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Subordination Results on Certain Subclasses of Analytic Functions Involving Generalized Differential and Integral Operators

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Abstract

In this paper, we define several interesting subordination results for some subclasses containing generalized differential and integral operators defined on the class of analytic functions in the unit disc involving k-th Hadamard product.

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1 Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} \ldots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1}$$

Given the functions $f, g \in \mathcal{A}$, $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$, their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$(f * g)(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad , z \in \mathbb{U}.$$

And for several functions $f_1, \ldots, f_m \in \mathcal{A}$

$$(f_1 * \ldots * f_m)(z) = f_1(z) * \ldots * f_m(z) = z + \sum_{n=2}^{\infty} (a_{1n} \ldots a_{mn}) z^n \qquad , z \in \mathbb{U}.$$

In [2], Darus and Ibrahim used the Hadamard product of k-ht order and defined the following generalized differential operator. For $f \in \mathcal{A}, \alpha \geq 0, \lambda \geq 0$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $D^k_{\alpha,\lambda} f(0) = 0$,

$$D^0 f(z) = f(z)$$

$$D^1_{\alpha,\lambda} f(z) = (\alpha - \lambda) f(z) + (\lambda - \alpha + 1) z f'(z)$$

$$\vdots$$

$$D^k_{\alpha,\lambda} f(z) = D^1_{\alpha,\lambda}(D^{k-1}_{\alpha,\lambda} f(z)) = z + \sum_{n=2}^{\infty} [(n-1)(\lambda - \alpha) + n]^k a_n z^n.$$

Also, the authors in [2] defined the following generalized integral operator. Let

$$\phi f(z) = \frac{(\lambda - \alpha) z}{(1-z)^2} - \frac{(\lambda - \alpha) z}{1 - z} + \frac{z}{(1-z)^2}$$

and

$$F(z) = \underbrace{\phi f(z) * \ldots * \phi f(z)}_{k - times}$$

$$= z + \sum_{n=2}^{\infty} [(n-1)(\lambda - \alpha) + n]^k z^n.$$

The integral operator $I^k_{\alpha,\lambda}$ such that $I^k_{\alpha,\lambda} = [F(z)]^{-1} * f(z), (z \in \mathbb{U})$ where $f \in \mathcal{A}$ and

$$F(z) * [F(z)]^{-1} = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$$

implies

$$[F(z)]^{-1} = z + \sum_{n=2}^{\infty} \frac{1}{[(n-1)(\lambda - \alpha) + n]^k} z^n$$

thus they have

$$I^k_{\alpha,\lambda}f(z) = z + \sum_{n=2}^{\infty} \frac{a_n}{[(n-1)(\lambda-\alpha)+n]^k} z^n.$$

Then they defined the following subclasses of $\mathcal{A}$ involving the above generalized differential operator in [2]. Let $\mathcal{M}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{\frac{z[D^k_{\alpha,\lambda}f(z)]'}{D^k_{\alpha,\lambda}f(z)}\right\} < \mu, \qquad , z \in \mathbb{U}$$

for some $\mu(\mu > 1)$. And let $\mathcal{N}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{\frac{z[D^k_{\alpha,\lambda}f(z)]''}{[D^k_{\alpha,\lambda}f(z)]'}\right\} < \mu, \qquad , z \in \mathbb{U}$$

for some $\mu(\mu > 1)$. Furthermore, in the same paper, they defined the following subclasses of $\mathcal{A}$ involving the above generalized integral operator. Let $\mathcal{S}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{\frac{z[I^k_{\alpha,\lambda}f(z)]'}{I^k_{\alpha,\lambda}f(z)}\right\} < \mu, \qquad , z \in \mathbb{U}$$

for some $\mu(\mu > 1)$. And let $\mathcal{K}^k_{\alpha,\lambda}(\mu)$ be the subclass of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\Re\left\{\frac{z[I^k_{\alpha,\lambda}f(z)]''}{[I^k_{\alpha,\lambda}f(z)]'}\right\} < \mu, \qquad , z \in \mathbb{U}$$

for some $\mu(\mu > 1)$.

2 Preliminaries

To prove our main results, we need the following definitions and lemmas.

Definition 2.1 *(Hadamard Product) Given two functions f and g in the class $\mathcal{A}$ where f is given by* (1) *and g is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) $f * g$ is defined as*

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{U}.$$

Definition 2.2 *(Subordination Principle) For analytic functions f and g in $\mathbb{U}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then the subordinate is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.*

Definition 2.3 *[5](Subordinating factor sequences) A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever f is analytic, univalent and convex in $\mathbb{U}$, we have the subordination is given by*

$$\sum_{n=1}^{\infty} a_n c_n z^n \prec f(z) \qquad\qquad z \in \mathbb{U},\ a_1 = 1.$$

Lemma 2.4 *[5] The sequence $\{c_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1 + 2\sum_{n=1}^{\infty} c_n z^n\right\} > 0 \qquad z \in \mathbb{U}.$$

We begin by recalling each of the following coefficient inequalities associated with the function classes $\mathcal{M}^k_{\alpha,\lambda}(\mu)$,$\mathcal{N}^k_{\alpha,\lambda}(\mu)$, $\mathcal{S}^k_{\alpha,\lambda}(\mu)$ and $\mathcal{K}^k_{\alpha,\lambda}(\mu)$, respectively.

