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Some Fixed Point Theorems in Cone Rectangular Metric Spaces

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Abstract

In this paper we establish some fixed point theorems in cone rectangular metric spaces setting. Our results improve and extend the recent known results.

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1 Introduction

Jungck [14] proved a common fixed point theorem for commuting mappings as generalizing the Banach's fixed point theorem. The concept of the commutativity has generalized in several ways. For this Sessa [26] introduced the concept of weakly commuting mappings, Jungck [15] extend this concept to compatible maps. In 1998, Jungck and Rhoades [16] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but the converse need not to be true for example see [23].

Later Huang and Zhang [11] introduced the notion of cone metric spaces. They replacing the set of real numbers by an ordered Banach space. They presented the notion of convergence of sequences in cone metric spaces and proved some fixed point theorems. After that, many authors established many fixed point theorems of contractive type mappings over a cone metric spaces. For some fixed point theorems in cone metric spaces we refer the reader to [1, 2], [4]-[8], [11]-[13], [17]-[19], [21, 22, 24, 25], [27]-[30].

Recently, Rezapour and Hamlbarani [24] proved that there are no normal cones with normal constant $K < 1$ and that for each $h > 1$ there are cones with normal constant $K > h$. Also, omitting the assumption of normality, they obtain generalizations of some results of [11].

Further Beiranvand, Moradi, Omid and Pazandeh [9] introduced the classes of T-contractive mappings, which are depending on another function. Moradi in [20] introduced the T-Kannan contractive mapping. Morales and Rojas [21], [22] have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Chatterjea T-Zamfirescu, T-weakly contraction mappings. Sumitra, Rhymend Uthariaraj and Hemavathy [29] proved a fixed point theorem in the setting of cone metric space for T-Hardy-Rogers type contraction condition.

In 2000 Branciari [10] introduced a class of generalized metric spaces by replacing triangular inequality by similar ones which involve four or more points instead of three and improved Banach contraction mapping principle.

Recently, Azam et.al [8] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a cone rectangular metric space setting.

The paper is a continuation of the study of some fixed point theorems in cone rectangular metric space setting. Our results improve and extend the results in [8].

2 Preliminary Notes

In the present article E stands for a real Banach space. Now, we present some necessary definitions and results, which will be needed in the sequel.

Definition 2.1. *[11] Let P be a subset of E, then P is called a* ***cone*** *if the following conditions are satisfied:*

(i) *P is closed, nonempty, and $P \neq \{\theta\}$,*

(ii) *$a,b \in R$, $a,b \geq 0$, and $x,y \in P$ imply that $ax + by \in P$,*

(iii) *$P \cap (-P) = \{\theta\}$.*

For a given cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by for $x,y \in E$, we say that $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, also, we write $x \ll y$ if $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 2.2. *[11] Let X be a nonempty set. Suppose that the map $d : X^2 \to E$ satisfies*

(i) $d(x,y) \geq 0, \forall x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

The cone P is called **normal** if there is a constant $K > 0$ such that for all $x,y \in E$

$$\theta \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the **normal constant** of P.

Proposition 2.3. *[24] Let (X,d) be a cone metric space with cone P not necessary to be normal. Then for $a,c,u,v,w \in E$, we have*

(i) If $a \leq ha$ *and* $h \in [0,1)$, *then* $a = \theta$.

(ii) If $\theta \leq u \ll c$ *for each* $\theta \ll c$, *then* $u = \theta$.

(iii) If $u \leq v$ *and* $v \ll w$, *then* $u \ll w$.

