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Finite difference scheme for singularly perturbed convection-diffusion problem with two small parameters

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Abstract

In this article a numerical method involving classical finite difference scheme on non-uniform grid is constructed for a singularly perturbed convection-diffusion boundary value problem with two small parameters affecting the convection and diffusion terms. The scheme has been analyzed for uniform convergence with respect to both singular perturbation parameters. To support the theoretical error bounds numerical results are presented.

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1 Introduction

The boundary value problems for ordinary differential equations in which one or more small positive parameter(s) multiplying the derivative(s), are known as singular perturbation problems. The solutions of such kind of problems change rapidly in a narrow region called boundary layer region. Classical numerical methods are inappropriate for singularly perturbed problems [1]. In this paper we consider second order two point singular perturbation problem with two small parameters multiplying to the highest and second highest derivative. Such kind of problems arise in chemical reactor theory, engineering, biology, lubrication theory etc. This type of problems were solved asymptotically by O'Malley [5–9] and numerically by Linβ and Roos [4], Gracia et al. [3], O'Riordan et al. [11], O'Riordan et al. [12] and Flaherty and O'Malley [2] etc.

O'Malley [5–9] examined the nature of asymptotic solution of the continuous problem where the ratio of μ^2 to ϵ was identified as significant. In [11,12], the standard upwind finite difference operator on two different choices of Shishkin mesh was shown to be parameter-uniform of first order. In [15] parameter-uniform methods on a uniform mesh were constructed. Vulanović [16], used the higher order finite difference scheme on a piecewise uniform mesh both of Shishkin and Bakhavalov type for solving quasi-linear boundary value problems with small parameters.

Roos and Uzelac [13] used streamline diffusion finite element method to generate a second order parameter-uniform scheme when μ is sufficiently small. In [3] Gracia et al. used classical finite difference analysis where the finite difference operator was a combination of the central difference, mid-point and standard upwind difference operators.

In this paper we construct a classical finite difference method on non-uniform grid for the two parameters singularly perturbed boundary value problems of the form

$$L_{\epsilon,\mu}y(x) \equiv -\epsilon y''(x) - \mu a(x)y'(x) + b(x)y(x) = f(x),\ x \in \Omega = (0,1), \quad (1)$$

$$y(0) = \alpha, \quad y(1) = \beta, \quad (2)$$

where $0 < \epsilon \ll 1$, $0 < \mu \ll 1$ are such that $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. The functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth satisfying

$$a(x) \geq a^* > 0, \quad b(x) \geq b^* > 0, \quad b(x)/a(x) \geq c^* > 0.$$

This problem includes both the reaction-diffusion problem when $\mu = 0$ and the convection-diffusion problem when $\mu = 1$. For this problem two boundary layers occur [5,8,9] at $x = 0$ and at $x = 1$.

Let $y_{\epsilon,\mu}$ be the solution of the continuous problem with two small parameters ϵ, μ and let $Y^N_{\epsilon,\mu}$ be a numerical approximation of $y_{\epsilon,\mu}$ obtained by using N mesh points. A numerical method is said to be parameter-uniform in the norm $\|.\|$ if

$$\|y_{\epsilon,\mu} - Y^N_{\epsilon,\mu}\| \leq C\vartheta(N), \quad \text{for } N \geq N_0,$$

where the error constant C is independent of any perturbation parameters and N. The function $\vartheta(N)$ and the natural number N_0, are independent of parameters ϵ and μ and

$$\lim_{N\to\infty} \vartheta(N) = 0.$$

Moreover a numerical method is said to be parameter-uniform of order p if

$$\|y_{\epsilon,\mu} - Y^N_{\epsilon,\mu}\| \leq CN^{-p}.$$

In other words, the numerical approximation $Y^N_{\epsilon,\mu}$ converges to $y_{\epsilon,\mu}$ for all values of ϵ and μ in the range $0 < \epsilon \ll 1$ and $0 < \mu \ll 1$.

It is well-known that in the case of the singularly perturbed reaction-diffusion problem ($\mu = 0$)

$$-\epsilon y'' + b(x)y = f(x), \quad b(x) \geq b^* > 0, \quad y(0) = \alpha, \quad y(1) = \beta,$$

the standard central difference operator on an appropriate Shishkin mesh produces almost second order parameter uniform convergence [10, 14] of the form

$$\|y_{\epsilon,\mu} - Y^N_{\epsilon,\mu}\| \leq C(N^{-1} \ln N)^2,$$

while in the case of the singularly perturbed convection-diffusion problem ($\mu = 1$)

$$-\epsilon y'' - a(x)y' = f(x), \quad a(x) \geq a^* > 0, \quad y(0) = \alpha, \quad y(1) = \beta,$$

the standard upwind difference operator on an appropriate Shishkin mesh produces almost first order parameter-uniform convergence [1]

$$\|y_{\epsilon,\mu} - Y^N_{\epsilon,\mu}\| \leq CN^{-1} \ln N.$$

2 The Continuous Problem

Consider the singularly perturbed boundary value problem (1)-(2). The operator $L_{\epsilon,\mu}$ satisfies the following maximum principle:

Lemma 2.1. *Let $\phi \in C^2(\bar{\Omega})$ be such that $\phi(0) \geq 0,\ \phi(1) \geq 0$ and $L_{\epsilon,\mu}\phi(x) \geq 0,\ \forall\, x \in \Omega$, then $\phi(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.*

Proof. Proof is by contradiction. Let $t \in \bar{\Omega}$ be such that $\phi(t) < 0$ and $\phi(t) = \min\limits_{x \in \bar{\Omega}} \phi(x)$. Then it is clear $t \notin \{0, 1\}$ and $\phi'(t) = 0,\ \phi''(t) \geq 0$. We have

$$L_{\epsilon,\mu}\phi(t) \equiv -\epsilon\phi''(t) - \mu a(t)\phi'(t) + b(t)\phi(t) < 0,$$

thus we get a contradiction. Therefore $\phi(x) \geq 0\ \forall\, x \in \bar{\Omega}$. □

An immediate consequence of this comparison principle is the following parameter-uniform bound on the solution y.

