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τ-curvature tensor in (k, μ)-contact manifolds

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Abstract

In this paper we study τ-curvature tensor in (k, μ) manifold. We study τ-flat and a ξ-τ- flat (k, μ)-contact metric manifolds and we obtain conditions for τ-flat (k, μ)-contact metric manifold to be ϕ-symmetric. We consider $\phi - \tau$-symmetric and $\phi - \tau$- Ricci recurrent (k, μ)-contact metric manifolds and (k, μ)-contact metric manifolds satisfying semi-symmetry condition $\tau.S = 0$.

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1. Introduction

M.M.Tripathi and et al. [7] introduced the τ-tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like M-projective curvature tensor, W_i-curvature tensor$(i = 0, ..., 9)$ and W_j^*-curvature tensors$(j = 0, 1)$. M.M.Tripathi and et al., [8],[9] studied τ curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. D.E.Blair, T.Koufogiogors and B.J.Papantoniou [2] studied the (k, μ)-nullity conditions on a contact metric manifold and gave several examples. The study of (k, μ)-contact manifolds is interesting as it contains both Sasakian and non-Sasakian manifolds. E.Boeckx[4] gave a complete classification of (k, μ)-contact manifolds. The authors U.C.De and et al [5],[6],[11] studied (k, μ)-contact manifolds with ϕ-recurrency and ϕ-symmetry conditions.
Motivated by the above studies, in this paper we study τ- curvature tensor in (k, μ)-contact metric manifolds. After preliminaries in section 2, we study τ-flat and a ξ-τ- flat (k, μ)-contact metric manifolds in section 3. We obtain conditions for τ-flat (k, μ)-contact metric manifold to be ϕ-symmetric.

We consider $\phi-\tau$-symmetric and $\phi-\tau$- Ricci recurrent (k,μ)-contact metric manifolds in section 4 and proved conditions for these manifolds to be Einstein and η-Einstein. The last section provides results on (k,μ)-contact metric manifolds satisfying semi-symmetry condition $\tau.S=0$.

2. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on M^n satisfying

$$\eta(\xi)=1, \quad \phi^2=-I+\eta\otimes\xi, \tag{1}$$

where I denotes the identity transformation. It follows from (1) that

$$\eta\circ\phi=0, \quad \phi(\xi)=0. \tag{2}$$

Let g be a compatible Riemannian metric with almost contact structure (ϕ,ξ,η), that is

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{3}$$

for all vector fields X,Y on M^n. Then the structure (ϕ,ξ,η,g) on M^n is called an almost contact metric structure and the manifold M^n equipped with such structure is called an almost contact metric manifold.
From (3), it follows that

$$i)g(\phi X,Y)=-g(X,\phi Y), \quad ii)g(X,\xi)=\eta(X), \tag{4}$$

for all vector fields X,Y on M^n.
An almost contact metric structure (ϕ,ξ,η,g) becomes a contact metric structure if

$$g(X,\phi Y)=d\eta(X,Y), \tag{5}$$

for all vector fields X,Y on M^n.
Given a contact metric manifold $M^n(\phi,\xi,\eta,g)$, we define a (1,1) tensor field h by
$h=\frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi=-\phi h$. Also we have $Tr.h=Tr.\phi h=0$ and $h\xi=0$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X\xi=-\phi X-\phi hX. \tag{6}$$

For a contact metric manifold $M^n(\phi,\xi,\eta,g)$, the (k,μ)-nullity distribution is

$$\begin{aligned} p\rightarrow N_p(k,\mu)=\{Z\in T_pM : R(X,Y)Z=&k[g(Y,Z)X-g(X,Z)Y]\\ &+\mu[g(Y,Z)hX-g(X,Z)hY]\}\end{aligned} \tag{7}$$

for any X,Y $\in T_pM$, and k, μ are real numbers and and R is the curvature tensor. Hence if the characteristic vector field ξ belongs to the (k, μ) -nullity distribution, then we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{8}$$

Thus a contact metric manifold satisfying relation (8) is called a (k, μ)-contact metric manifold [2]. In particular, if $\mu = 0$, then the notion of (k, μ) -nullity distribution reduces to the notion of k-nullity distribution introduced by S.Tanno [12]. In a (k, μ)-contact metric manifold the following relations hold [2, 6].

