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Vague Ideals of Po-Γ-Semigroups

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Abstract

In this paper the notion of vague ideals in po-Γ-semigroups has been introduced and several properties have been investigated. Characterization of regular po-Γ-semigroup in terms of vague ideals has been obtained. Operator po-semigroups of a po-Γ-semigroup have been made to work by obtaining various relationships between vague ideals of a po-Γ-semigroup and that of its operator po-semigroups.

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1 Introduction

In 1993 *W.L. Gau* and *D.J. Buehrer*[11] proposed the theory of vague sets as an improvement of the theory of fuzzy sets in approximating the real life situation. Vague sets are higher order fuzzy sets. A vague set A in the universe of discourse U is a pair (t_A, f_A) where t_A and f_A are fuzzy subsets of U satisfying the condition $t_A(u) \leq 1 - f_A(u)$ for all $u \in U$. *R. Biswas*[1] initiated the study of vague algebra by introducing the concepts of vague groups, vague normal groups. *H. Khan, M. Ahmad* and *R. Biswas*[21] introduced the notion of vague relations and studied some properties of them. *N. Ramakrishna*[29, 30] continued this study by studying vague cosets, vague products and several properties related to them. In 2008, *Y.B. Jun* and *C.H. Park*[16] introduced the notion of vague ideals in substraction algebra. *T. Eswarlal*[10] had introduced the concepts of vague ideals and normal vague ideals in semirings in 2008.

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[14]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by *Lofti Zadeh*[46] in his classic paper in 1965. *Azirel Rosenfeld*[31] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. *Nobuaki Kuroki*[22, 23, 24] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by *Kuroki*[22, 24]. In [23], *Kuroki* characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. *N. Kehayopulu* and *M. Tsingelis*[17, 18], *S.K Lee*[27] worked on po-semigroups. Authors who worked on fuzzy po-semigroup theory are *N. Kehayopulu* and *M. Tsingelis*[19, 20], *X.Y. Xie* and *F. Yan*[45], *M. Shabir* and *A. Khan*[43].

In 1986 the concept of Γ-semigroups was introduced by *M.K. Sen*[37] as a generalization of semigroups and ternary semigroups. Since then Γ-semigroups have been analyzed by lot of mathematicians, for instance *Sen*[38, 39, 40, 41], *Chattopadhay*[2, 3], *Dutta* and *Adhikari*[7, 8], *Hila*[12, 13], *Chinram*[4], *Saha*[32], *Seth*[42]. The Γ-semigroup introduced by *Sen* and *Saha*[37] may be called one sided Γ-semigroup. Later *Dutta* and *Adhikari*[7, 8] introduced both sided Γ-semigroup where the operation $\Gamma \times S \times \Gamma \to \Gamma$ was also taken into consideration. They defined operator semigroups for such type of Γ-semigroups. Many authors have studied semigroups in terms of fuzzy sets. Motivated by *Kuroki*[22, 23, 24] and others *S.K. Sardar* and *S.K. Majumder* studied Γ-semigroups in terms of fuzzy sets[33, 34, 35].

Y.I. Kwon and *S.K. Lee*[25, 26], *T.K. Dutta* and *N.C. Adhikari*[9], *Chinram* and *K. Tinpun*[5], *P. Dheena* and *B. Elavarasan*[6], *M. Siripitukdet* and *A. Iampan*[15, 44] are some authors who worked on po-Γ-semigroup theory. *T.K. Dutta* and *N.C. Adhikari*[9] have studied different properties of po-Γ-semigroup by defining operator po-semigroups of such type of po-Γ-semigroups.

In 1965 after the introduction of fuzzy sets by *L.A. Zadeh*[46], reconsideration of the concept of classical mathematics began. As an immediate result fuzzy algebra is an well established branch of mathematics at present. In a similar fashion after the commencement of the notion of vague sets by *W.L. Gau* and *D.J. Buehrer*[11] in 1993, many authors are presently trying to apply this concept in different algebraic structures. Concept of vague ideals of Γ-semigroups has been introduced by *S.K. Majumder*[28]. In this paper the concept of vague ideals in po-Γ-semigroups has been introduced and some of their important properties have been investigated. Here a regular po-Γ-semigroup has been characterized in terms of vague ideals. Various relationships between vague ideals of a po-Γ-semigroup and that of its operator po-semigroups have been obtained. Among other results an inclusion preserving bijection between the set of all vague ideals of a po-Γ-semigroup and that of its operator po-semigroups has obtained.

2 Preliminaries

Throughout S stands for po-Γ-semigroup unless or otherwise mentioned.

Let $S = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. The membership function for fuzzy sets can take values from the closed interval $[0, 1]$. A fuzzy set A in S is defined as the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in S\}$, where $\mu_A(x)$ is the grade of membership of the element x in the set A. The truth of the statement *the element x belongs to the set A* increases as the value of $\mu_A(x)$ closes to 1.

