A Short Note on Fuzzy Characteristic Interior Ideals of Po-Γ-Semigroups

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Abstract

In this paper the concept of fuzzy characteristic interior ideals in partially ordered Γ-semigroups(Po-Γ-semigroups) has been introduced. It is observed that it satisfies level subset criterion and characteristic function criterion.

1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[12]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by Lofti Zadeh[33] in his classic paper in 1965. Azirel Rosenfeld[21] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Nobuaki Kuroki[16, 17, 18, 20] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki[16, 18]. In [17], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Others who worked on fuzzy semigroup theory, such as X.Y. Xie[31, 32], Y.B. Jun[14], are mentioned in the bibliography. X.Y. Xie[31, 32] introduced the idea of extensions of fuzzy ideals in semigroups and in ordered semigroups.

The notion of a Γ-semigroup was introduced by Sen and Saha[26] as a generalization of semigroups and ternary semigroup. Γ-semigroup have been

analyzed by lot of mathematicians, for instance by Chattopadhyay[1, 2], Dutta and Adhikari[6, 7], Hila[10, 11], Chinram[3], Saha[24], Sen et al.[24, 25, 26], Seth[27]. S.K. Sardar and S.K. Majumder[8, 9, 22, 23] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ideals and ideal extensions of Γ-semigroups and studied them via its operator semigroups. Authors who worked on fuzzy partially ordered Γ-semigroup theory are Y.I. Kwon and S.K. Lee[19], Chinram[4], P. Dheena and B. Elavarasan[5], M. Siripitukdet and A. Iampan[28, 29, 30]. In [15], A.Khan, T. Mahmmod and M.I. Ali introduced the concept of fuzzy interior ideals in Po-Γ-semigroups. In this paper the concept of fuzzy characteristic ideals of a Po-Γ-semigroup has been introduced and it is observed that it satisfies level subset criterion as well as characteristic function criterion. Actually this paper is the extension of [15].

2   Preliminaries

In this section we discuss some elementary definitions which will be used in the sequel(see. [13, 22, 33]).

- Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mapping from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

- A Γ-semigroup S is called a po-Γ-semigroup if for any $a, b, c \in S$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

- Let S be a po-Γ-semigroup. By a subsemigroup of S we mean a non-empty subset A of S such that $A\Gamma A \subseteq A$.

- Let S be a po-Γ-semigroup. A subsemigroup A of S is called an interior ideal of S if (1) $S\Gamma A\Gamma S \subseteq A$, (2) $a \in A$ and $b \leq a$ imply $b \in A$.

- A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.

- Let μ be a fuzzy subset of a non-empty set X. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ for $t \in [0, 1]$, is called the t-cut of μ.

3   Main Results

In what follows $Aut(S)$ denote the set of all automorphisms of the po-Γ-semigroup S.

Definition 3.1 *A non-empty fuzzy subset μ of a po-Γ-semigroup S is called a fuzzy subsemigroup of S if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$ $\forall x, y \in S$, $\forall \gamma \in \Gamma$.*

Definition 3.2 *A fuzzy subsemigroup μ of a po-Γ-semigroup S is called a fuzzy interior ideal of S if (1) $\mu(x\alpha a\beta y) \geq \mu(a)$ $\forall x, a, y \in S$, $\forall \alpha, \beta \in \Gamma$, (2) $x \leq y$ implies $\mu(x) \geq \mu(y)$ $\forall x, y \in S$.*

Definition 3.3 *An interior ideal A of a po-Γ-semigroup S is called a characteristic interior ideal of S if $f(A) = A$ $\forall f \in Aut(S)$.*

Definition 3.4 *A fuzzy interior ideal μ of a po-Γ-semigroup S is called a fuzzy characteristic interior ideal of S if $\mu(f(x)) = \mu(x)$ $\forall x \in S$ and $\forall f \in Aut(S)$.*

Theorem 3.5 *A non-empty fuzzy subset μ of a po-Γ-semigroup S is a fuzzy interior ideal of S if and only if the t-cut of μ is an interior ideal of S for all $t \in [0,1]$, provided μ_t is non-empty.*

Proof: Let μ be a fuzzy interior ideal of S and $t \in [0,1]$ be such that μ_t is non-empty. Let $x, y \in \mu_t$ and $\gamma \in \Gamma$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Now since μ is a fuzzy interior ideal of S, so it is a fuzzy subsemigroup of S and hence $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$. Then $\mu(x\gamma y) \geq t$. Consequently, $x\gamma y \in \mu_t$. Hence μ_t is a subsemigroup of S. Now let $x, y \in S$, $\beta, \delta \in \Gamma$ and $a \in \mu_t$. Then $\mu(x\beta a\delta y) \geq \mu(a) \geq t$ and so $x\beta a\delta y \in \mu_t$.