$$h^2 = (k-1)\phi^2, \;\; k \leq 1 \tag{9}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{10}$$

$$\begin{aligned}(\nabla_X h)(Y) =&[(1-k)g(X, \phi Y) + g(X, h\phi Y)]\xi \\ &+ \eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY,\end{aligned} \tag{11}$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \tag{12}$$

$$\begin{aligned}\eta(R(X, Y)Z) =&k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &+ \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)],\end{aligned} \tag{13}$$

$$S(X, \xi) = 2nk\eta(X), \tag{14}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \tag{15}$$

$$\begin{aligned}S(X, Y) = &[2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ &+ [2(1-n) + n(2k + \mu)]\eta(X)\eta(Y) \;\; n \geq 1,\end{aligned} \tag{16}$$

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2k + \mu)]\eta(X)\xi, \tag{17}$$

$$r = 2n(2n - 2 + k - n\mu), \tag{18}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \tag{19}$$

where S is the Ricci tensor of type (0,2), Q is the Ricci-operator, i.e., $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. From (4), it follows that

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \tag{20}$$

τ-curvature tensor

In an n-dimensional Riemannian manifold M^n, the τ-curvature tensor is given by [7]

$$\begin{aligned}\tau(X, Y)Z =&a_0R(X, Y)Z + a_1S(Y, Z)X + a_2S(X, Z)Y + a_3S(X, Y)Z \\ &+ a_4g(Y, Z)QX + a_5g(X, Z)QY + a_6g(X, Y)QZ \\ &+ a_7r(g(Y, Z)X - g(X, Z)Y),\end{aligned} \tag{21}$$

where $a_0, \ldots\ldots, a_7$ are some smooth functions on M.
For different values of $a_0, \ldots.., a_7$, the τ curvature tensor reduces to the curvature tensor R , quasi-conformal curvature tensor, conformal curvature tensor , conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor , projective curvature tensor P, M-projective curvature tensor, W_i-curvature tensors $(i = 0, \ldots..9)$, W_j^*-curvature tensors (j=0,1).

3. τ-flat-(k, μ)-Contact manifolds

Definition 3.1. *A (k,μ)-contact metric manifold is said to be locally ϕ-symmetric* [13] *if the relation*

$$\phi^2((\nabla_W R)(X, Y)Z) = 0 \tag{22}$$

holds for all vector fields X,Y,Z,W orthogonal to ξ.

Taking $\tau = 0$ in (21) and using (16), (17), we get

$$\begin{aligned} -a_0 R(X, Y)Z =& a_1[\alpha g(Y, Z)X + \beta g(hY, Z)X + \gamma\eta(Y)\eta(Z)X] \\ &+ a_2[\alpha g(X, Z)Y + \beta g(hX, Z)Y + \gamma\eta(X)\eta(Z)Y] \\ &+ a_3[\alpha g(X, Y)Z + \beta g(hX, Y)Z + \gamma\eta(X)\eta(Y)Z] \\ &+ a_4 g(Y, Z)[\alpha X + \beta hX + \gamma\eta(X)\xi] \\ &+ a_5 g(X, Z)[\alpha Y + \beta hY + \gamma\eta(Y)\xi] \\ &+ a_6 g(X, Y)[\alpha Z + \beta hZ + \gamma\eta(Z)\xi] \\ &+ a_7 r[g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{23}$$

where $\alpha = [2(n-1) - n\mu], \quad \beta = [2(n-1) + \mu], \quad \gamma = [2(1-n) + n(2k + \mu)]$.
Differentiating (23) covariantly with respect to W, we get

(24)

$$\begin{aligned} -a_0((\nabla_W)R)(X, Y)Z =& a_1[\beta g((\nabla_W h)Y, Z)X + \gamma((\nabla_W\eta)Y\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X)] \\ &+ a_2[\beta g((\nabla_W h)X, Z)Y + \gamma((\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y)] \\ &+ a_3[\beta g((\nabla_W h)X, Y)Z + \gamma((\nabla_W\eta)(X)\eta(Y)Z + \eta(X)(\nabla_W\eta)(Y)Z)] \\ &+ a_4 g(Y, Z)[\beta(\nabla_W h)X + \gamma((\nabla_W\eta)(X)\xi + \eta(X)\nabla_W\xi)] \\ &+ a_5 g(X, Z)[\beta(\nabla_W h)Y + \gamma((\nabla_W\eta)(Y)\xi + \eta(Y)\nabla_W\xi)] \\ &+ a_6 g(X, Y)[\beta(\nabla_W h)Z + \gamma((\nabla_W\eta)(Z)\xi + \eta(Z)\nabla_W\xi)] \\ &+ a_7 dr(W)[g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