Gau and *Buehrer*[11] noticed that this single value of $\mu_A(x)$ combines the evidence for x and the evidence against x. It does not indicate the evidence for x and the evidence against x, and it does not also indicate how much there is of each. The necessity to introduce the concept of vague sets was originated from this point. Vague sets are different kind of fuzzy sets, which could be treated as a generalization of Zadeh's fuzzy sets[46].

Now we shall discuss some elementary concepts of po-Γ-semigroup theory which will be required in the sequel.

Definition 2.1 [*40*] *Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma....\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$(images to be denoted by $a\alpha b$) satisfying* (1) $x\gamma y \in S$, (2) $(x\beta y)\gamma z = x\beta(y\gamma z)$ *for all $x, y, z \in S, \alpha, \beta, \gamma \in \Gamma$.*

Definition 2.2 [*6*] *A Γ-semigroup S is said to be po-Γ-semigroup(partially ordered Γ-semigroup) if* (1) *S is a poset,* (2) *$a \le b$ in S implies that $a\alpha c \le b\alpha c$, $c\alpha a \le c\alpha b$ in S for all $c \in S$ and for all $\alpha \in \Gamma$.*

Remark 2.3 *Definition* 2.1 *and* 2.2 *are the definitions of one sided Γ-semigroup and one sided po-Γ-semigroup respectively. The following are the definitions of both sided Γ-semigroup*[*7*] *and both sided po-Γ-semigroup*[*9*] *given by T.K. Dutta and N.C. Adhikari. Throughout this paper unless or otherwise mentioned S stands for both sided po-Γ-semigroup.*

Definition 2.4 [*7*] *Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \longrightarrow a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \longrightarrow \alpha a\beta$ satisfying the following associative laws $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.*

Definition 2.5 [*9*] *A Γ-semigroup S is said to be a po-Γ-semigroup if* (1) *S and Γ are posets,* (2) *$a \le b$ in S implies that $a\alpha c \le b\alpha c, c\alpha a \le c\alpha b$ in S*

and $\gamma a\alpha \leq \gamma b\alpha$ in Γ for all $c \in S$ and for all $\alpha, \gamma \in \Gamma$, (3) $\alpha \leq \beta$ in Γ implies that $\alpha a\gamma \leq \beta a\gamma, \gamma a\alpha \leq \gamma a\beta$ in Γ and $a\alpha b \leq a\beta b$ in S for all $\gamma \in \Gamma$ and for all $a, b \in S$.

Remark 2.6 *The partial order relations on S and Γ are denoted by same symbol $\leq$.*

Example 2.7 [9] *Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over same set. Then S is a Γ-semigroup with respect to the usual matrix multiplication. Also S and Γ are posets with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then S is a po-Γ-semigroup.*

Now we discuss some elementary definitions on vague set theory which will be used in the sequel.

Definition 2.8 [10] *A vague set A in the universe of discourse S is a pair (t_A, f_A) where $t_A : S \to [0,1]$ and $f_A : S \to [0,1]$ are mappings(called truth membership function and false membership function respectively) where $t_A(x)$ is a lower bound of the grade of membership of x derived from the evidence for x and $f_A(x)$ is a lower bound on the negation of x derived from the evidence against x and $t_A(x) + f_A(x) \leq 1 \forall x \in S$.*

Definition 2.9 [10] *The interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A, and it is denoted by $V_A(x)$, i.e., $V_A(x) = [t_A(x), 1 - f_A(x)]$.*

Definition 2.10 [10] *A vague set A of S is said to be contained in another vague set B of S, i.e., $A \subseteq B$, if and only if $V_A(x) \leq V_B(x)$, i.e., $t_A(x) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x) \forall x \in S$.*

Definition 2.11 [10] *Two vague sets A and B of S are equal, i.e., $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$, i.e., $V_A(x) \leq V_B(x)$ and $V_B(x) \leq V_A(x) \forall x \in S$, which implies $t_A(x) = t_B(x)$ and $1 - f_A(x) = 1 - f_B(x)$.*

Definition 2.12 [10] *The union of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a vague set C of S, written as $C = A \cup B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \max(t_A, t_B)$ and $1 - f_C = \max(1 - f_A, 1 - f_B) = 1 - \min(f_A, f_B)$.*

Definition 2.13 [10] *The intersection of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a vague set C of S, written as $C = A \cap B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \min(t_A, t_B)$ and $1 - f_C = \min(1 - f_A, 1 - f_B) = 1 - \max(f_A, f_B)$.*

Definition 2.14 [*10*] *A vague set A of S with $t_A(x) = 0$ and $f_A(x) = 1 \forall x \in S$, is called the zero vague set of S.*