Lemma 2.2. *Let $y(x)$ be the solution of boundary value problem (1)-(2), then*

$$\|y\| \leq \max\{|\alpha|, |\beta|\} + {b^*}^{-1}\|f\|,$$

where $\|.\|$ denotes the pointwise maximum norm.

Proof. Define two barrier functions $\psi^{\pm}(x) = \max\{|\alpha|, |\beta|\} + b^{*-1}\|f\| \pm y(x)$, then it is easy to see $\psi^{\pm}(0) \geq 0$, $\psi^{\pm}(1) \geq 0$. Also we have

$$\begin{aligned} L_{\epsilon,\mu}\psi^{\pm}(x) &= b(x)(\max\{|\alpha|, |\beta|\} + b^{*-1}\|f\|) \pm L_{\epsilon,\mu}y(x) \\ &\geq b^*(\max\{|\alpha|, |\beta|\} + b^{*-1}\|f\|) \pm f(x) \\ &= b^* \max\{|\alpha|, |\beta|\} + \|f\| \pm f(x) \\ &\geq 0. \end{aligned}$$

A consequence of Lemma 2.1 gives the required estimate. ☐

Lemma 2.3. *Assuming that $a, b, f \in C^2(\bar{\Omega})$, the derivatives of the solution y of (1)-(2) satisfy the following bounds*

$$\begin{aligned} \|y^{(j)}\| &\leq \frac{C}{(\sqrt{\epsilon})^j}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon}}\right)^j\right)\max\{\|y\|, \|f\|\}, \quad j = 1, 2. \\ \|y^{(3)}\| &\leq \frac{C}{(\sqrt{\epsilon})^3}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon}}\right)^3\right)\max\{\|y\|, \|f\|, \|f'\|\}, \\ \|y^{(4)}\| &\leq \frac{C}{(\sqrt{\epsilon})^4}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon}}\right)^4\right)\max\{\|y\|, \|f\|, \|f'\|, \|f''\|\}, \end{aligned}$$

where $\|.\|$ denotes the pointwise maximum norm and C is a positive generic constant independent of ϵ and μ.

Proof. Given any $x \in \Omega$, we can construct the neighborhood $N_x = (c, c + \sqrt{\epsilon})$ of x, such that $N_x \in \Omega$. Then by mean value theorem $\exists$ a point $\eta \in N_x$ such that

$$y'(\eta) = \frac{y(c + \sqrt{\epsilon}) - y(c)}{\sqrt{\epsilon}}.$$

Taking the absolute values, we get

$$|y'(\eta)| \leq \frac{2\|y\|}{\sqrt{\epsilon}}.$$

Now integrating Eq. (1) from η to x, we get

$$\epsilon(y'(x) - y'(\eta)) = -\mu\left(a(\zeta)y(\zeta)|_{\eta}^{x} - \int_{\eta}^{x} a'(\zeta)y(\zeta)\, d\zeta\right) + \int_{\eta}^{x} b(\zeta)y(\zeta)\, d\zeta - \int_{\eta}^{x} f(\zeta)\, d\zeta.$$

Taking absolute values, we get

$$\begin{aligned} |y'(x)| &\leq |y'(\eta)| + \epsilon^{-1}(\mu(2\|a\|\|y\| + \|a'\|\|y\||x - \eta|) + \|b\|\|y\||x - \eta| + \|f\||x - \eta|) \\ &\leq \frac{C}{\sqrt{\epsilon}}\left(1 + \frac{\mu}{\sqrt{\epsilon}}\right)\|y\| + \frac{C}{\sqrt{\epsilon}}\|f\|, \quad \text{since} \quad x - \eta < \sqrt{\epsilon} \\ &= \frac{C}{\sqrt{\epsilon}}\left(1 + \frac{\mu}{\sqrt{\epsilon}}\right)\max\{\|y\|, \|f\|\}. \end{aligned}$$

Thus we get

$$\|y'\| \le \frac{C}{(\sqrt{\epsilon})}\left(1+\left(\frac{\mu}{\sqrt{\epsilon}}\right)\right)\max\{\|y\|,\|f\|\}.$$

Now using Eq. (1), we get

$$\begin{aligned}|y''(x)| &\le \epsilon^{-1}(\mu|a(x)||y'(x)|+|b(x)||y(x)|+|f(x)|)\\ &\le \epsilon^{-1}(\mu\|a\|\|y'\|+\|b\|\|y\|+\|f\|).\end{aligned}$$