We assume that all vector fields X,Y,Z,W are orthogonal to ξ. Now using (11), (20) in (24), we get

$$
\begin{aligned}
-a_0((\nabla_W)R)(X,Y)Z = & a_4\beta g(Y,Z)[(1-k)g(W,\phi X)+g(W,h\phi X)]\xi \\
& + a_4\gamma g(Y,Z)g(W+hW,\phi X)\xi \\
& + a_5\beta g(X,Z)[(1-k)g(W,\phi Y)+g(W,h\phi Y)]\xi \\
& + a_5\gamma g(X,Z)g(W+hW,\phi Y)\xi \\
& + a_6\beta g(X,Y)[(1-k)g(W,\phi Z)+g(W,h\phi Z)]\xi \\
& + a_6\gamma g(X,Y)g(W+hW,\phi Z)\xi \\
& + a_7 dr(W)[g(Y,Z)X-g(X,Z)Y].
\end{aligned} \tag{25}
$$

Applying ϕ^2 on both sides and using (1) and (2), we get

$$
\phi^2((\nabla_W R)(X,Y)Z) = \frac{a_7}{a_0} dr(w)[g(Y,Z)X-g(X,Z)Y]. \tag{26}
$$

Thus we can state that

Theorem 3.2. *A τ-flat $(k,\mu$)-contact manifold is ϕ-symmetric provided either r is a constant or $a_7=0$.*

As it is well known that [7] $a_7=0$ for W_i-curvature tensors $(i=0,.....9)$, W_j^*-curvature tensors $(j=0,1)$, Conharmonic curvature tensor, Projective curvature tensor, M-Projective curvature tensor, from theorem (3.2), we have

Corollary 3.3. *A (k,μ)-contact manifold (M^n,g) is ϕ-symmetric if either r is a constant or one of the following holds:*

(1) *(M^n,g) is W_i flat $(i=0,...9)$.*
(2) *(M^n,g) is W_j^* flat $(i=0,1)$.*
(3) *(M^n,g) is Conharmonically flat.*
(4) *(M^n,g) is Projectively flat.*
(5) *(M^n,g) is M projectively flat.*

4. ϕ -τ-symmetric (k,μ)-contact manifolds

Let (M^n,g) be a n dimensional (k,μ)-manifold. By using (16) (17) in (21), we obtain

$$
\begin{aligned}
\tau(X,Y)Z = & a_0R(X,Y)Z + a_1[\alpha g(Y,Z)X+\beta g(hY,Z)X+\gamma\eta(Y)\eta(Z)X] \\
& + a_2[\alpha g(X,Z)Y+\beta g(hX,Z)Y+\gamma\eta(X)\eta(Z)Y] \\
& + a_3[\alpha g(X,Y)Z+\beta g(hX,Y)Z+\gamma\eta(X)\eta(Y)Z] \\
& + a_4 g(Y,Z)[\alpha X+\beta hX+\gamma\eta(X)\xi] \\
& + a_5 g(X,Z)[\alpha Y+\beta hY+\gamma\eta(Y)\xi] \\
& + a_6 g(X,Y)[\alpha Z+\beta hZ+\gamma\eta(Z)\xi] \\
& + a_7 r[g(Y,Z)X-g(X,Z)Y].
\end{aligned} \tag{27}
$$