Definition 2.15 [*10*] *A vague set A of S with $t_A(x) = 1$ and $f_A(x) = 0 \forall x \in S$, is called the unit vague set of S.*

Definition 2.16 [*10*] *Let A be a vague set of the universe S with truth membership function t_A and false membership function f_A. For $\alpha, \beta \in [0, 1]$ with $\alpha \leq \beta$, the (α, β)-cut or vague cut of the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of S given by $A_{(\alpha,\beta)} = \{x \in S : V_A(x) \geq (\alpha, \beta)\}$, i.e., $A_{(\alpha,\beta)} = \{x \in S : t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta\}$.*

Definition 2.17 [*10*] *The α-cut, A_α of the vague set A is the (α, α)-cut of A and hence it is given by $A_\alpha = \{x \in S : t_A(x) \geq \alpha\}$.*

Definition 2.18 [*10*] *Let $\delta = (t_\delta, f_\delta)$ be a vague set of a po-Γ-semigroup S. For any subset T of S, the characteristic function of T taking values in $[0, 1]$ is a vague set $\delta_T = (t_{\delta_T}, f_{\delta_T})$ given by*

$$V_{\delta_T}(x) = \begin{cases} 1 & if\ x \in T \\ 0 & otherwise \end{cases}$$

i.e.,

$$t_{\delta_T}(x) = \begin{cases} 1 & if\ x \in T \\ 0 & otherwise \end{cases}$$

and

$$f_{\delta_T}(x) = \begin{cases} 0 & if\ x \in T \\ 1 & otherwise \end{cases}$$

Then δ_T is called the vague characteristic set of T in $[0, 1]$.

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. If $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$ be two elements of $I[0, 1]$, then we define *imax* to mean the maximum of two intervals as $imax\{I_1, I_2\} = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$. Similarly we can define *imin*. The concept of *imax* and *imin* could be extended to define *isup* and *iinf* of infinite numbers of elements of $I[0, 1]$[10].

3 Vague Ideals

Definition 3.1 [*9*] *Let S be a po-Γ-semigroup. A non-empty subset I of S is said to be a right ideal(left ideal) of S if* (1) *$I\Gamma S \subseteq I$(resp. $S\Gamma I \subseteq I$),* (2) *$a \in I$ and $b \leq a$ imply $b \in I$.*

Definition 3.2 [*9*] *Let S be a po-Γ-semigroup. A non-empty subset I of S is said to be an ideal of S if it is a right ideal as well as a left ideal of S.*

Definition 3.3 *A non-empty vague set A of a po-Γ-semigroup S is called a left vague ideal of S if it satisfies:*

(1) $V_A(x\gamma y) \geq V_A(y), i.e., t_A(x\gamma y) \geq t_A(y)$ and $1 - f_A(x\gamma y) \geq 1 - f_A(y)$ for all $x, y \in S$ and for all $\gamma \in \Gamma$,

(2) $x \leq y$ implies $V_A(x) \geq V_A(y), i.e., t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y)$ for all $x, y \in S$.

Definition 3.4 *A non-empty vague set A of a po-Γ-semigroup S is called a right vague ideal of S if it satisfies:*

(1) $V_A(x\gamma y) \geq V_A(x), i.e., t_A(x\gamma y) \geq t_A(x)$ and $1 - f_A(x\gamma y) \geq 1 - f_A(x)$ for all $x, y \in S$ and for all $\gamma \in \Gamma$,

(2) $x \leq y$ implies $V_A(x) \geq V_A(y), i.e., t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y)$ for all $x, y \in S$.

Definition 3.5 *A non-empty vague set A of a po-Γ-semigroup S is called a vague ideal of S if it is a left vague ideal of S and a right vague ideal of S.*

Example 3.6 *Let S be the set of all non positive integers without zero and Γ be the set of all non positive even integers without zero. Then S is a Γ-semigroup if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Again with respect to usual $\leq$ of Z, S becomes a po-Γ-semigroup. Let A be a vague set of S. The truth membership function t_A and false membership function f_A are defined as:*

$$t_A(x) = \begin{cases} 0.1 \; if \; x = -1 \\ 0.3 \; if \; x = -2 \\ 0.5 \; if \; x < -2 \end{cases}$$

and

$$f_A(x) = \begin{cases} 0.8 \; if \; x = -1 \\ 0.3 \; if \; x = -2 \\ 0.4 \; if \; x < -2 \end{cases}$$

Then A becomes a vague ideal of S.

The following two propositions can be verified by routine calculation.