Using the estimate for $\|y'\|$, we get

$$\begin{aligned}|y''(x)| &\le \epsilon^{-1}\left(\mu\|a\|\left(\frac{C}{(\sqrt{\epsilon})}\left(1+\left(\frac{\mu}{\sqrt{\epsilon}}\right)\right)\max\{\|y\|,\|f\|\}\right)+\|b\|\|y\|+\|f\|\right)\\ &\le \frac{C}{(\sqrt{\epsilon})^2}\left(1+\left(\frac{\mu}{\sqrt{\epsilon}}\right)^2\right)\max\{\|y\|,\|f\|\}.\end{aligned}$$

Thus we obtain

$$\|y\| \le \frac{C}{(\sqrt{\epsilon})^2}\left(1+\left(\frac{\mu}{\sqrt{\epsilon}}\right)^2\right)\max\{\|y\|,\|f\|\}.$$

Now differentiating Eq. (1), we get

$$-\epsilon y'''(x)-\mu(a(x)y''(x)+a'(x)y'(x))+b(x)y'(x)+b'(x)y(x)=f'(x). \tag{3}$$

Taking absolute values, we get

$$|y'''(x)| \le \epsilon^{-1}(\mu(|a(x)y''(x)|+|a'(x)y'(x)|)+|b(x)y'(x)|+|b'(x)y(x)|+|f'(x)|),$$

using the estimates for $\|y'\|$ and $\|y''\|$ we get the required estimate. Again differentiating Eq. (3) and taking absolute values and using the estimates for $\|y'\|$, $\|y''\|$ and $\|y'''\|$, we get the required estimate for $\|y^{(4)}\|$. □

In order to obtain the parameter-uniform error estimates the solution of the boundary value problem (1)-(2) can be decomposed into the regular and singular components

$$y = u + v + w, \tag{4}$$

where u is the regular component of the solution y and v and w are the left and right singular components respectively, satisfying

$$\begin{aligned}Lu &= f, \quad u(0),\ u(1) \text{ suitably chosen}, && (5)\\ Lv &= 0, \quad v(0)=y(0)-u(0),\ v(1)=0, && (6)\\ Lw &= 0, \quad w(0)=0, \quad w(1)=y(1)-u(1). && (7)\end{aligned}$$

The bounds on the derivatives of the regular and boundary layer components are given by following Lemma:

Lemma 2.4. *The following estimates [3] hold for the derivatives of the components u, v and w of the solution y of (1)*

$$\begin{aligned}
\|u^{(j)}\| &\leq C\left(1+\left(\frac{\epsilon}{\mu}\right)^{3-j}\right), \quad j=0(1)4,\\
|v(x)| &\leq C\exp(-(a^*\mu/\epsilon)x),\\
|w(x)| &\leq C\exp(-(c^*/2\mu)(1-x)),\\
\|v^{(j)}\| &\leq C(\mu/\epsilon)^j, \quad j=1(1)3,\\
\|w^{(j)}\| &\leq C\mu^{-j}, \qquad j=1(1)3.
\end{aligned}$$

3 The Discrete Problem

To approximate the solution of problem (1), we employ a finite difference scheme defined on a variable mesh. This mesh is defined as follows. Let N be the number (multiple of 4) of mesh points in the interval $[0,1]$. we divide the interval $[0,1]$ into three intervals $[0,\sigma_1]$, $[\sigma_1, 1-\sigma_2]$ and $[1-\sigma_2, 1]$, where the transition parameters σ_1 and σ_2 are given by

$$\sigma_1 = \min(1/4, (4\epsilon/a^*\mu)\ln(N)), \qquad \sigma_2 = \min(1/4, (4\mu/c^*)\ln(N)).$$

We place $N/4$, $N/2$ and $N/4$ mesh points respectively in $[0,\sigma_1]$, $[\sigma_1, 1-\sigma_2]$ and $[1-\sigma_2, 1]$. We set $h_{l,i} = x_i - x_{i-1}$ for $i = 1, 2\ldots, N/4$. Let $r_1 = \frac{h_{l,i}}{h_{l,i-1}}$ be the mesh ratio for the first interval. Then we have

$$\begin{aligned}
x_{N/4} - x_0 &= (x_{N/4} - x_{N/4-1}) + (x_{N/4-1} - x_{N/4-2}) + \ldots + (x_2 - x_1) + (x_1 - x_0)\\
&= r_1^{N/4-1}h_{l,1} + r_1^{N/4-2}h_{l,1} + \ldots + r_1 h_{l,1} + h_{l,1}\\
&= (r_1^{N/4-1} + r_1^{N/4-2} + \ldots + r_1 + 1)h_{l,1}\\
&= \frac{r_1^{N/4}-1}{r_1-1}h_{l,1}.
\end{aligned}$$

This gives

$$h_{l,1} = \sigma_1\left(\frac{r_1-1}{r_1^{N/4}-1}\right). \tag{8}$$

Therefore for given values of N and appropriate choice of r_1, we can choose $h_{l,1}$ from relation (8) and subsequent $h_{l,i}$'s can be obtained by $h_{l,i} = r_1 h_{l,i-1}$, $i = 2(1)N/4$, and hence x_i obtained for $i = 0, 1, \ldots, N/4$. We denote $\Omega_1^N = \{x_i\}_0^{N/4}$.