Differentiating (27) covariantly with respect to W, we get

(28)

$$((\nabla_W)\tau)(X,Y)Z = a_0((\nabla_W)R)(X,Y)Z + a_1[\beta g((\nabla_W h)Y,Z)X + \gamma((\nabla_W\eta)Y\eta(Z)X + \eta(Y)(\nabla_W\eta)(Z)X)] + a_2[\beta g((\nabla_W h)X,Z)Y + \gamma((\nabla_W\eta)(X)\eta(Z)Y + \eta(X)(\nabla_W\eta)(Z)Y)] + a_3[\beta g((\nabla_W h)X,Y)Z + \gamma((\nabla_W\eta)(X)\eta(Y)Z + \eta(X)(\nabla_W\eta)(Y)Z)] + a_4 g(Y,Z)[\beta(\nabla_W h)X + \gamma((\nabla_W\eta)(X)\xi + \eta(X)\nabla_W\xi)] + a_5 g(X,Z)[\beta(\nabla_W h)Y + \gamma((\nabla_W\eta)(Y)\xi + \eta(Y)\nabla_W\xi)] + a_6 g(X,Y)[\beta(\nabla_W h)Z + \gamma((\nabla_W\eta)(Z)\xi + \eta(Z)\nabla_W\xi)] + a_7 dr(W)[g(Y,Z)X - g(X,Z)Y].$$

We assume that all vector fields X,Y, Z,W are orthogonal to ξ. Using (11), (20) in (28), we obtain

(29)

$$\begin{aligned}((\nabla_W)\tau)(X,Y)Z =& a_0((\nabla_W)R)(X,Y)Z \\ &+ a_4\beta g(Y,Z)[(1-k)g(W,\phi X) + g(W,h\phi X)]\xi \\ &+ a_4\gamma g(Y,Z)g(W+hW,\phi X)\xi \\ &+ a_5\beta g(X,Z)[(1-k)g(W,\phi Y) + g(W,h\phi Y)]\xi \\ &+ a_5\gamma g(X,Z)g(W+hW,\phi Y)\xi \\ &+ a_6\beta g(X,Y)[(1-k)g(W,\phi Z) + g(W,h\phi Z)]\xi \\ &+ a_6\gamma g(X,Y)g(W+hW,\phi Z)\xi + a_7 dr(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned}$$

Applying ϕ^2 on both sides and using (1) and (2), we have

(30)

$$\phi^2((\nabla_W\tau)(X,Y)Z) = a_0\phi^2((\nabla_W R)(X,Y)Z) - a_7 dr(w)[g(Y,Z)X - g(X,Z)Y].$$

Thus we have

Theorem 4.1. *Let (M^n,g) be (k,μ)-contact manifold. If any two of the following statements hold then the remaining statement holds:*
(i)(M^n,g) is ϕ-τ-Symmetric.
(ii)(M^n,g) is ϕ-symmetric.
(iii)either $a_7 = 0$ or r is constant.

From (21), we have

(31) $$S_\tau(X,Y) = (a_0+na_1+a_2+a_3+a_5+a_6)S(X,Y)+(a_4+(n-1)a_7)rg(X,Y),$$

(32) $$Q_\tau X = (a_0+na_1+a_2+a_3+a_5+a_6)QX + (a_4+(n-1)a_7)rX,$$

where $S_\tau(X,Y)$ is the τ-Ricci tensor of type (0,2), Q_τ is the τ-Ricci-operator, i.e.,
$g(Q_\tau X,Y) = S_\tau(X,Y)$.
From (32) and (17), we have

(33) $$Q_\tau\xi = 2nk(a_0+na_1+a_2+a_3+a_5+a_6)\xi + (a_4+(n-1)a_7)r\xi.$$

Definition 4.2. *A Riemannian manifold (M^n, g) is called a ϕ-τ- Ricci-recurrent if its τ-Ricci curvature Q_τ satisfies the condition*

$$\phi^2(\nabla_W Q_\tau)X = A(W)Q_\tau(X). \tag{34}$$

Using (1) and (34), we get

$$-(\nabla_W Q_\tau)Y + \eta((\nabla_W Q_\tau)Y)\xi = A(W)Q_\tau(Y). \tag{35}$$

Taking Y=ξ and contracting with respect to Z, we obtain

$$-g((\nabla_W Q_\tau)\xi, Z) + \eta((\nabla_W Q_\tau)\xi)\eta(Z) = A(W)g(Q_\tau\xi, Z), \tag{36}$$

$$-g(\nabla_W Q_\tau\xi - Q_\tau(\nabla_W\xi), Z) + \eta((\nabla_W Q_\tau)\xi)\eta(Z) = A(W)g(Q_\tau\xi, Z). \tag{37}$$