Proposition 3.7 *Zero vague set, unit vague set and α-vague set of a po-Γ-semigroup S are left vague ideals(right vague ideals, vague ideals) of S respectively.*

Proposition 3.8 *If A and B are two left vague ideals(right vague ideals, vague ideals) of a po-Γ-semigroup S then $A \cap B$ is also a left vague ideal(right vague ideal, vague ideal) of S.*

Theorem 3.9 *A necessary and sufficient condition for a vague set A of a po-Γ-semigroup S to be a left vague ideal(right vague ideal, vague ideal) of S is that, t_A and $1 - f_A$ are fuzzy left ideals(resp. fuzzy right ideals, fuzzy ideals) of S.*

Proof: Let A be a left vague ideal of S and $x, y \in S, \gamma \in \Gamma$. Then $V_A(x\gamma y) \geq V_A(y)$ and $x \leq y$ implies $V_A(x) \geq V_A(y)$. Then (1) $t_A(x\gamma y) \geq t_A(y)$ and $1 - f_A(x\gamma y) \geq 1 - f_A(y)$, (2) $x \leq y$ implies $t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y)$. Hence t_A and $1 - f_A$ are fuzzy left ideals of S. The converse part of the theorem follows easily from definition. Similarly we can prove the other cases also.

Theorem 3.10 *Let S be a po-Γ-semigroup and A be a non-empty vague set of S. Then A is a left vague ideal(right vague ideal, vague ideal) of S if and only if the (α, β)-cut, $A_{(\alpha,\beta)}$ of A is a crisp left ideal(resp. crisp right ideal, crisp ideal) of S for all $\alpha, \beta \in [0, 1]$.*

Proof: Let A be a left vague ideal of S. Let $x \in S, \gamma \in \Gamma$ and $y \in A_{(\alpha,\beta)}$. Then $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$. By hypothesis $V_A(x\gamma y) \geq V_A(y)$ which implies $t_A(x\gamma y) \geq t_A(y) \geq \alpha$ and $1 - f_A(x\gamma y) \geq 1 - f_A(y) \geq \beta$. Consequently, $x\gamma y \in A_{(\alpha,\beta)}$.

Let $x, y \in S$ be such that $y \leq x$. Let $x \in A_{(\alpha,\beta)}$. Then $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$. Since A is a left vague ideal of S, so $t_A(y) \geq t_A(x) \geq \alpha$ and $1 - f_A(y) \geq 1 - f_A(x) \geq \beta$. Consequently, $y \in A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a crisp left ideal of S.

Conversely, let $A_{(\alpha,\beta)}$ be a crisp left ideal of S for all $\alpha, \beta \in [0, 1]$. Again let $s \in S, \gamma \in \Gamma$ and $x \in A_{(\alpha,\beta)}$. Let $t_A(x) = \alpha$ and $1 - f_A(x) = \beta$. Since $A_{(\alpha,\beta)}$ is a crisp left ideal of S, so we have $s\gamma x \in A_{(\alpha,\beta)}$. Then $t_A(s\gamma x) \geq \alpha = t_A(x)$ and $1 - f_A(s\gamma x) \geq \beta = 1 - f_A(x)$. This implies that $V_A(s\gamma x) \geq V_A(x)$.

Let $x, y \in S$ be such that $x \leq y$. Let $t_A(y) = \alpha$ and $1 - f_A(y) = \beta$. Then $y \in A_{(\alpha,\beta)}$. Since $A_{(\alpha,\beta)}$ is a left vague ideal of S, so $x \in A_{(\alpha,\beta)}$. Then $t_A(x) \geq \alpha = t_A(y)$ and $1 - f_A(x) \geq \beta = 1 - f_A(y)$. Hence A is a left vague ideal of S. Similarly we can prove the other cases also.

Theorem 3.11 *Let T be a non-empty subset of a po-Γ-semigroup S and δ_T is a vague characteristic set of T. Then δ_T is a left vague ideal(right vague ideal, vague ideal) of S if and only if T is a crisp left ideal(resp. crisp right ideal, crisp ideal) of S.*

Proof: Let δ_T be a left vague ideal of S. Let $x \in S, y \in T$ and $\gamma \in \Gamma$. Then $V_T(y) = 1$. Since δ_T is a left vague ideal of S, so we have $V_{\delta_T}(x\gamma y) \geq V_{\delta_T}(y) = 1$. But actually $V_{\delta_T}(x\gamma y) \leq 1$ whence $V_{\delta_T}(x\gamma y) = 1$. Consequently, $x\gamma y \in T$.

Let $x, y \in S$ be such that $y \leq x$. Let $x \in T$. Then $V_{\delta_T}(x) = 1$. Since δ_T is a left vague ideal of S, $V_{\delta_T}(y) \geq V_{\delta_T}(x) = 1$. Consequently, $y \in T$. Hence T is a crisp left ideal of S.

Conversely, let T be a crisp left ideal of S. Let $x, y \in S$ and $\gamma \in \Gamma$. If $y \in T$ then $x\gamma y \in T$ which implies $V_{\delta_T}(x\gamma y) = 1 = V_{\delta_T}(y)$. If $y \notin T$ then $x\gamma y \notin T$ which implies $V_{\delta_T}(x\gamma y) = 0 = V_{\delta_T}(y)$.