Again we set $h_{r,j-3N/4} = x_j - x_{j-1}$ for $j = 3N/4+1, 3N/4+2\ldots, N$. Let $r_2 = \frac{h_{r,j-3N/4}}{h_{r,j-3N/4-1}}$ be the mesh ratio for the interval $[1-\sigma_2, 1]$. Then we have

$$\begin{aligned}
x_N - x_{3N/4} &= (x_N - x_{N-1}) + (x_{N-1} - x_{N-2}) + \ldots + (x_{3N/4+2} - x_{3N/4+1}) + (x_{3N/4+1} - x_{3N/4}) \\
&= h_{r,N/4} + h_{r,N/4-1} + \ldots + h_{r,2} + h_{r,1} \\
&= r_2^{N/4-1} h_{r,1} + r_2^{N/4-2} h_{r,1} + \ldots + r_2 h_{r,1} + h_{r,1} \\
&= (r_2^{N/4-1} + r_2^{N/4-2} + \ldots + r_2 + 1) h_{r,1} \\
&= \frac{1 - r_2^{N/4}}{1 - r_2} h_{r,1}.
\end{aligned}$$

This gives

$$h_{r,1} = \sigma_2 \left(\frac{1 - r_2}{1 - r_2^{N/4}} \right). \tag{9}$$

Therefore for given values of N and appropriate choice of r_2, we can choose $h_{r,1}$ from relation (9) and subsequent $h_{r,j}$'s can be obtained by $h_{r,j} = r_2 h_{r,j-1},\ j = 2(1)N/4$ and hence x_i obtained for $i = 3N/4+1, 3N/4+2, \ldots, N$. We denote $\Omega_2^N = \{x_i\}_{3N/4+1}^N$.

For the middle interval $[\sigma_1, 1-\sigma_2]$, we set $h = x_j - x_{j-1} = 2(1-\sigma_1-\sigma_2)/N$ for $j = N/4+1, N/4+2\ldots, 3N/4$. Let $\Omega_0^N = \{x_i\}_{N/4+1}^{3N/4}$.

Let $\Omega^N = \Omega_1^N \cup \Omega_0^N \cup \Omega_2^N = \{x_i\}_0^N$. On Ω^N, we define the differential operators δ^2 and Δ corresponding to second and first derivative respectively as

$$\delta^2 Y_i = \frac{2r}{h_i^2(1+r)} [Y_{i+1} - (1+r)Y_i + rY_{i-1}],$$

and

$$\Delta Y_i = \frac{2}{h_i(1+r)} [Y_{i+1} - Y_i].$$

Then our finite difference scheme is

$$L_{\epsilon,\mu}^N Y_i \equiv r_i^- Y_{i-1} + r_i^c Y_i + r_i^+ Y_{i+1} = f_i, \tag{10}$$

where

$$
\begin{aligned}
r_i^- &= -\frac{2\epsilon r^2}{(1+r)h_i^2},\\
r_i^c &= \frac{2\epsilon r}{h_i^2} + \frac{2\mu a_i r}{(1+r)h_i} + b_i,\\
r_i^+ &= -\frac{2\epsilon r}{(1+r)h_i^2} - \frac{2\mu a_i r}{(1+r)h_i},\\
r &= \begin{cases} r_1 > 1 & \text{if} \quad x_i \in \Omega_1^N, \\ r_2 < 1 & \text{if} \quad x_i \in \Omega_2^N, \end{cases}\\
h_i &= \begin{cases} h_{l,i} & \text{if} \quad x_i \in \Omega_1^N, \\ h & \text{if} \quad x_i \in \Omega_0^N, \\ h_{r,i} & \text{if} \quad x_i \in \Omega_2^N. \end{cases}\\
\text{Let } H &= \max h_i.
\end{aligned}
$$

The difference scheme (10) satisfies the following discrete maximum principle and stability estimate

Lemma 3.1 (Discrete maximum principle)**.** *Let* ψ *be any mesh function such that* $\psi(0) \geq 0$, $\psi(1) \geq 0$, *then* $L_{\epsilon,\mu}^N \psi_i \geq 0$ *for* $1 \leq i \leq N-1$ *implies that* $\psi_i \geq 0$ *for all* $0 \leq i \leq N$.

Proof. Proof is by contradiction, suppose there is a positive integer k such that $\psi_k < 0$ for $1 \leq k \leq N-1$. Also suppose that $\psi_k = \min_{1 \leq i \leq N-1} \psi_i$. Then we have

$$
\begin{aligned}
L_{\epsilon,\mu}^N \psi_k &= r_k^- \psi_{k-1} + r_k^c \psi_i + r_k^+ \psi_{k+1}\\
&= -\frac{2\epsilon r}{(1+r)h_k^2}[r\psi_{k-1} - (1+r)\psi_k + \psi_{k+1}] - \frac{2\mu a_k r}{(1+r)h_k}(\psi_{k+1} - \psi_k) + b_k \psi_k\\
&= -\frac{2\epsilon r}{(1+r)h_k^2}[r(\psi_{k-1} - \psi_k) + (\psi_{k+1} - \psi_k)] - \frac{2\mu a_k r}{(1+r)h_k}(\psi_{k+1} - \psi_k) + b_k \psi_k.
\end{aligned}
\tag{11}
$$

Now since $\psi_k = \min_{1 \leq i \leq N-1} \psi_i$ therefore $\psi_{k-1} - \psi_k$ and $\psi_{k+1} - \psi_k$ are both positive and so from Equation (11) it is clear that $L_{\epsilon,\mu}^N \psi_k < 0$ which contradict the hypothesis and hence $\psi_i \geq 0$ for all $0 \leq i \leq N$. □

