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Solution for a Rectangular Plate on Elastic Foundation with Free Edges Using Reciprocal Theorem Method

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Abstract

In this paper, the reciprocal theorem method is used to obtain the theoretical solutions for a rectangular plate supported on the elastic foundation with free edges, which have widespread applications in the designs of civil engineering such as highway and airport pavement. Numerical examples are presented at the end to compare the results obtained by this method and those from conventional methods

Mathematics Subject Classification: 74Bxx

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1 Introduction

The theoretical analysis of highway and airport pavement is traditionally performed based on the Wassgarde methods [1]. However, this method considers the highway and airport pavement as an infinite plate supported on a Winkler foundation—an assumption that is not applicable in real practice because the model cannot determine the stresses when load is applied on the edges and at the corners of the plate. In practice, those stresses are of crucial importance to the design of highway and airport pavement [2]. The superimposition method has therefore been used to overcome the abovementioned problem [3]. Another method that is commonly used to calculate the response of concrete pavement is the finite element method [4]. In this paper, a difference approach using the reciprocal theorem is used to obtain the theoretical solutions for the rectangular plate supported on an elastic foundation with free edges. Initially, a basic solution is established for the simply supported plate along all edges and acted on by a unit concentrated force. Subsequently, by using the reciprocal theorem between the simply supported plate and the completely free plate, a theoretical solution for the latter plate is obtained. Finally, numerical results are presented for easy comparison with those reported in the relevant references.

2 Application of reciprocal theorem method and bending solutions for a rectangular plate

The simply supported rectangular plate on the Winkler foundation is subjected to a concentrated force of unit amplitude, as shown in Figure 1. The force is permitted to move on the surface of the plate freely. Its co-ordinates (ζ , η) are variable. The equation governing the equilibrium of the plate under the force can be written as:

$$\frac{\partial^4 w_1}{\partial x^4} + 2\frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{\partial^4 w_1}{\partial y^4} + Kw_1 = \frac{\delta(x-\zeta), y-\eta)}{D} \tag{1}$$

where $D = Eh^3/12(1-v^2)$ and h is the thickness of the plate, E is the Young's modulus of the plate material, ν is the Poisson ratio, K is the parameter of foundation reaction force and δ (x- ζ , y- η) is the Dirac Delta function.

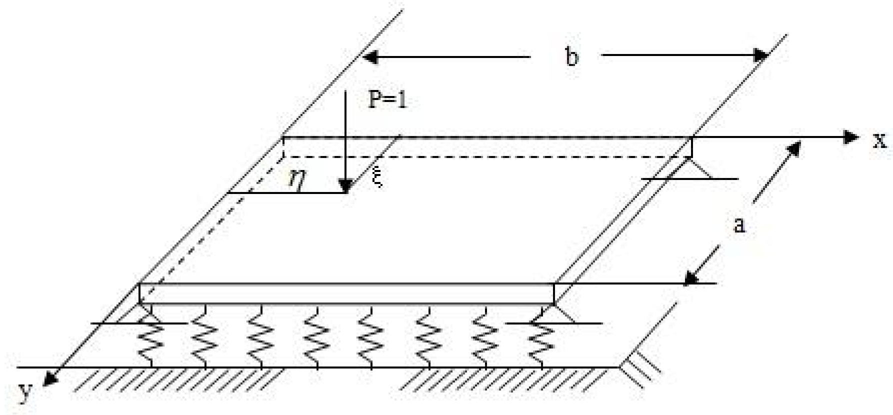

Figure 1. simply supported rectangular plate subjected to a concentrated unit amplitude force

Supposing that the plate in Fig.1 has a Navier's solution

$$w_1(x, y, \zeta, \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin k_m x \sin k_n y \tag{2}$$

Where $k_m = m\pi/a$; $k_n = n\pi/b$

The Dirac delta function $\delta(x-\zeta, y-\eta)$ in equation (1) can then be represented by a Fourier double sine series as follows:

$$\delta(x-\zeta, y-\eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin k_m x \sin k_n y \tag{3}$$