Using (33) and (4) in (37), also replacing Z by ϕZ, we have

$$\begin{aligned} &2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)g(\phi W + \phi hW, \phi Z) \\ &+ (a_4 + (n-1)a7)rg(\phi W + \phi hW, \phi Z) \\ &- S\tau(\phi W, \phi Z) - S_\tau(\phi hW, \phi Z) = 0. \end{aligned} \tag{38}$$

Using (1),(19),(31),(9),(14)in (38), we have

(39)

$$\begin{aligned} &[-2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) - 2(a_4 + (n-1)a_7)r + 2(k-1)(2n-2+\mu)]g(W, Z) \\ &+ [2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) + (a_4 + (n-1)a_7)r - 2(k-1)(2n-2+\mu)]\eta(w)\eta(Z) \\ &+ [2(2n-2+\mu) + (a_4 + (n-1)a_7)r]g(hW, Z) \\ &= [(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)][S(W, Z) - S(hW, Z)]. \end{aligned}$$

Replacing W by hW and also by using (1),(9), we get

(40)

$$\begin{aligned} &[-2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) - 2(a_4 + (n-1)a_7 r) + 2(k-1)(2n-2+\mu)]\, g(hW, Z) \\ &+ [2(2n-2+\mu) + (a_4 + (n-1)a_7)r](k-1)[-g(W, Z) + \eta(W)\eta(Z)] \\ &= (a_0 + na_1 + a_2 + a_3 + a_5 + a_6)[S(hW, Z) + (k-1)S(W, Z) - 2nk(k-1)\eta(W)\eta(Z)]. \end{aligned}$$

Subtracting (40) from (39), we obtain

(41)

$$\begin{aligned} &k(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)S(W, Z) = \\ &[-2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) - (k+1)(a_4 + (n-1)a_7)r]g(W, Z) \\ &+ [2nk^2(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) + k(a_4 + (n-1)a_7)r]\eta(W)\eta(Z) \\ &+ [4(k-1)(2n-2+\mu) - 2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) \\ &+ (k-3)(a_4 + (n-1)a_7)r]g(hW, Z). \end{aligned}$$

Taking W = hW and using (16), (9) in (41), we have

$$g(hW, Z) = \frac{F}{E}[-g(W, Z) + \eta(W)\eta(Z)], \tag{42}$$

where

$$E =[k(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)(2(2n - 1) - n\mu) + (k + 1)(a_4 + (n - 1)a_7)r], \tag{43}$$

$$F =[-k(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)(2(2n - 1) + \mu) + 4(k - 1)(2n - 2 + \mu) + (k - 3)(a_4 + (n - 1)a_7)r](k - 1). \tag{44}$$

From (41) and (42), we obtain

$$S(W, Z) = Lg(W, Z) + M\eta(W)\eta(Z), \tag{45}$$

where $L = \frac{BE-DF}{Ek(a_0+na_1+a_2+a_3+a_5+a_6)}$, $M = \frac{CE+DF}{Ek(a_0+na_1+a_2+a_3+a_5+a_6)}$,
$B = [-2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) - (k + 1)(a_4 + (n - 1)a_7)r]$,
$C = [2nk^2(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) + k(a_4 + (n - 1)a_7)r]$,
$D = [4(k - 1)(2n - 2 + \mu) - 2nk(a_0 + na_1 + a_2 + a_3 + a_5 + a_6)$
$+ (k - 3)(a_4 + (n - 1)a_7)r]$.
Thus we have

Theorem 4.3. *A ϕ-τ-Ricci recurrent (k, μ)-contact manifold is η-Einstein provided* $(a_0 + na_1 + a_2 + a_3 + a_5 + a_6) \neq 0$.

