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Exact soliton solutions of the Huxley equation by the modified $(\frac{G'}{G})$-expansion method

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Abstract

The modified $(\frac{G'}{G})$-expansion method is one of the effective methods to find exact travelling wave of nonlinear evolution equations. In this paper, we look for exact solutions of the Huxley equation by the modified $(\frac{G'}{G})$-expansion method.

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1 Introduction

In this paper, we will find exact solutions of the Huxley equation, in the form

$$u_t = u_{xx} + u(k-u)(u-1)$$

which is a core mathematical framework for modern biophysically based neural modeling. There are many methods to solve nonlinear partial differential equations (NLPDEs), such as tanh-sech method [1−4], extended tanh method [5 − 8], hyperbolic function method [9], sine-cosine method [10 − 12], Jacobi elliptic function expansion method [13], F-expansion method [14], and the transformed rational function method [15].
Very recently, Wang et al. [16] introduced a new method called the $(\frac{G'}{G})$-expansion method to look for traveling wave solutions of nonlinear evolution equations. The $(\frac{G'}{G})$-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$, and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation(LODE). By using the $(\frac{G'}{G})$-expansion method, Wang et al. successfully obtain more traveling wave solutions of four nonlinear evolution equations.
Lately, work has been done on the extensions of the $(\frac{G'}{G})$-expansion method. For example, in [17], the method was modified to deal with the mKdV equation with variable coefficients. In [18], the method was modified to find more types of non-traveling wave and coefficient function solutions.

2 Modified $(\frac{G'}{G})$-expansion method

Considering the nonlinear partial differential in the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G'}{G})$-expansion method.
Step 1. Combining the independent variables x and t into one variable $\xi = x - vt$, we suppose that

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{2}$$

the traveling wave variable (2) permits us reducing Eq.(1) to an ordinary differential equation(ODE) for $u = u(\xi)$

$$P(u, -vu', u', v^2u'', -vu'', u'', ...) = 0, \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i (\frac{G'}{G})^i + \sum_{i=1}^{m} \beta_i (\frac{G'}{G})^{-i}, \tag{4}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

$\alpha_0, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m, \lambda$ and μ are constants to be determined later, $\alpha_m \neq 0$, or $\beta_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).
Step 3. By substituting (4) into (3) and using second order LODE (5),collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq.(3) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero,yields a set of algebraic equations for $\alpha_0, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m, \lambda$ and μ.
Step4. Assuming that the constants $\alpha_0, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m, \lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting $\alpha_0, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m, v$ and the general solutions of Eq.(5) into (4) we have more travelling wave solutions of the nonlinear evolution equation (1).

3 Huxley equation

We consider the Huxley equation

$$u_t = u_{xx} + u(k - u)(u - 1), \tag{6}$$

where $k \neq 0$.
By making the transformation

$$u(x,t) = u(\xi), \quad \xi = x - vt,$$

the Eq.(6) becomes

$$-vu' - u'' + u^3 - (k+1)u^2 + ku = 0. \tag{7}$$

Suppose that the solution of ODE (7) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i (\frac{G'}{G})^i + \sum_{i=1}^{m} \beta_i (\frac{G'}{G})^{-i}, \tag{8}$$

where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{9}$$

Considering the homogeneous balance between $u''(\xi)$ and $u^3(\xi)$ in Eq.(7), we required that

$$m + 2 = 3m$$

then $m = 1$, so we can write (8) as

$$u(\xi) = \alpha_1(\frac{G'}{G}) + \alpha_0 + \beta_1(\frac{G'}{G})^{-1}, \tag{10}$$

and therefore

$$u^2(\xi) = \alpha_1^2(\frac{G'}{G})^2 + 2\alpha_0\alpha_1(\frac{G'}{G}) + (\alpha_0^2 + 2\alpha_1\beta_1) + 2\alpha_0\beta_1(\frac{G'}{G})^{-1} + \beta_1^2(\frac{G'}{G})^{-2}, \tag{11}$$

$$\begin{aligned} u^3(\xi) &= \alpha_1^3(\frac{G'}{G})^3 + 3\alpha_0\alpha_1^2(\frac{G'}{G})^2 + 3\alpha_1(\alpha_1\beta_1 + \alpha_0^2)(\frac{G'}{G}) + \alpha_0^3 + 6\alpha_0\alpha_1\beta_1 \\ &+ 3\beta_1(\alpha_0^2 + \alpha_1\beta_1)(\frac{G'}{G})^{-1} + 3\alpha_0\beta_1^2(\frac{G'}{G})^{-2} + \beta_1^3(\frac{G'}{G})^{-3}, \end{aligned} \tag{12}$$

$$u'(\xi) = -\alpha_1(\frac{G'}{G})^2 - \alpha_1\lambda(\frac{G'}{G}) + (\beta_1 - \mu\alpha_1) + \beta_1\lambda(\frac{G'}{G})^{-1} + \beta_1\mu(\frac{G'}{G})^{-2}, \tag{13}$$

$$\begin{aligned} u''(\xi) &= 2\alpha_1(\frac{G'}{G})^3 + 3\alpha_1\lambda(\frac{G'}{G})^2 + \alpha_1(\lambda^2 + 2\mu)(\frac{G'}{G}) + \alpha_1\mu\lambda + \lambda\beta_1 \\ &+ \beta_1(\lambda^2 + 2\mu)(\frac{G'}{G})^{-1} + 3\lambda\mu\beta_1(\frac{G'}{G})^{-2} + 2\beta_1\mu^2(\frac{G'}{G})^{-3}. \end{aligned} \tag{14}$$