Substituting equation (2) and (3) into (1), the following expression can be obtained

$$w_1(x, y, \zeta, \eta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{abDK_{mn}} \sin k_m \zeta \sin k_m x \sin k_n \eta \sin k_n y \tag{4}$$

where $K_{mn} = (k_m^2 + k_n^2) + K$

Now, for the rectangular plate supported on Winkler foundation with free edges, as shown in Figure 2, its boundary displacements can be represented as follows:

$$W(x,0) = D_1 + (D_2 + D_1)\, x/a + \sum_{m=1}^{\infty} B_{1m} \sin k_m x \tag{5}$$

$$W(x,b) = D_3 + (D_4 + D_3)\, x/a + \sum_{m=1}^{\infty} B_{2m} \sin k_m x \tag{6}$$

$$W(0,y) = D_1 + (D_3 + D_1)\, y/b + \sum_{n=1}^{\infty} B_{1n} \sin k_n y \tag{7}$$

$$W(a,y) = D_2 + (D_4 + D_2)\, y/b + \sum_{n=1}^{\infty} B_{2n} \sin k_n y \tag{8}$$

where D_1,D_2,D_3 and D_4 are the displacements at the corners (0,0), (a,0), (0,b) and (a,b), respectively.

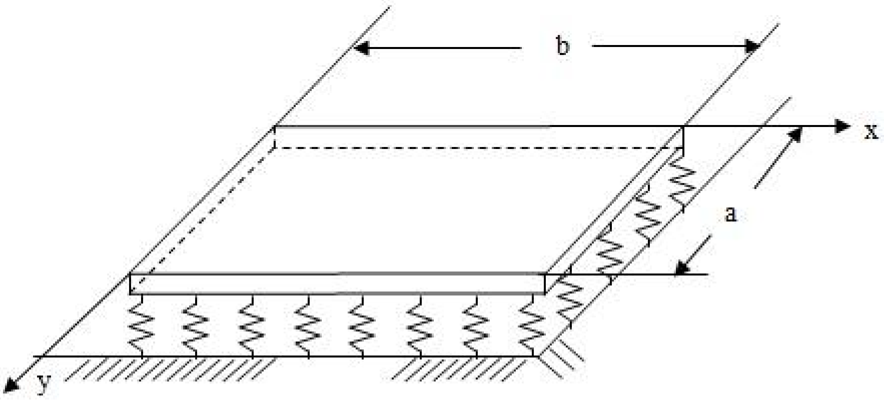

Figure 2. A rectangular plate with free edges on Winkler foundation

For a better understanding of the use of the reciprocal theorem in obtaining a solution for a rectangular plate with free edges on Winkler foundation, it is easier to use a beam instant of rectangular plate as an example to demonstrate how the reciprocal theorem works. As shown in Figure 3, a simply supported beam is subjected to a concentrated force p1 at coordinate x= ξ .The displacement along the direction of this force is indicated by $\triangle_1$.Another simply supported beam is shown in Figure 4, with a concentrated force p_2 , the corresponding displacement is $\triangle_2$.