Example 2.4. *[11] Let $E = R^2$, $P = \{(x,y) \in E : x,y \geq 0\} \subset R^2$, $X = R$, and $d : X^2 \to E$ defined by*

$$d(x,y) = (|x-y|, \alpha|x-y|),$$

where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 2.5. *[8] Let X be a nonempty set and the mapping $d : X^2 \to E$ satisfies:*

(i) $d(x,y) \geq \theta, \forall x,y \in X$ *and* $d(x,y) = \theta$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$, *for all* $x,y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$ *for all* $x,y \in X$ *and for all distinct points* $z,w \in X \setminus \{x,y\}$, *(rectangular inequality).*

Then d is called a cone rectangular metric and (X,d) is called a cone rectangular metric space.

Definition 2.6. *[8] Let (X,d) be a cone rectangular metric space and $\{x_n\}$ be a sequence in (X,d). Then* *(i)* $\{x_n\}$ *converges to* $x \in X$ *whenever for every* $c \in E$ *with* $\theta \ll c$, *there is a natural number* n_0 *such that* $d(x_n, x) \ll c$ *for all* $n \geq n_0$, *we denote this by*

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x,$$

(ii) $\{x_n\}$ *is a Cauchy sequence whenever for every* $c \in E$ *with* $\theta \ll c$, *there is a natural number* n_0 *such that* $d(x_n, x_{n+m}) \ll c$ *for all* $n \geq n_0$,

(iii) (X,d) *is called a complete cone rectangular metric space if every Cauchy sequence in* (X,d) *is convergent in* (X,d).

Notice that any cone metric space is a cone rectangular metric space but the converse is not true in general.

Example 2.7. *Let* $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x,y \geq 0\}$, $X = \mathbb{R}$, $d : X^2 \to E$ *such that*

$$d(x,y) = \begin{cases} (0,0), & \text{if } x = y, \\ (3\alpha, 3), & \text{if } x \text{ and } y \text{ are in } \{1,2\},\ x \neq y, \\ (\alpha, 1), & \text{if } x \text{ and } y \text{ can not both at a time in } \{1,2\},\ x \neq y,, \end{cases}$$

where $\alpha > 0$ *is a constant. Then* (X,d) *is a cone rectangular metric space but it is not a cone metric space since we have* $d(1,2) = (3\alpha,3) > d(1,3) + d(3,2) = (2\alpha,2)$.

Definition 2.8. *Let* P *be a cone defined as above and let* Φ *be the set of non-decreasing continuous functions* $\varphi : P \to P$ *satisfying:*

(i) $\theta < \varphi(t) < t$ *for all* $t \in P \setminus \{\theta\}$,

(ii) the series $\sum_{n\geq 0} \varphi^n(t)$ *converge for all* $t \in P \setminus \{\theta\}$

From (i), *we have* $\varphi(\theta) = \theta$, *and from* (ii), *we have* $\lim_{n\to 0} \varphi^n(t) = \theta$ *for all* $t \in P \setminus \{\theta\}$.

Definition 2.9. *Let* T *and* S *be self maps of a nonempty set* X. *If* $w = Tx = Sx$ *for some* $x \in X$, *then* x *is called a coincidence point of* T *and* S *and* w *is called a point of coincidence of* T *and* S.

Definition 2.10. *[16] Two self mappings* T *and* S *are said to be weakly compatible if they commute at their coincidence points, that is,* $Tx = Sx$ *implies that* $TSx = STx$.

Lemma 2.11. *[1] Let* T *and* S *be weakly compatible self mappings of nonempty set* X. *If* T *and* S *have a unique point of coincidence* $w = Tx = Sx$, *then* w *is the unique common fixed point of* T *and* S.

3 Main Results

We start with the following theorem.