Lemma 3.2 (Stability)**.** *Let* ψ_i *be any mesh function such that* $\psi_0 = \psi_N = 0$. *Then*

$$
|\psi_i| \leq \frac{1}{b^*} \max_{1 \leq j \leq N-1} |L_{\epsilon,\mu}^N \psi_j|, \qquad 0 \leq i \leq N.
$$

Proof. Suppose $M = \dfrac{1}{b^*} \max\limits_{1\le j\le N-1} |L^N_{\epsilon,\mu}\psi_j|$. Define two mesh functions $\varphi_i^\pm$ such that

$$\varphi_i^\pm = M \pm \psi_i.$$

Then it is clear that $\varphi_0^\pm \ge 0$ and $\varphi_N^\pm \ge 0$ and for $1 \le i \le N-1$ we have

$$\begin{aligned} L^N_{\epsilon,\mu}\varphi_i^\pm &= r_i^-\varphi_{i-1} + r_i^c\varphi_i + r_i^+\varphi_{i+1} \\ &= b_iM \pm (r_i^-\phi_{i-1} + r_i^c\phi_i + r_i^+\phi_{i+1}) \\ &\ge Mb^* \pm L^N_{\epsilon,\mu}\psi_i \\ &\ge 0. \end{aligned}$$

By using Lemma 3.1 we get $\varphi_i^\pm \ge 0$ for $0 \le i \le N$ and hence

$$|\psi_i| \le \frac{1}{b^*} \max_{1\le j\le N-1} |L^N_{\epsilon,\mu}\psi_j| \quad \forall\, 0 \le i \le N.$$

□

Lemma 3.3. *For every $\psi \in C^3(0,1)$, we have*

$$\begin{aligned} \left\| \left(\delta^2 - \frac{d^2}{dx^2}\right)\psi \right\| &\le Ch_i\|\psi^{(3)}\|, \quad x_i \in \Omega_1^N \cup \Omega_2^N, \\ \left\| \left(\delta^2 - \frac{d^2}{dx^2}\right)\psi \right\| &\le Ch_i^2\|\psi^{(4)}\|, \quad x_i \in \Omega_0^N, \\ \left\| \left(\Delta - \frac{d}{dx}\right)\psi \right\| &\le Ch_i\|\psi^{(2)}\|, \quad x_i \in \Omega^N, \end{aligned}$$

where

$$\|\psi^{(j)}\| = \sup_{x_i\in\Omega^N} \|\psi^{(j)}(x_i)\|, \quad j = 1, 2.$$

Proof. Let r and h_i are defined as above then for $x_i \in \Omega_1^N \cup \Omega_2^N$, taking Taylor series expansion and neglecting the term of fourth and higher order, we get the following expansions for ψ_{i+1} and ψ_{i-1}

$$\psi_{i+1} \simeq \psi_i + h_i\psi_i' + \frac{h_i^2}{2}\psi_i'' + \frac{h_i^3}{6}\psi'''(\xi_1),\ x_i < \xi_1 < x_{i+1}, \tag{12}$$

$$\psi_{i-1} \simeq \psi_i - h_{i-1}\psi_i' + \frac{h_{i-1}^2}{2}\psi_i'' - \frac{h_{i-1}^3}{6}\psi'''(\xi_2),\ x_{i-1} < \xi_2 < x_i. \tag{13}$$

Multiplying Equation (13) by r and adding it to Equation (12), we get the following approximation

$$\left(\delta^2 - \frac{d^2}{dx^2}\right)\psi(x_i) \simeq \frac{h_i}{3r(1+r)}\left(r^2\psi'''(\xi_1) - \psi'''(\xi_2)\right).$$

Taking absolute value and using intermediate value theorem, we obtain

$$\left\| \left(\delta^2 - \frac{d^2}{dx^2} \right) \psi \right\| \leq Ch_i \|\psi^{(3)}\|.$$

Now for $x_i \in \Omega_0^N$, taking Taylor series expansion and neglecting the term of fifth and higher order, we get the following expansions for ψ_{i+1} and ψ_{i-1}

$$\psi_{i+1} \simeq \psi_i + h\psi_i' + \frac{h^2}{2}\psi_i'' + \frac{h^3}{6}\psi_i''' + \frac{h^4}{24}\psi^{(4)}(\xi_1), \; x_i < \xi_1 < x_{i+1}, \quad (14)$$

$$\psi_{i-1} \simeq \psi_i - h\psi_i' + \frac{h^2}{2}\psi_i'' - \frac{h^3}{6}\psi_i''' + \frac{h^4}{24}\psi^{(4)}(\xi_2), \; x_{i-1} < \xi_2 < x_i. \quad (15)$$

On adding (14) and (15), we get the following approximation

$$\left(\delta^2 - \frac{d^2}{dx^2} \right) \psi(x_i) \simeq \frac{h^2}{12} \left(\psi^{(4)}(\xi_1) + \psi^{(4)}(\xi_2) \right).$$

Taking absolute value and using intermediate value theorem, we obtain

$$\left\| \left(\delta^2 - \frac{d^2}{dx^2} \right) \psi \right\| \leq Ch^2 \|\psi^{(4)}\|.$$