Let $x, y \in S$ be such that $x \leq y$. If $y \in T$ then $V_{\delta_T}(y) = 1$. Since T is a crisp left ideal of S, so $x \in T$. Then $V_{\delta_T}(x) = 1 = V_{\delta_T}(y)$. If $y \notin T$ then $V_{\delta_T}(y) = 0$. So $V_{\delta_T}(x) \geq 0 = V_{\delta_T}(y)$. Hence δ_T is a left vague ideal of S. Similarly we can prove the other cases also.

Remark 3.12 *Propositions* 3.7, 3.8, *Theorems* 3.9, 3.10 *and* 3.11 *are true in case of po-semigroup also.*

4 Composition Vague Ideals

Definition 4.1 *Let A and B be two vague sets of a po-Γ-semigroup S. Then the product of A and B, denoted by $A \circ B$ and defined by* $V_{A\circ B} = \sup[\min\{V_A(y), V_B(z)\} : x \leq y\gamma z]$ *for* $y, z \in S, \gamma \in \Gamma$, *i.e.,* $t_{A\circ B} = \sup[\min\{t_A(y), t_B(z)\} : x \leq y\gamma z]$ *and* $f_{A\circ B} = \inf[\max\{f_A(y), f_B(z)\} : x \leq y\gamma z]$, *for* $y, z \in S, \gamma \in \Gamma$.

Theorem 4.2 *In a po-Γ-semigroup S for a non-empty vague set A of S the following are equivalent:* (1) *A is a left vague ideal(right vague ideal) of S,* (2) *$B \circ A \subseteq A$(resp. $A \circ B \subseteq A$) and $x \leq y$ implies $V_A(x) \geq V_A(y)$, i.e., $t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y) \forall x, y \in S$, where B is the vague characteristic set of S.*

Proof: (1) $\Rightarrow$ (2) : Let A be a left vague ideal of S. So by Definition 3.3, $x \leq y$ implies that $V_A(x) \geq V_A(y)$, *i.e.,* $t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y) \forall x, y \in S$. Let $a \in S$. Suppose there exists $x, y \in S$ and $\delta \in \Gamma$ such that $a \leq x\delta y$. Since A is a left vague ideal of S, we have $V_{B\circ A}(a) = \sup[\min\{V_B(x), V_A(y)\} : a \leq x\delta y] = \sup[\min\{1, V_A(y)\} : a \leq x\delta y] = \sup[V_A(y) : a \leq x\delta y]$. Now since A is a left vague ideal of S, so we have $V_A(x\gamma y) \geq V_A(y) \forall x, y \in S$ and $\forall \gamma \in \Gamma$. So in particular, $V_A(y) \leq V_A(a) \forall a \leq x\gamma y$. Hence $\sup[V_A(y) : a \leq x\delta y] \leq V_A(a)$. Thus $V_A(a) \geq V_{B\circ A}(a)$. Hence $B \circ A \subseteq A$. If there do not exist $x, y \in S$ and $\gamma \in \Gamma$ such that $a \leq x\gamma y$ then $V_{B\circ A}(a) = 0 \leq V_A(a)$. Hence $B \circ A \subseteq A$. By similar argument we can show that $A \circ B \subseteq A$ when A is a right vague ideal of S.

(2) $\Rightarrow$ (1) : Let $B \circ A \subseteq A$. Let $x, y \in S, \gamma \in \Gamma$ and $a := x\gamma y$. Then clearly $a \leq x\gamma y$. Now $V_A(x\gamma y) = V_A(a) \geq V_{B\circ A}(a)$. Now , we have $V_{B\circ A}(a) = \sup[\min\{V_B(u), V_A(v)\} : a \leq u\alpha v] \geq \min\{V_B(x), V_A(y)\} = \min\{1, V_A(y)\} = V_A(y)$. Hence $V_A(x\gamma y) \geq V_A(y)$. Hence A is a left vague ideal of S. By a similar argument we can show that if $A \circ B \subseteq A$, then A is a right vague ideal of S.

In view of above theorem we can have the following theorem.