Theorem 4.4. *A ϕ-τ-Ricci recurrrnt (k, μ)-contact manifold is Einstein if CE+DF=0.*

5. ξ-τ-flat (k, μ)-contact manifolds

Using (1) in (21), we have

(46)
$$\tau(X, \xi, \xi, V) =a_0R(X, \xi, \xi, V) + a_1S(\xi, \xi)g(X, V) + a_2S(X, \xi)\eta(V) + a_3S(X, \xi)\eta(V) + a_4S(X, V) + a_5S(\xi, V)\eta(X) + a_6S(\xi, V)\eta(X) + a_7r[g(X, V) - \eta(X)\eta(V)].$$

Using (12),(14) in (46), we get

$$S(X, V) = -\left[\frac{a_0k + 2nka_1 + a_7r}{a_4}\right] g(X, V) - \frac{\mu a_0}{a_4}g(hX, V) - \left[\frac{a_0k + 2nk(a_2 + a_3 + a_5 + a_6) - a_7r}{a_4}\right] \eta(X)\eta(V). \tag{47}$$

Changing V to hV in (46) and using (1), (9), (16), we obtain

$$g(X, hV) = P[g(X, V) - \eta(X)\eta(V)], \tag{48}$$

where $P = \left[\frac{(k-1)\mu a_0+(2(n-1)+\mu)a_4}{(2(n-1)-n\mu)a_4+a_0k+2nka_1+a_7r}\right]$.
In view of (47)and (48), we get

$$S(X, V) = -\left[\frac{A_1 + \mu a_0P}{a_4}\right] g(X, V) + \left[\frac{\mu a_0P - A_2}{a_4}\right] \eta(X)\eta(V), \tag{49}$$

where
$A_1 = [a_0k + 2nka_1 + a_7r]$, $A_2 = [a_0k + 2nk(a_2 + a_3 + a_5 + a_6) - a_7r]$.
We can state that

Theorem 5.1. *A ξ-τ-flat (k,μ)-contact manifold is η-Einstein provided $a_4 \neq 0$.*

If $a_4 = 0$ and $a_7 \neq 0$, then by using (8), (14) and the fact that M^n is $\xi - \tau$ -flat in (46), we get

$$\begin{aligned} 0 =&(a_0k + 2nka_1 + a_7r)g(X,V) + \mu a_0 g(hX,V) \\ &+ [-a_0k + 2nk(a_2 + a_3 + a_5 + a_6 - a_7r)]\eta(X)\eta(V). \end{aligned} \tag{50}$$

Contracting the above equation, we have

$$r = \frac{a_0k(1-n) - 2nk(na_1 + a_2 + a_3 + a_5 + a_6)}{a_7(n-1)}. \tag{51}$$

Thus we have

Theorem 5.2. *If $a_4 = 0$ and $a_7 \neq 0$ in a $\xi - \tau$ -flat (k,μ)-contact manifold then the scalar curvature r is given by* (51) *.*

If $a_4 = 0$, $a_7 = 0$ then by using (8), (14) in (46) and the fact that M is $\xi - \tau$ -flat, also contracting the resultant expression, we obtain

$$k[a_0(n-1) + 2n(na_1 + a2 + a_3 + a5 + a_6)] = 0. \tag{52}$$

Since $k \neq 0$, we have $(n-1)a_0 + 2n(na_1 + a2 + a_3 + a5 + a_6) = 0$.
Thus we can state that

Theorem 5.3. *A ξ-τ -flat (k,μ)-contact manifold, $(n-1)a_0 + 2n(na_1 + a2 + a_3 + a5 + a_6) = 0$ if $a_4 = 0$ and $a_7 = 0$.*

6. (k,μ)- Contact manifold satisfying the condition τ.S=0

Let (M^n, g) be a (k,μ) manifold. Suppose τ.S=0. Then we have

$$S(\tau(U,V)Y,\xi) + S(Y,\tau(U,V)\xi). \tag{53}$$

Taking $Y = \xi$ in (53), we have

$$S(\tau(U,V)\xi,\xi) = 0. \tag{54}$$

In view of (21), (14), (8),(17)in (54),we have

$$S(U,V) = \frac{-2nk(a_1 + a_2 + a_4 + a_5)}{a_3}\eta(U)\eta(V) - \frac{2nk}{a_3}g(U,V) \quad a_3 \neq 0. \tag{55}$$

We can state the above result as follows:

Theorem 6.1. *A (k,μ)-contact manifold with condition τ.S=0 is η-Einstein manifold, provided $a_3 \neq 0$.*

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