Similarly one can easily show that

$$\left\| \left(\Delta - \frac{d}{dx} \right) \psi \right\| \leq Ch_i \|\psi^{(2)}\|, \quad \text{for} \quad x_i \in \Omega^N.$$

□

Now we can deduce the following truncation error bounds for the difference operator $L_{\epsilon,\mu}^N$ on Ω^N, as

$$\|(L_{\epsilon,\mu}^N - L_{\epsilon,\mu})y\| \leq \epsilon h_i \|y^{(3)}\| + \mu h_i \|a\| \|y^{(2)}\|, \text{ for } x_i \in \Omega_1^N \cup \Omega_2^N, \quad (16)$$

$$\|(L_{\epsilon,\mu}^N - L_{\epsilon,\mu})y\| \leq \epsilon h^2 \|y^{(4)}\| + \mu h \|a\| \|y^{(2)}\|, \text{ for } x_i \in \Omega_0^N. \quad (17)$$

Like the solution of continuous problem (1), the solution of discrete problem (10) can be decomposed into the following sum

$$Y = U + V + W, \quad (18)$$

where U is the regular component of the solution Y, V and W are left and right singular component respectively, satisfying

$$L_{\epsilon,\mu}^N U = f, \quad U(0) = u(0), \; U(1) = u(1), \quad (19)$$

$$L_{\epsilon,\mu}^N V = 0, \quad V(0) = v(0), \; V(1) = 0, \quad (20)$$

$$L_{\epsilon,\mu}^N W = 0, \quad W(0) = w(0), \; W(1) = w(1). \quad (21)$$

Lemma 3.4. *Left and right layer components V and W satisfies the following bounds*

$$\begin{aligned} |V(x_i)| &\leq C\prod_{j=1}^{i}(1+\lambda_1 h_i)^{-1},\ |V(0)| \leq C, \\ |W(x_i)| &\leq C\prod_{j=i+1}^{N}(1+\lambda_2 h_i)^{-1},\ |W(1)| \leq C. \end{aligned}$$

The parameters λ_1 and λ_2 are defined as $\lambda_1 = a^*\mu/2\epsilon,\ \lambda_2 = c^*/2\mu$.

Proof. For the proof the readers are refer to [11]. □

It can be noted that both the functions $V(x_i)$ and $W(x_i)$ are decreasing.

Lemma 3.5. *The error in the regular component satisfies the following error estimate*

$$\|(U-u)\| \leq \begin{cases} CN^{-1}, & \text{if } \ x_i \in \Omega_1^N \cup \Omega_2^N, \\ CN^{-2}, & \text{if } \ x_i \in \Omega_0^N, \end{cases}$$

where u is the solution of (5) and U is the solution of (19).

Proof. For $x_i \in \Omega_1^N \cup \Omega_2^N$, We have

$$\begin{aligned} |L^N_{\epsilon,\mu}(U-u)(x_i)| &= |L^N_{\epsilon,\mu}U(x_i) - L^N_{\epsilon,\mu}u(x_i)|, \\ &= |f - L^N_{\epsilon,\mu}u(x_i)|, \\ &= |(L_{\epsilon,\mu} - L^N_{\epsilon,\mu})u(x_i)|, \\ &\leq CH(\epsilon\|u^{(3)}\| + \mu\|a\|\|u^{(2)}\|), \quad \text{using (16)}, \\ &\leq CH, \\ &\leq CN^{-1}. \end{aligned}$$

Similarly for $x_i \in \Omega_0$, we have

$$\begin{aligned} |L^N_{\epsilon,\mu}(U-u)(x_i)| &= |L^N_{\epsilon,\mu}U(x_i) - L^N_{\epsilon,\mu}u(x_i)|, \\ &= |f - L^N_{\epsilon,\mu}u(x_i)|, \\ &= |(L_{\epsilon,\mu} - L^N_{\epsilon,\mu})u(x_i)|, \\ &\leq CH(\epsilon H\|u^{(4)}\| + \mu\|a\|\|u^{(2)}\|), \quad \text{using (17)}, \\ &\leq CH^2, \quad \text{provided } \ \mu\|a\| \leq H, \\ &\leq CN^{-2}. \end{aligned}$$

A consequence of stability Lemma (3.2) gives

$$\|(U-u)\| \leq \begin{cases} CN^{-1}, & \text{if } \ x_i \in \Omega_1^N \cup \Omega_2^N, \\ CN^{-2}, & \text{if } \ x_i \in \Omega_0^N. \end{cases}$$

□

Lemma 3.6. *The left layer component of the error satisfies the following estimate*

$$\|(V - v)\| \leq CN^{-1}(\ln N)^2,$$

where v is the solution of (6) and V is the solution of (20).