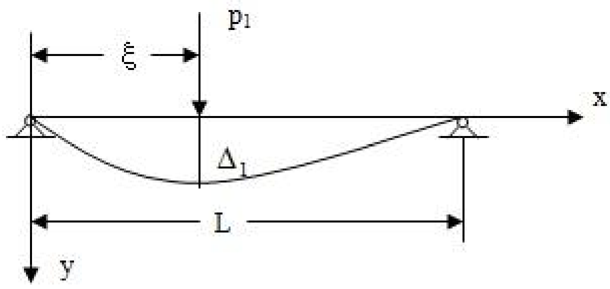

Figure 3 simply supported beam subjected to a concentrated force p_1

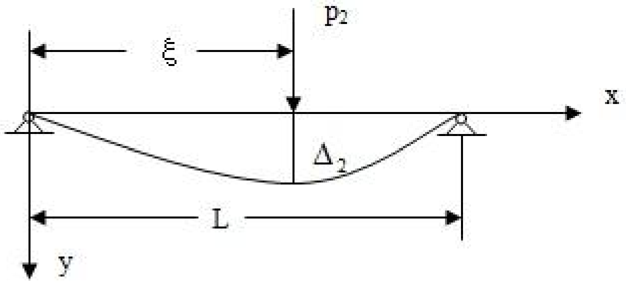

Figure 4 Simply supported beam subjected to a concentrated force p_2

With reference to the reciprocal theorem, Figure 4 demonstrates that the work done by the force p_1 moving along the displacement $\triangle_2$ is equal to the work performed by the force p_2 moving along the displacement $\triangle_1$. This relationship is expressed mathematically as follows:

$$P_1\Delta_2 = P_2\Delta_1 \tag{9}$$

Let p_1 be unity then equation (9) becomes

$$\Delta_2 = P_2\Delta_1 \tag{10}$$

equation (10) shows that the displacement $\triangle_2$ can be expressed by the multiplication of the force p2 and the displacement $\triangle_1$. Because the force p1 is permitted to move along the beam, the displacement $\triangle_1$ is the function of the coordinate ξ . Therefore, the displacement $\triangle_2$ in equation (9) is a function of the same coordinate ξ and in fact represents the flexural displacement of the beam in Figure 4.

$$\begin{aligned} W(\zeta,\eta) = & \int_0^a [V_{1y}(x,0,\zeta,\eta)W(x,0) - V_{1y}(x,b,\zeta,\eta)W(x,b)]dx \\ & + \int_0^b [V_{1x}(0,y,\zeta,\eta)W(0,y) - V_{1x}(a,y,\zeta,\eta)W(a,y)]dy \\ & + [R_1(a,b,\zeta,\eta)D_4 - R_1(0,b,\zeta,\eta)D_3] \\ & + [R_1(0,0,\zeta,\eta)D_1 - R_1(a,0,\zeta,\eta)D_2] \end{aligned} \tag{11}$$

where

$$V_{1y}(x,y,\zeta,\eta) = -D[\frac{\partial^3 w_1(x,y,\zeta,\eta)}{\partial x^3} + (2-\nu)\frac{\partial^3 w_1(x,y,\zeta,\eta)}{\partial x^2 \partial y}] \tag{12}$$

and

$$V_{1x}(x,y,\zeta,\eta) = -D[\frac{\partial^3 w_1(x,y,\zeta,\eta)}{\partial y^3} + (2-\nu)\frac{\partial^3 w_1(x,y,\zeta,\eta)}{\partial y^2 \partial x}] \tag{13}$$

are the distributed vertical edge reactions along the edges perpendicular to the x axis and y axis as shown in Figure 1 and

$$R_1(x,y,\zeta,\eta) = -2D(1-\nu)\frac{\partial^2 w_1(x,y,\zeta,\eta)}{\partial y \partial x} \tag{14}$$

is the concentrated force acting at the corner of the plate.
As shown in equation (11), the displacement w(ξ , η) is expressed as a function of the coordinates (ξ , η). It can be also expressed as a function of coordinates (x , y) by replacing ξ and η in equation (11) with x and y after the integration is completed- it does not matter which coordinate is used because either of them gives the same displacement function.
Substituting equations (5) through (8) into equation (11) and completing integration, a Navier's type solution for the plate of Figure 2 is obtained. This solution has to be transformed into Levy's type using the method mentioned in reference [7]. As a result, the following equation is obtained.