Theorem 4.3 *In a po-Γ-semigroup S for a non-empty vague set A of S the following are equivalent:* (1) *A is a vague ideal of S*, (2) *$B \circ A \subseteq A$, $A \circ B \subseteq A$ and $x \leq y$ implies $V_A(x) \geq V_A(y)$, i.e., $t_A(x) \geq t_A(y)$ and $1 - f_A(x) \geq 1 - f_A(y) \forall x, y \in S$, where B is the vague characteristic set of S.*

Definition 4.4 [*9*] *A po-Γ semigroup S is called regular if for any $x \in S$ there exist $a \in S, \alpha, \beta \in \Gamma$ such that $x \in x\alpha a\beta x$.*

Definition 4.5 [*36*] *Let A be a subset of a po-Γ semigroup S. Then we define $(A] := \{x \in S : x \leq y$ for some $y \in A\}$.*

Proposition 4.6 [*36*] *In a po-Γ-semigroup S, if A and B be is any sided ideals of S, then* (1) $(A] = A, (B] = B$, (2) $(A \cap B] = (A] \cap (B]$, (3) $(A]\Gamma(B] \subseteq (A\Gamma B]$

Theorem 4.7 [*36*] *A po-Γ semigroup S is regular if and only if $A \cap B = (A\Gamma B]$ for any right ideal A and for any left ideal B of S.*

Proposition 4.8 *Let A be a right vague ideal and B be a left vague ideal of a po-Γ semigroup S. Then $A \circ B \subseteq A \cap B$.*

Proof: Let A be a right vague ideal and B be a left vague ideal of S. Let $x \in S$. Suppose there exist $u, v \in S$ and $\gamma \in \Gamma$ such that $x \leq u\gamma v$. Then

$$\begin{aligned} V_{A\circ B}(x) &= \sup[\min\{V_A(u), V_B(v)\} : x \leq u\gamma v] \\ &\leq \sup[\min\{V_A(u\gamma v), V_B(u\gamma v)\} : x \leq u\gamma v] \\ &\leq \min\{V_A(x), V_B(x)\} = V_{A\cap B}(x). \end{aligned}$$

Suppose, there do not exist $u, v \in S$ and $\gamma \in \Gamma$ such that $x \leq u\gamma v$. Then $V_{A\circ B}(x) = 0 \leq V_{A\cap B}(x)$. Hence $A \circ B \subseteq A \cap B$.

Proposition 4.9 *Let S be a regular po-Γ semigroup and A and B be two vague sets of S. Then $A \circ B \supseteq A \cap B$.*

Proof: Let $c \in S$. Since S is regular, then there exists an element $x \in S$ and $\alpha, \beta \in \Gamma$ such that $c \leq c\alpha x\beta c = c\gamma c$ where $\gamma := \alpha x\beta \in \Gamma$. Then $V_{A\circ B}(c) = \sup[\min\{v_A(u), V_B(v)\} : c \leq u\delta v] \geq \min\{V_A(c), V_B(c)\} = V_{A\cap B}(c)$. Hence $A \circ B \supseteq A \cap B$.

Theorem 4.10 *In a po-Γ semigroup S, the following conditions are equivalent: (1) S is regular, (2) $A \circ B = A \cap B$ for every right vague ideal A and every left vague ideal B of S.*

Proof: (1) $\Rightarrow$ (2) : Let S be a regular. Then by Proposition 4.8 and 4.9 we have $A \circ B = A \cap B$.

(2) $\Rightarrow$ (1) : Suppose (2) holds. Let R be a crisp right ideal and L be a crisp left ideal of S. Let $x \in R \cap L$. Then $x \in (R \cap L]$. this implies that $x \in (R]$ and $x \in (L]$. Hence $V_{\delta_{(R]}}(x) = V_{\delta_{(L]}}(x) = 1$(where $\delta_{(R]}$ and $\delta_{(L]}$ are the vague characteristic sets of $(R]$ and $(L]$ respectively). Then $V_{\delta_{(R]} \cap \delta_{(L]}}(x) = 1$. Now by Theorem 3.11, $\delta_{(R]}$ is a right vague ideal and $\delta_{(L]}$ is a left vague ideal of S. Hence by hypothesis $\delta_{(R]} \circ \delta_{(L]} = \delta_{(R]} \cap \delta_{(L]}$. So $V_{\delta_{(R]} \circ \delta_{(L]}}(x) = 1, i.e., \sup[\min\{V_{\delta_{(R]}}(y), V_{\delta_{(L]}}(z)\} : x \leq y\gamma z] = 1$. This implies that there exist some $r, s \in S$ and $\gamma_1 \in \Gamma$ such that $x \leq r\gamma_1 s$ and $V_{\delta_R}(r) = 1 = V_{\delta_L}(s)$. Hence $r \in R$ and $s \in L$. Hence $x \in (R\Gamma L]$. Consequently, $R \cap L \subseteq (R\Gamma L]$. Also $(R\Gamma L] \subseteq R \cap L$. Hence $R \cap L = (R\Gamma L]$. Consequently, S is regular.

