \\
&\qquad - 3\phi(2^n(x-y)) - \phi(2^n(2y)) + 2\phi(2^n y) \Big| \\
&\leq \sum_{n=k}^{\infty} \frac{2^n+1}{4^n} 15\mu = \frac{10}{3}\mu \times \frac{15}{2^k} = 50\mu\left(|x| + |y|\right).
\end{aligned}$$

Thus f satisfies (2.32) for all $x, y, z \in \mathbb{R}$ with $0 < |x| + |y| < 1$.

We claim that an additive-quadratic functional equation (1.4) is not stable for $s = 1$ in Theorem 2.3. Suppose on the contrary that there exist an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\gamma > 0$ satisfying (2.10). Since f is bounded and continuous for all $x \in \mathbb{R}$, A and Q are bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.3, A must have the form $A(x) = cx$ and $Q(x) = cx^2$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f(x)| \leq (\gamma + |c|)\,|x + x^2|. \tag{2.35}$$

But we can choose a positive integer m with $m\mu > \gamma + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{(2^n+1)}{(2^n)^2}\phi(2^n x) \geq \sum_{n=0}^{m-1} \frac{\mu(2^n+1)}{(2^n)^2}(x+x^2) = m\mu(x+x^2) > (\gamma + |c|)\,(x+x^2)$$

which contradicts (2.35). Therefore the additive functional equation (1.4) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality (2.25).

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