Let $x, y \in S$ be such that $y \leq x$. Let $x \in \mu_t$. Then $\mu(x) \geq t$. Since μ is a fuzzy interior ideal of S, so $\mu(y) \geq \mu(x) \geq t$. Consequently, $y \in \mu_t$. Hence μ_t is an interior ideal of S.

In order to prove the converse, we have to show that μ satisfies the condition of Definition 3.1 and the condition of Definition 3.2. If the condition of Definition 3.1 is false, then there exist $x_0, y_0 \in S$, $\gamma \in \Gamma$ such that $\mu(x_0\gamma y_0) < \min\{\mu(x_0), \mu(y_0)\}$.

Taking
$$t_0 := \frac{1}{2}[\mu(x_0\gamma y_0) + \min\{\mu(x_0), \mu(y_0)\}],$$

we see that $\mu(x_0\gamma y_0) < t_0 < \min\{\mu(x_0), \mu(y_0)\}$. This implies that $x_0, y_0 \in \mu_{t_0}$ and $x_0\gamma y_0 \notin \mu_{t_0}$, which is a contradiction. Hence the condition of Definition 3.1 is true. Similarly we can prove the other condition also.

Let $x, y \in S$ be such that $x \leq y$. Let $\mu(y) = t$, then $y \in \mu_t$. Since μ_t is an interior ideal of S, so $x \in \mu_t$. Then $\mu(x) \geq t = \mu(y)$. Consequently, μ is a fuzzy interior ideal of S.

Using Theorem 3.5 we can prove the following.

Theorem 3.6 *A non-empty fuzzy subset μ of a po-Γ-semigroup S is a fuzzy characteristic interior ideal of S if and only if the t-cut of μ is an characteristic interior ideal of S for all $t \in [0,1]$, provided μ_t is non-empty.*

Theorem 3.7 *Let A be a non-empty subset of a po-Γ-semigroup S. Then A is a characteristic interior ideal of S if and only if its characteristic function χ_A is a fuzzy characteristic interior ideal of S.*

Proof: Let A be a characteristic interior ideal of S. Then A is an interior ideal of S. Hence by Theorem 1[15], χ_A is a fuzzy interior ideal of S. Let $x \in S$. If $x \in A$, then $\chi_A(x) = 1$. Then for all $f \in Aut(S)$, $f(x) \in f(A) = A$ whence $\chi_A(f(x)) = 1$. If $x \notin A$, then $\chi_A(x) = 0$. Then for all $f \in Aut(S)$, $f(x) \notin f(A)$ whence $\chi_A(f(x)) = 0$. Thus we see that $\chi_A(f(x)) = \chi_A(x)$ for all $x \in S$. Hence χ_A is a fuzzy characteristic interior ideal of S.

Conversely, let χ_A be a fuzzy characteristic interior ideal of S. Then χ_A is a fuzzy interior ideal of S. Hence by Theorem 1[15], A is an interior ideal of S. Now, let $f \in Aut(S)$ and $a \in A$. Then $\chi_A(a) = 1$ and so $\chi_A(f(a)) = \chi_A(a) = 1$. Hence $f(a) \in A$. Thus, we obtain $f(A) \subseteq A$ for all $f \in Aut(S)$. Again $a \in A$, $f \in Aut(S) \Rightarrow \exists b \in S \ni f(b) = a$. If possible, suppose that $b \notin A$. Then $\chi_A(b) = 0$. So $\chi_A(f(b)) = \chi_A(b) = 0$. Hence $f(b) \notin A$, i.e., $a \notin A$-a contradiction. Hence $b \in A$, i.e., $f(b) \in A$. Thus we obtain $A \subseteq f(A)$ $\forall f \in Aut(S)$. Hence the proof.

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