5 Corresponding Vague Ideals

Many results of po-semigroups could be extended to po-Γ-semigroups directly and via operator po-semigroups[9](left, right) of a po-Γ-semigroup. In order to make operator po-semigroups of a po-Γ-semigroup work in the context of vague sets as it worked in the study of po-Γ-semigroups[9], we obtain various relationships between vague ideals of a po-Γ-semigroup and that of its operator po-semigroups. Here, among other results we obtain an inclusion preserving bijection between the set of all vague ideals of a po-Γ-semigroup and that of its operator po-semigroups.

Definition 5.1 [9] *Let S be a Γ-semigroup. Let us define a relation ρ on $S \times \Gamma$ as follows :$(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of Γ-semigroup S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha a\beta, b]$.*

Let $((S, \Gamma), \leq)$ be a po-Γ-semigroup. We define a relation $\leq$ on L by $[a, \alpha] \leq [b, \beta]$ if and only if $a\alpha s \leq b\beta s$ for all $s \in S$ and $\gamma a\alpha \leq \gamma b\beta$ for all $\gamma \in \Gamma$. With respect to this relation L becomes a po-semigroup. In a similar way R can be made into a po-semigroup.

If there exists an element $[e, \delta] \in L([\gamma, f] \in R)$ such that $e\delta s = s$(resp. $s\gamma f = s$) for all $s \in S$ then $[e, \delta]$(resp. $[\gamma, f]$) is called the left(resp. right) unity of S.

Now let S be a Γ-semigroup with unities and L and R are po-semigroups. We define a relation $\leq$ in S by $a \leq b$ if and only if $[a,\alpha] \leq [b,\alpha]$ in L and $[\alpha,a] \leq [\alpha,b]$ in R for all $\alpha \in \Gamma$. We also define a relation $\leq$ in Γ by $\alpha \leq \beta$ if and only if $[a,\alpha] \leq [a,\beta]$ in L and $[\alpha,a] \leq [\beta,a]$ in R for all $a \in S$. With respect to these relations S becomes a po-Γ-semigroup.

Definition 5.2 [28] *For a vague set A of R we define a vague set A^* of S by $(V_A)^*(a) = \inf\limits_{\gamma\in\Gamma} V_A([\gamma,a]), i.e., (t_A)^*(a) = \inf\limits_{\gamma\in\Gamma} t_A([\gamma,a])$ and $1-(f_A)^*(a) = 1-\inf\limits_{\gamma\in\Gamma} f_A([\gamma,a])$, where $a \in S$. For a vague set B of S we define a vague set $B^{*'}$ of R by $(V_B)^{*'}([\alpha,a]) = \inf\limits_{s\in S} V_B(s\alpha a), i.e., (t_B)^{*'}([\alpha,a]) = \inf\limits_{s\in S} t_B(s\alpha a)$ and $1-(t_B)^{*'}([\alpha,a]) = 1-\inf\limits_{s\in S} t_B(s\alpha a)$, where $[\alpha,a] \in R$. For a vague set C of L, we define a vague set $C^{+})$ of S by $(V_C)^+(a) = \inf\limits_{\gamma\in\Gamma} V_C([a,\gamma]), i.e., (t_C)^+(a) = \inf\limits_{\gamma\in\Gamma} t_C([a,\gamma])$ and $1-(f_C)^+(a) = 1-\inf\limits_{\gamma\in\Gamma} f_C([a,\gamma])$, where $a \in S$. For a vague set D of S we define a vague set $D^{+'})$ of L by $(V_D)^{+'}([a,\alpha]) = \inf\limits_{s\in S} V_D(a\alpha s), i.e., (t_D)^{+'}([a,\alpha]) = \inf\limits_{s\in S} t_D(a\alpha s)$ and $1-(f_D)^{+'}([a,\alpha]) = 1-\inf\limits_{s\in S} f_D(a\alpha s)$, where $[a,\alpha] \in L$.*

Now we recall the following propositions from [9] which will be required in the sequel.

Proposition 5.3 [9] *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and P be a crisp ideal of L. Then P^+ is a crisp ideal of S.*

Proposition 5.4 [9] *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and Q is a crisp ideal of S. Then $Q^{+'}$ is a crisp ideal of L.*

Theorem 5.5 [9] *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup. Then there exist an inclusion preserving bijection $P \mapsto P^{+'}$ between the set of all crisp ideals of S and set of all crisp ideals of L.*

Remark 5.6 *The right operator analogues of Proposition* 5.3, 5.4 *and Theorem* 5.5 *are also true.*

For convenience of the readers, we may note that for a po-Γ-semigroup S and its left, right operator semigroups L, R respectively four mappings namely $()^+$, $()^{+'}, ()^+, ()^{*'}$ occur. They are defined as follows: For $I \subseteq R, I^* = \{s \in S, [\alpha,s] \in I \forall \alpha \in \Gamma\}$; for $P \subseteq S, P^{*'} = \{[\alpha,x] \in R : s\alpha x \in P \forall s \in S\}$; for $J \subseteq L, J^+ = \{s \in S, [s,\alpha] \in J \forall \alpha \in \Gamma\}$; for $Q \subseteq S, Q^{+'} = \{[x,\alpha] \in L : x\alpha s \in Q \forall s \in S\}$.