$$\begin{aligned}
W(\zeta,\eta) = & \sum_{m=1}^{\infty}\frac{B_{1m}}{2\lambda^2}\left[\frac{\Phi(\alpha_m,m)\sinh\alpha_m(b-\eta)}{\sinh\alpha_m b} - \frac{\Phi(\beta_m,m)\sinh\beta_m(b-\eta)}{\sinh\beta_m b}\right]\sin k_m\zeta \\
& \sum_{m=1}^{\infty}\frac{B_{2m}}{2\lambda^2}\left[\frac{\Phi(\alpha_m,m)\sinh\alpha_m\eta}{\sinh\alpha_m b} - \frac{\Phi(\beta_m,m)\sinh\beta_m\eta}{\sinh\beta_m b}\right]\sin k_m\zeta \\
& \sum_{n=1}^{\infty}\frac{B_{1n}}{2\lambda^2}\left[\frac{\Phi(\alpha_n,n)\sinh\alpha_n(a-\zeta)}{\sinh\alpha_n a} - \frac{\Phi(\beta_n,n)\sinh\beta_n(a-\zeta)}{\sinh\beta_n a}\right]\sin k_n\eta \\
& \sum_{n=1}^{\infty}\frac{B_{2n}}{2\lambda^2}\left[\frac{\Phi(\alpha_n,n)\sinh\alpha_n\zeta}{\sinh\alpha_n a} - \frac{\Phi(\beta_n,n)\sinh\beta_n\zeta}{\sinh\beta_n b}\right]\sin k_n\eta \qquad (15) \\
& + D_1\left\{\frac{(a-\zeta)(b-\eta)}{ab} - \sum_{m=1}^{\infty}\frac{\lambda^2}{m\pi}\left[\frac{\sinh\alpha_m(b-\eta)}{\alpha_m^2\sinh\alpha_m b} - \frac{\sinh\beta_m(b-\eta)}{\beta_m^2\sinh\beta_m b} + \frac{2\lambda^2(b-\eta)}{\alpha_m^2\beta_m^2 b}\right]\sin k_m\zeta\right\} \\
& + D_2\left\{\frac{\zeta(b-\eta)}{ab} - \sum_{m=1}^{\infty}\frac{\lambda^2(-1)^m}{m\pi}\left[\frac{\sinh\alpha_m(b-\eta)}{\alpha_m^2\sinh\alpha_m b} - \frac{\sinh\beta_m(b-\eta)}{\beta_m^2\sinh\beta_m b} + \frac{2\lambda^2(b-\eta)}{\alpha_m^2\beta_m^2 b}\right]\sin k_m\zeta\right\} \\
& + D_3\left\{\frac{(a-\zeta)\eta}{ab} - \sum_{m=1}^{\infty}\frac{\lambda^2}{m\pi}\left[\frac{\sinh\alpha_m\eta}{\alpha_m^2\sinh\alpha_m b} - \frac{\sinh\beta_m\eta}{\beta_m^2\sinh\beta_m b} + \frac{2\lambda^2\eta}{\alpha_m^2\beta_m^2 b}\right]\sin k_m\zeta\right\} \\
& + D_4\left\{\frac{\zeta\eta}{ab} - \sum_{m=1}^{\infty}\frac{\lambda^2(-1)^m}{m\pi}\left[\frac{\sinh\alpha_m\eta}{\alpha_m^2\sinh\alpha_m b} - \frac{\sinh\beta_m\eta}{\beta_m^2\sinh\beta_m b} + \frac{2\lambda^2\eta}{\alpha_m^2\beta_m^2 b}\right]\sin k_m\zeta\right\}
\end{aligned}$$

Where $\Phi(x,y) = x^2 - (2-\nu)y^2$; $\alpha_m = \sqrt{k_m^2+\lambda^2}$; $\beta_m = \sqrt{k_m^2-\lambda^2}$

$\alpha_n = \sqrt{k_n^2+\lambda^2}$; $\beta_n = \sqrt{k_n^2-\lambda^2}$; $\lambda = K^{1/4}$