Theorem 2.5 *[2] Let $f \in \mathcal{A}$. Then $f \in \mathcal{M}^k_{\alpha,\lambda}(\mu)$ if*

$$\sum_{n=2}^{\infty} |[(n-1)(\lambda-\alpha)+n]^k| \left\{(n-\kappa) + |n+\kappa-2\mu|\right\} |a_n| \leq 2(\mu-1)$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$.

Theorem 2.6 *Let $f \in \mathcal{A}$. Then $f \in \mathcal{N}^k_{\alpha,\lambda}(\mu)$ if*

$$\sum_{n=2}^{\infty} n|[(n-1)(\lambda-\alpha)+n]^k| \left\{(n-\kappa) + |n+\kappa-2\mu|\right\} |a_n| \leq 2(\mu-1)$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$.

Theorem 2.7 *[2] Let $f \in \mathcal{A}$. Then $f \in \mathcal{S}^k_{\alpha,\lambda}(\mu)$ if*

$$\sum_{n=2}^{\infty} \frac{\{(n-\kappa) + |n+\kappa-2\mu|\}}{|[(n-1)(\lambda-\alpha)+n]^k|} |a_n| \leq 2(\mu-1)$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$.

Theorem 2.8 *Let $f \in \mathcal{A}$. Then $f \in \mathcal{K}^k_{\alpha,\lambda}(\mu)$ if*

$$\sum_{n=2}^{\infty} \frac{n\left\{(n-\kappa)+|n+\kappa-2\mu|\right\}}{|[(n-1)(\lambda-\alpha)+n]^k|}|a_n| \leq 2(\mu-1)$$

for some $0 \leq \kappa \leq 1$ and $\mu > 1$.

In the present paper,we obtain the several subordination results associated with the classes $\mathcal{M}^k_{\alpha,\lambda}(\mu)$, $\mathcal{N}^k_{\alpha,\lambda}(\mu)$,$\mathcal{S}^k_{\alpha,\lambda}(\mu)$ and $\mathcal{K}^k_{\alpha,\lambda}(\mu)$ by employing the tecnique used earlier by Attiya [1], Srivastava and Attiya [4].

3 Main Results

Theorem 3.1 *Let the function f defined by* (1) *be in the class $\mathcal{M}^k_{\alpha,\lambda}(\mu)$. Also let $\mathbb{C}$ denote the familiar class of functions $f \in \mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then*

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}(f*g)(z) \prec g(z) \quad (2)$$

for every function g in $\mathbb{C}$, and

$$\Re\{f(z)\} > -\frac{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}. \quad (3)$$

The constant factor $\frac{|(2+\lambda-\alpha)^k|\{(2-\kappa)+|2(\mu-1)-\kappa|\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\{(2-\kappa)+|2(\mu-1)-\kappa|\}\right]}$ is the best estimate.

Proof. Let $f \in \mathcal{M}^k_{\alpha,\lambda}(\mu)$ and let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathbb{C}$. Then we have

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}(f*g)(z)$$

$$= \frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}(z+\sum_{n=2}^{\infty} a_n b_n z^n).$$

Thus, by Definition 2.3, the assertion of the theorem will holds if the sequence

$$\left\{\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}a_n\right\}_{n=1}^{\infty} \quad (4)$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 2.4, this is equivalent to the inequality.

$$\Re\left\{1+\sum_{n=1}^{\infty} \frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}a_n z^n\right\} > 0. \quad (5)$$

Now

$$\Re\left\{1+\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}\sum_{n=1}^{\infty}a_nz^n\right\}$$

$$=\Re\{1+\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}z$$

$$+\frac{1}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}\times$$

$$\sum_{n=2}^{\infty}|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}a_nz^n\}$$

$$\geq 1-\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}r$$

$$-\frac{1}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}\times$$

$$\sum_{n=2}^{\infty}|((n-1)(\lambda-\alpha)+n)^k|\left\{(n-\kappa)+|2+\kappa-2\mu|\right\}a_nz^n\}$$

$$\geq 1-\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}r$$

$$-\frac{2(\mu-1)}{2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}r$$

$$=1-r>0.$$

So, (5) holds true in $\mathbb{U}$. This proves the inequality (2). The inequality (3) follows from (5) by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n$. To prove the sharpness of the constant