Theorem 3.1. *Let (X,d) be a complete cone rectangular metric space and let the mapping $T: X \to X$ satisfy the following:*

$$d(Tx,Ty) \leq \varphi(d(x,y)) \tag{1}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence (x_n) in X such that $x_{n+1} = Tx_n$, for all $n = 0,1,2,\ldots$. We assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, by (1) we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \varphi(d(x_{n-1},x_n)) = \varphi(d(Tx_{n-2},Tx_{n-1})) \\ &\leq \varphi^2(d(x_{n-2},x_{n-1})) \\ &\vdots \\ &\leq \varphi^n(d(x_0,x_1)). \end{aligned}$$

Similarly for $k = 1,2,3,\ldots$, we get

$$d(x_n,x_{n+2k}) \leq \varphi^n(d(x_0,x_{2k})), \tag{2}$$

$$d(x_n,x_{n+2k+1}) \leq \varphi^n(d(x_0,x_{2k+1})). \tag{3}$$

By using rectangular inequality and (2), we have

$$\begin{aligned} d(x_0,x_4) &\leq d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_4) \\ &\leq d(x_0,x_1) + \varphi(d(x_0,x_1)) + \varphi^2(d(x_0,x_2)). \end{aligned}$$

$$\begin{aligned} d(x_0,x_6) &\leq d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_4) + d(x_4,x_6) \\ &\leq \sum_{i=0}^{3} \varphi^i(d(x_0,x_1)) + \varphi^4(d(x_0,x_2)). \end{aligned}$$

By indiction we have for each $k = 2,3,4,\ldots$

$$d(x_0,x_{2k}) \leq \sum_{i=0}^{2k-3} \varphi^i(d(x_0,x_1)) + \varphi^{2k-2}(d(x_0,x_2)). \tag{4}$$

Also, by using rectangular inequality and (3) we have

$$\begin{aligned} d(x_0,x_5) &\leq d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_3) + d(x_3,x_4) + d(x_4,x_5) \\ &\leq \sum_{i=0}^{4} \varphi^i(d(x_0,x_1)). \end{aligned}$$

By indiction we have for each $k = 0, 1, 2, 3, 4, \ldots$

$$d(x_0, x_{2k+1}) \leq \sum_{i=0}^{2k} \varphi^i(d(x_0, x_1)). \tag{5}$$

Using (2) and (4) we have for each $k = 2, 3, 4, \ldots,$

$$\begin{aligned} d(x_n, x_{n+2k}) &\leq \varphi^n(d(x_0, x_{2k})) \\ &\leq \varphi^n(\sum_{i=0}^{2k-3} \varphi^i(d(x_0, x_1)) + \varphi^{2k-2}(d(x_0, x_2))) \\ &\leq \varphi^n(\sum_{i=0}^{2k-3} \varphi^i(d(x_0, x_1) + d(x_0, x_2)) \\ &+ \varphi^{2k-2}(d(x_0, x_1) + d(x_0, x_2))) \\ &\leq \varphi^n(\sum_{i=0}^{2k-2} \varphi^i(d(x_0, x_1) + d(x_0, x_2))) \\ &\leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2))). \end{aligned}$$

Similarly, using (3) and (5) we have for each $k = 0, 1, 2, 3, 4, \ldots,$

$$\begin{aligned} d(x_n, x_{n+2k+1}) &\leq \varphi^n(d(x_0, x_{2k+1})) \\ &\leq \varphi^n((\sum_{i=0}^{2k} \varphi^i(d(x_0, x_1))) \\ &\leq \varphi^n(\sum_{i=0}^{2k} \varphi^i(d(x_0, x_1) + d(x_0, x_2))) \\ &\leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2))). \end{aligned}$$

Hence, for each m,

$$d(x_n, x_{n+m}) \leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2))). \tag{6}$$

Since $\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2))$ converge, where $d(x_0, x_1) + d(x_0, x_2) \in P \setminus \{\theta\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2)) \in P \setminus \{\theta\}$. Hence

$$\varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(x_0, x_1) + d(x_0, x_2))) \to \theta \quad \text{as } n \to \infty.$$

Hence, for given $c \gg \theta$ there is a natural number $n_0 \in \mathbb{N}$ such that

$$\varphi^n(\sum_{i=0}^{\infty}\varphi^i(d(x_0,x_1)+d(x_0,x_2))) \ll c \qquad \forall n \geq n_0. \tag{7}$$