Equation (11) applies to the terms for which $k_m^2 \geq \lambda^2$ and $k_n^2 \geq \lambda^2$. In the case of $k_m^2 \leq \lambda^2$ and $k_n^2 \leq \lambda^2$,the expression of the terms in equation (11 can be modified by replacing $\beta_m = \sqrt{k_m^2-\lambda^2}$ and $\beta_n = \sqrt{k_n^2-\lambda^2}$ with $\beta_m = i\sqrt{k_m^2-\lambda^2}$ and

$\beta_n = i\sqrt{k_n^2 - \lambda^2}$ ($i = \sqrt{-1}$), respectively, and by making use of the identities $\sinh ix = i\sin x$ and $\cosh ix = \cos x$.

In order to determine the unknown coefficients $B_{1m}; B_{2m}; A_{1n}; A_{2n}; D_1; D_2; D_3$ and D_4 in equation (12), it is made to satisfy the following boundary conditions:

$$\frac{\partial^3 W(\zeta,0)}{\partial \eta^3} + (2-\nu)\frac{\partial^3 W(\zeta,0)}{\partial \eta \partial \zeta^2} = 0; \quad \frac{\partial^3 W(\zeta,b)}{\partial \eta^3} + (2-\nu)\frac{\partial^3 W(\zeta,b)}{\partial \eta \partial \zeta^2} = 0 \qquad (16,17)$$

$$\frac{\partial^3 W(0,\eta)}{\partial \zeta^3} + (2-\nu)\frac{\partial^3 W(0,\eta)}{\partial \eta^2 \partial \zeta} = 0; \quad \frac{\partial^3 W(a,\eta)}{\partial \zeta^3} + (2-\nu)\frac{\partial^3 W(a,\eta)}{\partial \eta^2 \partial \zeta} = 0 \qquad (18\text{-}19)$$

$$\frac{\partial^2 W(0,0)}{\partial \zeta \partial \eta} = 0; \quad \frac{\partial^2 W(a,a)}{\partial \zeta \partial \eta} = 0; \quad \frac{\partial^2 W(0,b)}{\partial \zeta \partial \eta} = 0; \quad \frac{\partial^2 W(a,b)}{\partial \zeta \partial \eta} = 0 \qquad (20\text{-}23)$$

Equations (16)-(23) indicate that the distributed vertical edge reactions along four edges of the plate in Figure 2 are zero and equations (20)-(23) show that the vertical concentrated force acting at the four corners of the plate are zero. It is easy to prove that the boundary conditions can be satisfied automatically, provided that bending moments distributed along the plate edges are zero,

Substituting equation (15) into (16)-(23), eight homogeneous algebraic equations relating the unknown displacement coefficients $B_{1m}; B_{2m}; A_{1n}; A_{2n}; D_1; D_2;$ and D_4 can be obtained as follows.

$$\begin{aligned}
& B_{1m}[-\alpha_m \Phi^2(\alpha_m,k_m) c\tanh \alpha_m b + \beta_m \Phi^2(\beta_m,k_m) c\tanh \beta_m b] \\
& + B_{2m}\left[\frac{\alpha_m \Phi^2(\alpha_m,k_m)}{\sinh \alpha_m b} - \frac{\beta_m \Phi^2(\beta_m,k_m)}{\sinh \beta_m b}\right] \\
& - \sum_{n=1}^{\infty} A_{1n} \frac{k_m k_n}{K_{mn}} C_{mn} + \sum_{n=1}^{\infty} (-1)^m A_{1n} \frac{k_m k_n}{K_{mn}} C_{mn} \\
& - D_1\left(\frac{2\lambda^4}{m\pi}\right)\left[\frac{\Phi(\alpha_m,k_m)}{\alpha_m} c\tanh \alpha_m b - \frac{\Phi(\beta_m,k_m)}{\beta_m} c\tanh \beta_m b - \frac{2\lambda^2(2-\nu)k_m^2}{\alpha_m^2 \beta_m^2}\right] \\
& - (-1)^m D_2\left(\frac{2\lambda^4}{m\pi}\right)\left[\frac{\Phi(\alpha_m,k_m)}{\alpha_m} c\tanh \alpha_m b - \frac{\Phi(\beta_m,k_m)}{\beta_m} c\tanh \beta_m b - \frac{2\lambda^2(2-\nu)k_m^2}{\alpha_m^2 \beta_m^2}\right] \\
& + D_3\left(\frac{2\lambda^4}{m\pi}\right)\left[\frac{\Phi(\alpha_m,k_m)}{\alpha_m \sinh \alpha_m b} - \frac{\Phi(\beta_m,k_m)}{\beta_m \sinh \beta_m b} - \frac{2\lambda^2(2-\nu)k_m^2}{\alpha_m^2 \beta_m^2}\right]
\end{aligned} \qquad (24)$$