By substituting $(10)-(14)$ into ODE.(7) and collecting all terms with the same power of $(\frac{G'}{G})$ together, the left-hand side of ODE.(7) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for $\alpha_0, \alpha_1, \beta_1, k, v, \mu$ and λ as follows:

$$\begin{aligned}
(\frac{G'}{G})^3 &: \quad -2\alpha_1 + \alpha_1^3 = 0, \\
(\frac{G'}{G})^2 &: \quad v\alpha_1 - 3\alpha_1\lambda + 3\alpha_0\alpha_1^2 - (k+1)\alpha_1^2 = 0, \\
(\frac{G'}{G})^1 &: \quad v\alpha_1\lambda - (\lambda^2\alpha_1 + 2\alpha_1\mu) + 3\alpha_0^2\alpha_1 + 3\alpha_1^2\beta_1 - 2(k+1)\alpha_0\alpha_1 + k\alpha_1 = 0, \\
(\frac{G'}{G})^0 &: \quad -v\beta_1 + v\alpha_1\mu - \alpha_1\lambda\mu - \beta_1\lambda + \alpha_0^3 + 6\alpha_0\alpha_1\beta_1 - (k+1)\alpha_0^2 - 2(k+1)\alpha_1\beta_1 + k\alpha_0 = 0, \\
(\frac{G'}{G})^{-1} &: \quad -v\beta_1\lambda - (2\beta_1\mu + \beta_1\lambda^2) + 3\alpha_0^2\beta_1 + 3\alpha_1\beta_1^2 - 2(k+1)\alpha_0\beta_1 + k\beta_1 = 0, \\
(\frac{G'}{G})^{-2} &: \quad -v\beta_1\mu - 3\beta_1\mu\lambda + 3\alpha_0\beta_1^2 - (k+1)\beta_1^2 = 0, \\
(\frac{G'}{G})^{-3} &: \quad -2\beta_1\mu^2 + \beta_1^3 = 0.
\end{aligned}$$

Solving the algebraic equations above with aid Maple, yields

Case I:

$$\alpha_1 = \sqrt{2}, \quad \alpha_0 = \frac{\sqrt{2}(k+1)+3\lambda - v}{3\sqrt{2}}, \quad \beta_1 = 0, \quad \mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}, \tag{15}$$

λ, v and k are arbitrary constants.

By using (15), expression (10) can be written as

$$u(\xi) = \sqrt{2}(\frac{G'}{G}) + \frac{\sqrt{2}(k+1)+3\lambda - v}{3\sqrt{2}}, \tag{16}$$

where $\xi = x - vt$ and $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$. Eq.(16) is the formula of a solution of Eq.(7).

Substituting the general solutions of Eq.(9) into Eq.(16) we have three types of traveling wave solutions of the Huxley equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_{1,1}(\xi) = \frac{\sqrt{2}}{2}(\frac{c_1 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_2 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_1 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}) + \frac{2(k+1) - v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.

In particular, if $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_{1,1} = u_{1,1}(\xi)$ can be written as:

$$u_{1,1}(\xi) = \frac{\sqrt{2}}{2}\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + \xi_0) + \frac{2(k+1) - v\sqrt{2}}{6},$$

where $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$.

When $\lambda^2 - 4\mu < 0$,

$$u_{1,2}(x,t) = \frac{\sqrt{2}}{2}(\frac{-c_1 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}) + \frac{2(k+1) - v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$,

$$u_{1,3}(x,t) = \frac{\sqrt{2}c_2}{c_1 + c_2\xi} + \frac{2(k+1) - v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.

Case II:

$$\alpha_1 = -\sqrt{2}, \quad \alpha_0 = \frac{\sqrt{2}(k+1)-3\lambda + v}{3\sqrt{2}}, \quad \beta_1 = 0, \quad \mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}, \tag{17}$$