Thus from (6) and (7), we have

$$d(x_n,x_{n+m}) \ll c \qquad \forall n \geq n_0.$$

Then (x_n) is a Cauchy sequence in X. Since X is a complete, there exists a point q in X such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = q$. We prove that $Tq = q$. Given $c \gg \theta$ we choose a natural numbers k_1, k_2 such that

$$d(q,x_n) \ll \frac{c}{3} \quad \forall n \geq k_1, \quad d(x_{n+1},x_n) \ll \frac{c}{3} \quad \forall n \geq k_2.$$

By rectangular inequality we have

$$\begin{aligned} d(Tq,q) &\leq d(Tq,Tx_n)+d(Tx_n,Tx_{n-1})+d(Tx_{n-1},q) \\ &\leq \varphi(d(q,x_n))+d(x_{n+1},x_n)+d(x_n,q) \\ &< d(q,x_n)+d(x_{n+1},x_n)+d(x_n,q) \\ &\ll \frac{c}{3}+\frac{c}{3}+\frac{c}{3} = c, \end{aligned}$$

for all $n \geq k$ where $k = max\{k_1,k_2\}$. Since c is arbitrary we have

$$d(Tq,q) \ll \frac{c}{m} \qquad \forall m \in \mathbb{N}.$$

Since $\frac{c}{m} \to \theta$ as $m \to \infty$, we conclude

$$\frac{c}{m} - d(Tq,q) \to -d(Tq,q) \quad \text{as} \quad m \to \infty.$$

Since P is closed, $-d(Tq,q) \in P$. Hence $d(Tq,q) \in P \cap -P$. Then $d(Tq,q) = \theta$. Therefore $Tq = q$. Hence q is a fixed point of T. Now, we prove the uniqueness of the fixed point. Let p be another fixed point of T, that is $p = Tp$, then

$$d(q,p) = d(Tq,Tp) \leq \varphi(d(q,p)) < d(q,p).$$

Hence $q = p$. □

Corollary 3.2. *Let (X,d) be a complete cone rectangular metric space, and let the mapping $T : X \to X$ satisfy the following:*

$$d(T^m x, T^m y) \leq \phi(d(x,y)) \tag{8}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Then T has a unique fixed point in X.

Proof. From Theorem (3.1), we conclude that T^m has a unique fixed point say q. Hence

$$Tq = T(T^m q) = T^{m+1}q = T^m(Tq).$$

Then Tq is also a fixed point to T^m. By uniqueness of q, we have $Tq = q$. □

Corollary 3.3. ***(Theorem 3 [8])*** *Let (X,d) be a complete cone rectangular metric space, and let the mapping $T : X \to X$ satisfy the following:*

$$d(Tx,Ty) \leq \lambda d(x,y) \tag{9}$$

for all $x,y \in X$, where $\lambda \in [0,1)$. Then T has a unique fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition (2.8). Hence the result follows from Theorem (3.1). □

Theorem 3.4. *Let (X,d) be a cone rectangular metric space, and let the mappings $S,T : X \to X$ satisfy the following:*

$$d(Tx,Ty) \leq \varphi(d(Sx,Sy)) \tag{10}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Suppose that $T(X) \subseteq S(X)$, and $S(X)$ or $T(X)$ is a complete subspace of X, then the mappings S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X since $T(X) \subseteq S(X)$ we can choose $x_1 \in X$ such that $Tx_0 = Sx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} such that $Tx_n = Sx_{n+1}$, for all $n = 0,1,2,\ldots$. We assume that $Tx_n \neq Tx_{n-1}$ for all $n \in \mathbb{N}$. Then, by (10) we have

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &\leq \varphi(d(Sx_n,Sx_{n+1})) = \varphi(d(Tx_{n-1},Tx_n)) \\ &\leq \varphi^2(d(Sx_{n-1},Sx_n)) \\ &\vdots \\ &\leq \varphi^n(d(Tx_0,Tx_1)). \end{aligned}$$