Proof. First suppose that $\sigma_1 = 1/4$, then $\mu/\epsilon \leq C \ln N$, using the classical argument in order to obtain the following truncation error bounds

$$\begin{aligned}
|L^N_{\epsilon,\mu}(V - v)(x_i)| &= |L^N_{\epsilon,\mu}V(x_i) - L^N_{\epsilon,\mu}v(x_i)|, \\
&= |0 - L^N_{\epsilon,\mu}v(x_i)|, \\
&= |(L_{\epsilon,\mu} - L^N_{\epsilon,\mu})v(x_i)|, \\
&\leq CN^{-1}(\epsilon\|v^{(3)}\| + \mu\|v^{(2)}\|), \quad \text{using (16)}, \\
&\leq CN^{-1}(\mu^3/\epsilon^2), \quad \text{using Lemma 2.4}, \\
&\leq CN^{-1}(\ln N)^2,
\end{aligned}$$

using Lemma 3.2, we get

$$\|(V - v)\| \leq CN^{-1}(\ln N)^2.$$

Now suppose that $\sigma_1 < 1/4$, we firstly analyze the error in the fine mesh region $[0, \sigma_1]$ and then we proceed to analyze the coarse mesh on $[\sigma_1, 1]$. Suppose $x_i \in [0, \sigma_1]$, in this case, we calculate a bound on the truncation error of the form

$$\begin{aligned}
|L^N_{\epsilon,\mu}(V - v)(x_i)| &\leq CH(\epsilon\|v^{(3)}\| + \mu\|v^{(2)}\|), \quad \text{using (16)}, \\
&\leq CN^{-1}(\sigma_1\epsilon\|v^{(3)}\| + \sigma_1\mu\|v^{(2)}\|), \\
&\leq C(\mu^2/\epsilon)(N^{-1}\ln N), \quad \text{since} \quad \sigma_1 = O((\epsilon/\mu)\ln N).
\end{aligned}$$

Now consider the barrier function

$$\psi_i = C(N^{-1} + (N^{-1}\ln N)(\sigma_1 - x_i)(\mu/\epsilon)),$$

then we have

$$L^N_{\epsilon,\mu}\psi_i = C(N^{-1}\ln N)(\mu^2/\epsilon) + b_i\psi_i \geq |L^N_{\epsilon,\mu}(V - v)(x_i)| \geq 0,$$

using discrete maximum principle we obtain,

$$\begin{aligned}
|(V - v)(x_i)| \leq \psi_i &= C(N^{-1} + (N^{-1}\ln N)(\sigma_1 - x_i)(\mu/\epsilon)) \\
&\leq C(N^{-1} + (N^{-1}\ln N)\sigma_1(\mu/\epsilon)) \\
&\leq CN^{-1}(\ln N)^2. \qquad (22)
\end{aligned}$$

Now suppose $x_i \in [\sigma_1, 1]$, using triangle inequality we have

$$|(V - v)(x_i)| \leq |V(x_i)| + |v(x_i)|. \tag{23}$$

From Lemma 3.4 we have

$$|V(x_i)| \leq C \prod_{j=1}^{i} (1 + \lambda_1 h_i)^{-1}.$$

Therefore

$$\begin{aligned} |V(x_{N/4})| &\leq C \prod_{j=1}^{N/4} (1 + \lambda_1 h_i)^{-1} \\ &\leq C(1 + \lambda_1 h_1)^{-N/4}. \end{aligned}$$

Now we have

$$\begin{aligned} \lambda_1 h_1 &= \lambda_1 \sigma_1 \frac{(r-1)}{(r^{N/4} - 1)} \\ &= \frac{\lambda_1 \sigma_1}{1 + r + r^2 + \ldots + r^{N/4-1}} \\ &\geq 4N^{-1} \lambda_1 \sigma_1 \\ &= 4N^{-1} \left(\frac{a^* \mu}{2\epsilon} \right) \left(\frac{4\epsilon}{a^* \mu} \ln N \right) \\ &= 8N^{-1} \ln N. \end{aligned}$$

Therefore

$$|V(x_{N/4})| \leq C(1 + 8N^{-1} \ln N)^{-N/4}.$$

Now using the inequality $\ln(1+x) > x(1-x/2)$ and taking $x = 8N^{-1} \ln N$, we can show that $(1 + 8N^{-1} \ln N)^{-N/4} \leq 4N^{-1}$, and therefore we conclude that on the interval $[\sigma_1, 1]$ we have $|V(x_i)| \leq CN^{-1}$. Again we have

$$\begin{aligned} |v(x_i)| &\leq C \exp((-a^* \mu/\epsilon) x_i) \\ &\leq C \exp((-a^* \mu/\epsilon) \sigma_1) \\ &= C \exp(-4 \ln N) \leq CN^{-4}. \end{aligned}$$

Therefore from (23) on the interval $[\sigma_1, 1]$, we have

$$|(V - v)(x_i)| \leq CN^{-1}. \tag{24}$$

Combining (22) and (24), we have

$$\|(V - v)\| \leq CN^{-1} (\ln N)^2.$$

□

Lemma 3.7. *The right layer component of the error satisfies the following estimate*

$$\|(W - w)\| \leq CN^{-1} \ln N,$$

where w is the solution of (7) and W is the solution of (21).