λ, v and k are arbitrary constants.
By using (17), expression (10) can be written as

$$u(\xi) = -\sqrt{2}(\frac{G'}{G}) + \frac{\sqrt{2}(k+1) - 3\lambda + v}{3\sqrt{2}}, \tag{18}$$

where $\xi = x - vt$ and $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$. Eq.(18) is the formula of a solution of Eq.(7).
Substituting the general solutions of Eq.(9) into Eq.(18) we have three types of traveling wave solutions of the Huxley equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$u_{2,1}(\xi) = -\frac{\sqrt{2}}{2}(\frac{c_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{c_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + c_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}) + \frac{2(k+1) + v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
If $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_{2,1} = u_{2,1}(\xi)$ can be written as:

$$u_{2,1}(\xi) = -\frac{\sqrt{2}}{2}\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + \xi_0) + \frac{2(k+1) + v\sqrt{2}}{6},$$

where $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$.
When $\lambda^2 - 4\mu < 0$,

$$u_{2,2}(x,t) = -\frac{\sqrt{2}}{2}(\frac{-c_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{c_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}) + \frac{2(k+1) + v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu = 0$,

$$u_{2,3}(x,t) = -\frac{\sqrt{2}c_2}{c_1 + c_2\xi} + \frac{2(k+1) + v\sqrt{2}}{6},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
Case III:

$$\alpha_1 = 0, \quad \alpha_0 = \frac{\sqrt{2}(k+1) + 3\lambda + v}{3\sqrt{2}}, \quad \beta_1 = \sqrt{2}\mu, \quad \mu = \frac{2(-k^2+k-1) + v^2 + 3\lambda^2}{12}, \tag{19}$$

λ, v and k are arbitrary constants.
By using (19), expression (10) can be written as

$$u(\xi) = \frac{\sqrt{2}(k+1) + 3\lambda + v}{3\sqrt{2}} + \sqrt{2}\mu(\frac{G'}{G})^{-1}, \tag{20}$$

where $\xi = x - vt$ and $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$. Eq.(20) is the formula of a solution of Eq.(7).
Substituting the general solutions of Eq.(9) into Eq.(20) we have three types of traveling wave solutions of the Huxley equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$u_{3,1}(\xi) = \frac{2(k+1)+3\lambda+v}{3\sqrt{2}} + 2\sqrt{2}\mu\left(\frac{c_1 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_2 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_1 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi} - \lambda\right)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
If $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_{3,1} = u_{3,1}(\xi)$ can be written as:

$$u_{3,1}(\xi) = \frac{2(k+1)+3\lambda+v}{3\sqrt{2}} + 2\sqrt{2}\mu(\tanh(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + \xi_0) - \lambda)^{-1},$$

where $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$.
When $\lambda^2 - 4\mu < 0$,

$$u_{3,2}(x,t) = \frac{2(k+1)+3\lambda+v}{3\sqrt{2}} + 2\sqrt{2}\mu\left(\frac{-c_1 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi} - \lambda\right)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu = 0$,

$$u_{3,3}(x,t) = \frac{2(k+1)+3\lambda+v}{3\sqrt{2}} + \sqrt{2}\mu(\frac{c_2}{c_1+c_2\xi} - \frac{1}{2}\lambda)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
Case IV:

$$\alpha_1 = 0, \quad \alpha_0 = \frac{\sqrt{2}(k+1)-3\lambda-v}{3\sqrt{2}}, \quad \beta_1 = -\sqrt{2}\mu, \quad \mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}, \tag{21}$$

λ, v and k are arbitrary constants.
By using (21), expression (10) can be written as

$$u(\xi) = \frac{\sqrt{2}(k+1)-3\lambda-v}{3\sqrt{2}} - \sqrt{2}\mu(\frac{G'}{G})^{-1}, \tag{22}$$

where $\xi = x - vt$ and $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$. Eq.(22) is the formula of a solution of Eq.(7).
Substituting the general solutions of Eq.(9) into Eq.(22) we have three types of traveling wave solutions of the Huxley equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$u_{4,1}(\xi) = \frac{2(k+1)-3\lambda-v}{3\sqrt{2}} - 2\sqrt{2}\mu\left(\frac{c_1 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_2 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_1 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi} - \lambda\right)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
If $c_1 > 0$ and $c_1^2 > c_2^2$, then $u_{4,1} = u_{4,1}(\xi)$ can be written as:

$$u_{4,1}(\xi) = \frac{2(k+1) - 3\lambda - v}{3\sqrt{2}} - 2\sqrt{2}\mu(\tanh(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + \xi_0) - \lambda)^{-1},$$

where $\xi_0 = \tanh^{-1}(\frac{c_2}{c_1})$.
When $\lambda^2 - 4\mu < 0$,

$$u_{4,2}(x,t) = \frac{2(k+1) - 3\lambda - v}{3\sqrt{2}} - 2\sqrt{2}\mu(\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{c_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + c_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} - \lambda)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.
When $\lambda^2 - 4\mu = 0$,

$$u_{4,3}(x,t) = \frac{2(k+1) - 3\lambda - v}{3\sqrt{2}} - \sqrt{2}\mu(\frac{c_2}{c_1 + c_2\xi} - \frac{1}{2}\lambda)^{-1},$$

where $\xi = x - vt$, $\mu = \frac{2(-k^2+k-1)+v^2+3\lambda^2}{12}$ and c_1, c_2 are arbitrary constants.

4 Conclusion

In this paper, the modified $(\frac{G'}{G})$-expansion method was applied successfully for solving the Huxley equation. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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