Similarly for $k = 1,2,3,\ldots$, we get

$$d(Tx_n,Tx_{n+2k}) \leq \varphi^n(d(Tx_0,Tx_{2k})), \tag{11}$$

$$d(Tx_n,Tx_{n+2k+1}) \leq \varphi^n(d(Tx_0,Tx_{2k+1})). \tag{12}$$

By using rectangular inequality and (11), we have

$$\begin{aligned} d(Tx_0,Tx_4) &\leq d(x_0,Tx_1) + d(Tx_1,Tx_2) + d(Tx_2,Tx_4) \\ &\leq d(Tx_0,Tx_1) + \varphi(d(Tx_0,Tx_1)) + \varphi^2(d(Tx_0,Tx_2)). \end{aligned}$$

$$d(Tx_0,Tx_6) \le d(Tx_0,Tx_1)+d(Tx_1,Tx_2)+d(Tx_2,Tx_3)+d(Tx_3,Tx_4)+d(Tx_4,Tx_6)$$
$$\le \sum_{i=0}^{3}\varphi^i(d(Tx_0,Tx_1))+\varphi^4(d(Tx_0,Tx_2)).$$

By indiction we have for each $k=2,3,4,\dots$

$$d(Tx_0,Tx_{2k}) \le \sum_{i=0}^{2k-3}\varphi^i(d(Tx_0,Tx_1))+\varphi^{2k-2}(d(Tx_0,Tx_2)). \tag{13}$$

Also, by using rectangular inequality and (12) we have

$$d(Tx_0,Tx_5) \le d(Tx_0,Tx_1)+d(Tx_1,Tx_2)+d(Tx_2,Tx_3)+d(Tx_3,Tx_4)+d(Tx_4,Tx_5)$$
$$\le \sum_{i=0}^{4}\varphi^i(d(Tx_0,Tx_1)).$$

By indiction we have for each $k=0,1,2,3,4,\dots$

$$d(Tx_0,Tx_{2k+1}) \le \sum_{i=0}^{2k}\varphi^i(d(Tx_0,Tx_1)). \tag{14}$$

Using (11) and (13) we have for each $k=2,3,4,\dots,$ we have

$$\begin{aligned}
d(Tx_n,Tx_{n+2k}) &\le \varphi^n(d(Tx_0,Tx_{2k}))\\
&\le \varphi^n(\sum_{i=0}^{2k-3}\varphi^i(d(Tx_0,Tx_1))+\varphi^{2k-2}(d(Tx_0,Tx_2)))\\
&\le \varphi^n(\sum_{i=0}^{2k-3}\varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2))\\
&+\varphi^{2k-2}(d(Tx_0,Tx_1)+d(Tx_0,Tx_2)))\\
&\le \varphi^n(\sum_{i=0}^{2k-2}\varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2)))\\
&\le \varphi^n(\sum_{i=0}^{\infty}\varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2))).
\end{aligned}$$

Similarly, using (12) and (14) we have for each $k=0,1,2,3,4,\dots,$ we have

$$\begin{aligned}
d(Tx_n,Tx_{n+2k+1}) &\le \varphi^n(d(Tx_0,Tx_{2k+1}))\\
&\le \varphi^n((\sum_{i=0}^{2k}\varphi^i(d(Tx_0,Tx_1)))\\
&\le \varphi^n(\sum_{i=0}^{2k}\varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2)))\\
&\le \varphi^n(\sum_{i=0}^{\infty}\varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2))).
\end{aligned}$$

Hence, for each m we conclude

$$d(Tx_n, Tx_{n+m}) \leq \varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(Tx_0, Tx_1) + d(Tx_0, Tx_2))). \tag{15}$$