where $C_{mn} = \dfrac{4}{a\lambda^2}[\lambda^2(2-\nu) + k_m^2 k_n^2 (1-\nu)^2]$

If N terms are taken for each of B_{1m}, B_{2m}, A_{1n} and A_{2n} in equation (24), a set of 4N+4 homogenous algebraic equations can be established after enforcing the other seven boundary conditions, (16)-(23). From algebra theory, it is easy to solve these algebraic equations to determine the unknown quantities B_{1m}, B_{2m}, A_{1n} , A_{2n} , D_1, D_2, D_3 and D_4

3 The computed results and comparison

To test the validity of the aforementioned method, we take the thin plate as shown in Fig.1 as an example. The parameters are taken from reference [1] a=b=1m,c=0.5m, h=0.18m, $E=300MN/m^2$,$k=50MN/m^2$, ν=0.35,$q=1N/m^2$.The calculated results are listed in table 1.

Table 1 the deflections and stresses of thin plate

Location m		Deflection (10^{-8} m)					Stress (N/m^2)				
x	y	[1]	[2]	[3]	[4]	[5]	[1]	[2]	[3]	[4]	[5]
0.5	0.5	0.561	0.581	0.576	0.591	0.585	0.842	0.821	0.928	0.823	0.843
0.5	0.4	0.556	0.574	0.568	0.583	0.577	0.778	0.753	0.866	0.752	0.780
0.5	0.3	0.541	0.550	0.546	0.559	0.555	0.582	0.553	0.668	0.576	0.581
0.5	0.2	0.520	0.528	0.530	0.523	0.521	0.253	0.267	0.352	0.253	0.267
0.5	0.1	0.500	0.501	0.501	0.494	0.494	0.062	0.027	0.154	0.061	0.067
0.5	0.0	0.484	0.477	0.483	0.473	0.478	0.009	0.000	0.091	0.000	0.000

In table 1, [1] ,[2],[3],[4] and[5] represent the results from references [1];[2]; [3];[4] and that in the papers respectively. The data in table 1 show that the calculated results tally with those obtained by other methods.

In the above analytical procedure, the governing differential equation is satisfied throughout, and the boundary conditions can be met to any degree of accuracy by taking more terms in equation (24). However, after conducting convergence tests, it is shown that only 35 terms (e.g. K=35) need to be taken in order to provide an accuracy of up to four significant digits in the displacement of the plate investigated

4 Conclusions

The reciprocal theorem method used for obtaining analytical type solutions of rectangular plates supported on a foundation with free edges has been introduced and described. For illustrative purposes it has been utilized to calculate the response of a plate on a foundation with four free edges. Only simple definite integration is needed to obtain a solution for this problem and the governing differential equation can be satisfied throughout the domain of the plate. Boundary conditions are satisfied to any desired degree of accuracy. It is a known shortcoming of solution of the conventional Rayleigh- Ritz method that it is not possible to select shape functions which exactly satisfy the free edge conditions. The problem is completely eliminated here, since, as is the case with the superposition method, no function needs to be chosen. The mathematical technique described here is applicable to other complicated problems, such as point-supported plates and plates supported with elastic edges.

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