Proof. The proof is similar to the proof given for the left boundary layer component. First suppose that $\sigma_1 = 1/4$, then $1/\mu \leq C \ln N$, using the classical argument in order to obtain the following truncation error bounds

$$\begin{aligned} |L^N_{\epsilon,\mu}(W - w)(x_i)| &= |L^N_{\epsilon,\mu} W(x_i) - L^N_{\epsilon,\mu} w(x_i)|, \\ &= |0 - L^N_{\epsilon,\mu} w(x_i)|, \\ &= |(L_{\epsilon,\mu} - L^N_{\epsilon,\mu}) w(x_i)|, \\ &\leq CN^{-1}(\epsilon \|w^{(3)}\| + \mu \|w^{(2)}\|), \quad \text{using (16)}, \\ &\leq CN^{-1}(1/\mu), \qquad \text{using Lemma 2.4} \\ &\leq CN^{-1}(\ln N). \end{aligned}$$

Now suppose that $\sigma_2 < 1/4$, we firstly analyze the error in the coarse mesh region $[0, 1 - \sigma_2]$ and then we proceed to analyze the fine mesh on $[\sigma_2, 1]$. Suppose $x_i \in [0, 1 - \sigma_2]$, using triangle inequality we have

$$|(W - w)(x_i)| \leq |W(x_i)| + |w(x_i)|. \tag{25}$$

From Lemma 3.4 we have

$$|W(x_i)| \leq C \prod_{j=i+1}^{N} (1 + \lambda_2 h_i)^{-1}.$$

So we have

$$\begin{aligned} |W(x_{3N/4})| &\leq C \prod_{j=3N/4+1}^{N} (1 + \lambda_2 h_i)^{-1} \\ &\leq C(1 + \lambda_2 h_N)^{-N/4}. \end{aligned}$$

Now we have

$$\begin{aligned}
\lambda_2 h_N &= \lambda_2 r^{N/4-1} h_{3N/4} \\
&= \lambda_2 r^{N/4-1} \sigma_2 \frac{(1-r)}{(1-r^{N/4})} \\
&= \frac{\lambda_2 \sigma_2 r^{N/4-1}}{1+r+r^2+\ldots+r^{N/4-1}} \\
&= \frac{\lambda_2 \sigma_2}{1+r^{-1}+r^{-2}+\ldots+r^{-(N/4-1)}} \\
&> 4N^{-1}\lambda_2\sigma_2 \\
&= 4N^{-1}(c^*/2\mu)((4\mu/c^*)\ln N) \\
&= 8N^{-1}\ln N.
\end{aligned}$$

Therefore

$$|W(x_{3N/4})| \leq C(1+8N^{-1}\ln N)^{-N/4}.$$

Now using the inequality $\ln(1+x) > x(1-x/2)$ and taking $x = 8N^{-1}\ln N$, we can show that $(1+8N^{-1}\ln N)^{-N/4} \leq 4N^{-1}$, and therefore we conclude that on the interval $[0, 1-\sigma_2]$ we have

$$|W(x_i)| \leq CN^{-1}.$$

Again we have

$$\begin{aligned}
|w(x_i)| &\leq C\exp(-(c^*/2\mu)(1-x_i)) \\
&\leq C\exp(-(c^*/2\mu)\sigma_2) \\
&= C\exp(-(c^*/2\mu)(4\mu/c^*)\ln N) \\
&= CN^{-2}.
\end{aligned}$$

Therefore from (25) we have

$$\|(W-w)\| \leq CN^{-1}. \tag{26}$$

Now suppose $x_i \in [1-\sigma_2, 1]$, then using classical analysis we can obtain the following truncation error bounds

$$|L^N_{\epsilon,\mu}(W-w)(x_i)| \quad \leq \quad CH(\epsilon\|w^{(3)}\| + \mu\|w^{(2)}\|).$$

Using the bounds on w in Lemma 2.4, we find that this simplifies to

$$|L^N_{\epsilon,\mu}(W-w)(x_i)| \leq \frac{C}{\mu}(h_i + h_{i+1}). \tag{27}$$

Since we are in the fine mesh region, we have $h_{i+1} = h_i = \frac{16\mu}{c*} N^{-1} \ln N$. Using (27) we now obtain

$$|L^N_{\epsilon,\mu}(W - w)(x_i)| \leq CN^{-1} \ln N.$$

Use maximum principle to obtain

$$|W - w(x_i)| \leq CN^{-1} \ln N.$$

□

Theorem 3.8. *At each mesh point $x_i \in \Omega^N$ the maximum pointwise error satisfies the following parameter-uniform error bound*

$$\|Y - y(x_i)\|_{\Omega^N} \leq CN^{-1} \ln N.$$

Proof. The proof follows from Lemmas 3.5, 3.6 and 3.7. □

4 Numerical Results

A numerical method for solving singularly perturbed convection-diffusion boundary value problem with two small parameters is considered. It is a practical method and can easily be implemented on a computer to solve such problems. To validate the theoretical results, we apply the proposed numerical scheme to a test problem with two small parameters having two boundary layers. The maximum absolute error $E^N_\epsilon = \max_i \mid y(x_i) - y_i \mid$ at the nodal points are tabulated in the table for different values of perturbation parameters ϵ and μ by using $N = 128$.

Example 4.1. *Consider the problem*

$$-\epsilon y'' - \mu y' + y = x; \quad y(0) = 1, y(1) = 0.$$

Table 1: Maximum absolute error for Example 4.1

$\epsilon \backslash \mu$	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
10^{-1}	1.543E-02	1.542E-02	1.542E-02	1.542E-02	1.542E-02	1.542E-02
10^{-2}	7.358E-04	7.274E-04	7.265E-04	7.264E-04	7.264E-04	7.264E-04
10^{-3}	2.348E-03	2.350E-03	2.350E-03	2.350E-03	2.350E-03	2.350E-03
10^{-4}	1.487E-02	9.261E-03	8.641E-03	8.578E-03	8.572E-03	8.571E-03
10^{-5}	8.409E-02	4.606E-02	4.131E-02	4.083E-02	4.078E-02	4.077E-02

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