Since $\sum_{i=0}^{\infty} \varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2))$ converge, where $d(Tx_0,Tx_1)+d(Tx_0,Tx_2) \in P \setminus \{\theta\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i(d(Tx_0,Tx_1)+d(Tx_0,Tx_2)) \in P \setminus \{\theta\}$. Hence

$$\varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(Tx_0, Tx_1) + d(Tx_0, Tx_2))) \to \theta \quad \text{ as } n \to \infty.$$

Hence, for given $c \gg \theta$ there is a natural number $n_0 \in \mathbb{N}$ such that

$$\varphi^n(\sum_{i=0}^{\infty} \varphi^i(d(Tx_0, Tx_1) + d(Tx_0, Tx_2))) \ll c \qquad \forall n \geq n_0. \tag{16}$$

Thus from (15) and (16), we have

$$d(Tx_n, Tx_{n+m}) \ll c \qquad \forall n \geq n_0.$$

Then (Tx_n) is a Cauchy sequence in X. Suppose $T(X)$ is a complete subspace of X, then there exists a point q in $T(X)$ such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = q$. Also, we can find a point $p \in X$ such that $Sp = q$.

We prove that $Tp = q$. Given $c \gg \theta$ we choose a natural numbers k_1, k_2 such that

$$d(q, Sx_n) \ll \frac{c}{3} \quad \forall n \geq k_1, \quad d(Tx_n, Tx_{n-1}) \ll \frac{c}{3} \quad \forall n \geq k_2.$$

By rectangular inequality we have

$$\begin{aligned} d(Tp, q) &\leq d(Tp, Tx_n) + d(Tx_n, Sx_n) + d(Sx_n, q) \\ &\leq \varphi(d(Sp, Sx_n)) + d(Tx_n, Tx_{n-1}) + d(Sx_n, q) \\ &< d(q, Sx_n) + d(Tx_n, Tx_{n-1}) + d(Sx_n, q) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c, \end{aligned}$$

for all $n \geq k$ where $k = max\{k_1, k_2\}$. Since c is arbitrary we have $d(Tp, q) = \theta$. Therefore $Tp = Sp = q$. Hence q is a coincidence point of S and T. Now, we prove the uniqueness of the coincidence point. Let u be another coincidence point of S and T, that is $u = Sv = Tv$, then

$$d(q, u) = d(Tp, Tv) \leq \varphi(d(Sp, Sv)) = \varphi(d(q, u)) < d(q, u),$$

a contradiction. Hence $q = u$. Since S and T are weakly compatible, by Lemma(2.11) q is the unique common fixed point of S and T. $\square$

Corollary 3.5. *Let (X,d) be a cone rectangular metric space, and let the mappings $S,T : X \to X$ satisfy the following:*

$$d(Tx,Ty) \leq \lambda d(Sx,Sy) \tag{17}$$

for all $x,y \in X$, where $\lambda \in [0,1)$. Suppose that $T(X) \subseteq S(X)$, and $S(X)$ or $T(X)$ is a complete subspace of X, then the mappings S and T have a unique coincidence point in X. Moreover, if S and T are weakly compatible then S and T have a unique common fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition (2.8). Hence the result follows from Theorem (3.4). ☐

Remark 3.6. *If we put $S = I$, in Theorem (3.4) where I is the identity mapping, we have Theorem (3.1).*

Theorem 3.7. *Let (X,d) be a cone rectangular metric space, and let the mappings $f,T : X \to X$ satisfy the following:*

$$d(Tfx,Tfy) \leq \varphi(d(Tx,Ty)) \tag{18}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Suppose that T is one to one, $T(X)$ is a complete subspace of X, then the mapping f has a unique fixed point in X. Moreover, if f and T are commuting at the fixed point of f, then f and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence (x_n) in X such that $x_{n+1} = fx_n$, for all $n = 0,1,2,\ldots$. We assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, by (18) we have

$$\begin{aligned} d(Tx_n,Tx_{n+1}) &= d(Tfx_{n-1},Tfx_n) \\ &\leq \varphi(d(Tx_{n-1},Tx_n)) \\ &\leq \varphi^2(d(Tx_{n-2},Tx_{n-1})) \\ &\vdots \\ &\leq \varphi^n(d(Tx_0,Tx_1)). \end{aligned}$$