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]},$$

we consider the function $f_0\in\mathcal{M}^k_{\alpha,\lambda}(\mu)$ given by

$$f_0(z)=z-\frac{2(\mu-1)}{|(2+\lambda-\alpha)^k|(2-\kappa)+|2(\mu-1)-\kappa|}z^2,$$

thus from (2), we have

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}f_0(z)\prec\frac{z}{1-z},\quad z\in\mathbb{U}$$

it can be easily verified that

$$\min_{|z|<1}\left\{\Re\left(\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}f_0(z)\right)\right\}=-\frac{1}{2}.$$

This shows that the constant

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2\left[2(\mu-1)+|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}$$

is the best possible and so this completes the proof of the theorem. ■

The proofs of the following Theorems 3.2, 3.3 and 3.4 are much akin to that of Theorem 3.1.

Theorem 3.2 *Let the function f defined by* (1) *be in the class $\mathcal{N}^k_{\alpha,\lambda}(\mu)$. Also let $\mathbb{C}$ denote the familiar class of functions $f\in\mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then*

$$\frac{|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}{2(\mu-1)+2|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}(f*g)(z)\prec g(z) \quad (6)$$

for every function g in $\mathbb{C}$, and

$$\Re\{f(z)\}>-\frac{\left[2(\mu-1)+2|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}\right]}{2|(2+\lambda-\alpha)^k|\left\{(2-\kappa)+|2(\mu-1)-\kappa|\right\}}. \quad (7)$$

The constant factor $\frac{|(2+\lambda-\alpha)^k|\{(2-\kappa)+|2(\mu-1)-\kappa|\}}{2(\mu-1)+2|(2+\lambda-\alpha)^k|\{(2-\kappa)+|2(\mu-1)-\kappa|\}}$ *is the best estimate.*

Theorem 3.3 *Let the function f defined by* (1) *be in the class $\mathcal{S}^k_{\alpha,\lambda}(\mu)$. Also let $\mathbb{C}$ denote the familiar class of functions $f\in\mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then*

$$\frac{(2-\kappa)+|2(\mu-1)-\kappa|}{2\left[2(\mu-1)|(2+\lambda-\alpha)^k|+(2-\kappa)+|2(\mu-1)-\kappa|\right]}(f*g)(z)\prec g(z) \quad (8)$$

for every function g in $\mathbb{C}$, and

$$\Re\{f(z)\}>-\frac{2(\mu-1)|(2+\lambda-\alpha)^k|+(2-\kappa)+|2(\mu-1)-\kappa|}{(2-\kappa)+|2(\mu-1)-\kappa|}\quad z\in\mathbb{U}. \quad (9)$$

The constant factor $\frac{(2-\kappa)+|2(\mu-1)-\kappa|}{2\left[2(\mu-1)|(2+\lambda-\alpha)^k|+(2-\kappa)+|2(\mu-1)-\kappa|\right]}$ *is the best estimate.*

Theorem 3.4 *Let the function f defined by (1) be in the class $\mathcal{K}^k_{\alpha,\lambda}(\mu)$. Also let $\mathbb{C}$ denote the familiar class of functions $f \in \mathcal{A}$ which are univalent and convex in $\mathbb{U}$. Then*

$$\frac{(2-\kappa)+|2(\mu-1)-\kappa|}{2(\mu-1)|(2+\lambda-\alpha)^k|+2\{(2-\kappa)+|2(\mu-1)-\kappa|\}}(f*g)(z) \prec g(z) \quad (10)$$

for every function g in $\mathbb{C}$, and

$$\Re\{f(z)\} > -\frac{2(\mu-1)|(2+\lambda-\alpha)^k|+2\{(2-\kappa)+|2(\mu-1)-\kappa|\}}{(2-\kappa)+|2(\mu-1)-\kappa|}. \quad (11)$$

The constant factor $\frac{(2-\kappa)+|2(\mu-1)-\kappa|}{2(\mu-1)|(2+\lambda-\alpha)^k|+2\{(2-\kappa)+|2(\mu-1)-\kappa|\}}$ is the best estimate.

For $\lambda=\alpha, k=0, \kappa=1$ and $\mu>\frac{3}{2}$, we have

Remark 3.5 [3] If $f \in \mathcal{M}^k_{\alpha,\lambda}(\mu)$ then, $\frac{1}{3}(f*g)(z) \prec g(z),\ g \in \mathbb{C}$ and

$$\Re f(z) > -\frac{3}{2},\ z \in \mathbb{U}.$$

The constant $\frac{1}{3}$ is the best estimate.

Remark 3.6 If $f \in \mathcal{N}^k_{\alpha,\lambda}(\mu)$ then, $\frac{2}{5}(f*g)(z) \prec g(z),\ g \in \mathbb{C}$ and

$$\Re f(z) > -\frac{5}{4},\ z \in \mathbb{U}.$$

The constant $\frac{2}{5}$ is the best estimate.

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