Proposition 5.7 *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and A be a vague ideal L. Then A^+ is a vague ideal of S.*

Proof: Let $a, b \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} t_A^+(a\gamma b) &= \inf_{\alpha\in\Gamma} t_A([a\gamma b, \alpha]) = \inf_{\alpha\in\Gamma} t_A([a,\gamma][b,\alpha]) \\ &\geq \inf_{\alpha\in\Gamma} t_A([b,\alpha])(\text{since } A \text{ is a vague ideal of } L) \\ &= t_A^+(b). \end{aligned}$$

Similarly we can show that $t_A^+(a\gamma b) \geq t_A^+(a)$. By using similar argument we can show that $1 - f_A^+(a\gamma b) \geq 1 - f_A^+(b)$ and $1 - f_A^+(a\gamma b) \geq 1 - f_A^+(a)$. Let $a, b \in S$ be such that $a \leq b$. Then $[a, \alpha] \leq [b, \alpha]$ in L for all $\alpha \in \Gamma$. Then $t_A([a, \alpha]) \geq t_A([b, \alpha])$ and $1 - f_A([a, \alpha]) \geq 1 - f_A([b, \alpha])$ for all $\alpha \in \Gamma$. Now

$$t_A^+(a) = \inf_{\alpha\in\Gamma} t_A([a,\alpha]) \geq \inf_{\alpha\in\Gamma} t_A([b,\alpha])(\text{since } A \text{ is a fuzzy ideal of } L) = t_A^+(b).$$

By similar argument we can show that $1 - f_A^+(a) \geq 1 - f_A^+(b)$. Hence A^+ is a vague ideal of S.

Proposition 5.8 *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup and B be a vague ideal of S. Then $B^{+'}$ is a vague ideal of L.*

Proof: Let $[a, \alpha], [b, \beta] \in L$. Then

$$\begin{aligned} t_B^{+'}([a,\alpha][b,\beta]) &= t_B^{+'}([a\alpha b, \beta]) = \inf_{s\in S} t_B(a\alpha b\beta s) = \inf_{s\in S} t_B(a\alpha(b\beta s)) \\ &\geq \inf_{s\in S} t_B(b\beta s)(\text{since } B \text{ is a vague ideal of } S) \\ &= t_B^{+'}([b,\beta]). \end{aligned}$$

Similarly we can show that $t_B^{+'}([a,\alpha][b,\beta]) \geq t_B^{+'}([a,\alpha])$. By similar argument we can show that $1 - f_B^{+'}([a,\alpha][b,\beta]) \geq 1 - f_B^{+'}([b,\beta])$ and $1 - f_B^{+'}([a,\alpha][b,\beta]) \geq 1 - f_B^{+'}([a,\alpha])$. Let $[a,\alpha], [b,\alpha] \in L$ be such that $[a,\alpha] \leq [b,\alpha]$. Then $a \leq b$ in S, implies $a\alpha s \leq b\alpha s\ \forall \alpha \in \Gamma, \forall s \in S$. So $t_B(a\alpha s) \geq t_B(b\alpha s)$ and $1 - f_B(a\alpha s) \geq 1 - f_B(b\alpha s)\ \forall \alpha \in \Gamma, \forall s \in S$. Then

$$\begin{aligned} t_B^{+'}([a,\alpha]) &= \inf_{s\in S} t_B(a\alpha s) \\ &\geq \inf_{s\in S} t_B(b\alpha s)(\text{since } B \text{ is a vague ideal of } S) \\ &= t_B^{+'}([b,\alpha]). \end{aligned}$$

By similar argument we can show that $1 - f_B^{+'}([a,\alpha]) \geq 1 - f_B^{+'}([b,\alpha])$. Hence $B^{+'}$ is a vague ideal of L.

Theorem 5.9 *Let S be a po-Γ-semigroup with unities and L be its left operator po-semigroup. Then there exist an inclusion preserving bijection $A \mapsto A^{+'}$ between the set of all vague ideals of S and set of all vague ideals of L.*

Proof: Let A be a vague ideal of a po-Γ-semigroup S. Then by Proposition 5.8, $A^{+'}$ is a vague ideal of the po-semigroup L. So by Proposition 5.7, $(A^{+'})^{+}$ is a vague ideal of S. From Theorem V.1[28], it is clear that $(A^{+'})^{+} = A$ and $(A^{+})^{+'} = A$. Also the inclusion preserving property follows from Theorem V.1[28]. Hence the proof.

Remark 5.10 *The right operator analogues of Propositions* 5.7, 5.8 *and Theorem* 5.9 *are also true.*

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