Similarly for $k = 1,2,3,\ldots$, we get

$$d(Tx_n,Tx_{n+2k}) \leq \varphi^n(d(Tx_0,Tx_{2k})), \tag{19}$$

$$d(Tx_n,Tx_{n+2k+1}) \leq \varphi^n(d(Tx_0,Tx_{2k+1})). \tag{20}$$

Using the same argument in the proof of theorem (3.4) to prove that (Tx_n) is a Cauchy sequence in X.

Since $T(X)$ is a complete subspace of X, then there exists a point q in $T(X)$ such that $\lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Tfx_n = q$. Also, we can find a point $p \in X$ such that $Tp = q$.

We prove that $Tfp = Tp$. Given $c \gg \theta$ we choose a natural number k_1, k_2 such that

$$d(q,Sx_n) \ll \frac{c}{3} \quad \forall n \geq k_1, \quad d(Tx_n,Tx_{n-1}) \ll \frac{c}{3} \quad \forall n \geq k_2.$$

By rectangular inequality we have

$$\begin{aligned} d(Tp,Tfp) &\leq d(Tp,Tx_n) + d(Tx_n,Tfx_n) + d(Tfx_n,Tfp) \\ &\leq d(q,Tx_n) + d(Tx_n,Tx_{n+1}) + \varphi(d(Tx_n,Tp)) \\ &< d(q,Tx_n) + d(Tx_n,Tx_{n+1}) + d(Tx_n,q) \\ &\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c, \end{aligned}$$

for all $n \geq k$ where $k = max\{k_1,k_2\}$. Since c is arbitrary we have $d(Tp,Tfp) = \theta$. Therefore $Tp = Tfp = q$. Since T is one to one, $p = fp$. Hence p is a fixed point of f. Now, we prove the uniqueness of the fixed point of f. Let r be another fixed point of f that is $r = fr$ then

$$d(Tp,Tr) = d(Tfp,Tfr) \leq \varphi(d(Tp,Tr)) < d(Tp,Tr).$$

Hence $Tp = Tr$. Since T is one to one we conclude $p = r$.

Since f and T are commuting at the fixed point of f, $Tfp = fTp = Tp$. Therefore Tp is a fixed point of f. Since f has a unique fixed point, $Tp = p$. Hence $Tp = fp = p$. □

Remark 3.8. *If we put $T = I$, in Theorem (3.7), where I is the identity mapping, we have Theorem (3.1).*

Example 3.9. *Let $X = \{1,2,3,4\}$, $E = \mathbb{R}^2$ and $P = \{(x,y) : x,y \geq 0\}$ is a cone in E. Define $d : X^2 \to E$ as following:*

$$\begin{aligned} &d(1,2) = d(2,1) = (3,6), \\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = (1,2), \\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = (2,4). \end{aligned}$$

Then (X,d) is a complete cone rectangular metric space. We define the mappings $S,T : X \to X$ as following

$$Tx = \begin{cases} 3, & if\, x \neq 4, \\ 1, & if\, x = 4, \end{cases}, \qquad Sx = \begin{cases} 2, & if\, x = 1, \\ 1, & if\, x = 2, \\ 3, & if\, x = 3, \\ 4, & if\, x = 4. \end{cases}$$

Clearly $T(X) \subseteq S(X)$, $S(X)$ is a complete subspace of X, and the pairs (T,S) is weakly compatible. The inequality (10) holds for all $x,y \in X$, where $\varphi(t) = \frac{1}{2}t$, and 3 is the unique common fixed point of the mappings S and T.

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