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Solving Polynomial Equations

Mircea Ion Cîrnu
Department of Mathematics,
Faculty of Applied Sciences,
Polytechnic University of Bucharest,
Romania
cirnumircea@yahoo.com

Abstract

In this paper are given simple methods for calculating approximate values of the extreme roots of polynomials - roots dominant and dominated in modulus. They are obtained by improving old methods, namely the Newton's radical method and the Daniel Bernoulli's ratio method. The eigenvalues of a square matrix can be also calculated, even if it is not known its characteristic polynomial. Unlike the old methods, the present methods can calculate multiple and complex roots. By suitable variable changes, can be solved polynomials which initially have not extreme roots. In this way can be calculated complex roots of the polynomials with real coefficients and radicals of real or complex numbers. Using the results from a previous Author's work, finally shown how the present methods can be used to solve the nonlinear algebraic equations. Throughout the paper are given illustrative examples.

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1 Introduction

The fact that the roots of the polynomials can be obtained immediately using computer programs as MATLAB, does not diminish the importance of finding new methods for solving the polynomial equations, simpler that the current ones. In the paper [3], the Author gave an automatic formula - without algebraic calculus, based on discrete deconvolution algorithm, for numerical solutions of the linear difference equations with constant coefficients. Using this formula, in [4] the deconvolution was applied to Daniel Bernoulli's method, [13], in which the dominant root of a polynomial is calculated as limit of the ratio of two consecutive solutions of the difference equation for which the polynomial is characteristic.

In the present paper, we give a simplified method for approximate calculus of the polynomial roots, computing both dominant and dominated (extreme) roots as limit of the ratio of the sums of consecutive integer powers of the polynomial roots or, alternately, of the radical of such sum. Therefore the use of the difference equation used in Bernoulli's method is eliminated. This simplifies both the theory and applications of the method. The sums of the power roots will be calculated as traces of powers of the companion matrix of the polynomial or by some automatic deconvolution formulas deduced by the Newton's identities. The first possibility can be also used to calculate the extreme eigenvalues of a square matrix, without knowing its characteristic polynomial. Unlike the Bernoulli's method, by our method can be calculated the multiple and complex extreme roots. By suitable changes of variables, it can be applied to the polynomials without extreme roots. In this way can be calculated complex roots of polynomials with real coefficients and radicals. The second method mentioned above, to determine the extreme roots as limits of radicals of sums of power roots, the limit being taken after the radical index which is the same with the exponent of the power roots, was given by Newton and is unusable, because it requires the calculation of high index radicals. However, it has an historic significance because led on Newton to the discovery of his famous identities about the sums of power roots. In some examples below the roots will be calculated, for completeness, both by ratio and radical methods.

Structure of the paper is the following: After the Introduction, in Section 2 we give the *ratio and radical formulas* to approximate calculus of the extreme (dominant and dominated in modulus) numbers from a finite set, if known the sums of integer powers of the numbers from that set. In the following Sections we apply these formulas to the set of the roots of a polynomial. In Section 3, we calculate the extreme eigenvalues of a matrix using the formulas of Section 2 and the traces of matrix powers. In Section 4 we compute the roots of a polynomial, by method presented in the previous Section, considering these roots as the eigenvalues of the companion matrix. As already presented methods require laborious calculations of matrix powers, we will give in Section 5 another possibility for calculus of sums of integer powers of polynomial roots, with less numerical calculus, by some simple automatic formulas deduced from the Newton's identities and based on discrete deconvolution. For this, in Section 5.1 we present the notions of discrete convolution and deconvolution. These notions were used by the Author in a series of papers, [3] - [8], to solve various types of equations, especially with recurrence. In Section 5.2 we recall the Newton's identities, which were demonstrated in [2] by Laplace transform method. In the papers cited in [2] one can find other proofs of these identities. In Section 5.3 we obtain the above mentioned deconvolution formulas for calculus of sums of integer powers, both positive and negative, of polynomial roots. In Section 5.4, we apply those shown in the previous Sections to solve polynomial equations with extreme roots. Various situations are presented. In the Section 5.5 are given cases of polynomial equations initially without extreme roots, but which may have such roots by a convenient change of variable and can be solved by the methods presented here. In this manner both complex roots of polynomials with real coefficients and radicals can be calculated. In the Section 6, we make a final comparison between Bernoulli's method and our methods presented here.

In the paper [9] has shown that the variation of two consecutive approximate values of the exact solution of an algebraic equation $f(x) = 0$ is a root of the Taylor polynomial of the function $f(x)$ in the known approximate value, named *resolving polynomial*, in the cited work being also obtained different forms for the constant term of this polynomial. In this way the algebraic equations are reduced to polynomial equations, obtaining more accurate values of the solutions of the algebraic equations, than those deduced by Newton-Raphson method. Based on these facts, the present methods can be used to approximate solve the algebraic equations. We illustrate this possibility in the last Section of the paper. Throughout the paper are given examples, the calculations being performed using MATLAB.

2 Preliminary results

In the Theorem below we give a general result that will be used in the integer paper. With several variations it is well-known. However, for completion, we give here this result and its elementary proof.
Let M be the set of complex numbers $x_j \neq 0$, $j = 1, 2, \ldots, n$, not necessarily distinct. If there is $x_D \in M$, respective $x_d \in M$, such that $|x_j| < |x_D|$, respective $|x_j| > |x_d|$, for all x_j distinct by x_D, respective x_d, then we call x_D *dominant number* (in modulus), respective x_d *dominated number* (in modulus) of the set M. These numbers are named *extreme numbers* of the set M. We denote

$$S_k = \sum_{j=1}^{n} x_j^k, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{1}$$

Theorem 2.1. *The extreme numbers of the set M are given, if exist, by the formulas*

$$x_D = \lim_{k\to\infty} \frac{S_{k+1}}{S_k} = \lim_{k\to\infty} \sqrt[2k+1]{\frac{1}{m} S_{2k+1}}\ , \tag{2}$$

respective

$$x_d = \lim_{k\to\infty} \frac{S_{-k}}{S_{-k-1}} = \frac{1}{\lim_{k\to\infty} \sqrt[2k+1]{\frac{1}{m} S_{-2k-1}}} . \tag{3}$$

Here $m \in \{1, 2, \ldots, n\}$, *denotes the multiplicity of the extreme number, namely the number of its occurrences in the set* M.

Proof. Let $x_1 = x_2 = \cdots = x_m = x_D$ be the dominant root. Then we have $\left|\frac{x_j}{x_D}\right| < 1$, hence $\lim_{k\to\infty} \left(\frac{x_j}{x_D}\right)^k = 0$, for $j = m+1, m+2, \ldots, n$, therefore

$$\lim_{k\to\infty} \frac{S_{k+1}}{S_k} = \lim_{k\to\infty} \frac{\sum_{j=1}^n x_j^{k+1}}{\sum_{j=1}^n x_j^k} = \lim_{k\to\infty} \frac{m x_D^{k+1} + \sum_{j=m+1}^n x_j^{k+1}}{m x_D^k + \sum_{j=m+1}^n x_j^k} =$$

$$= x_D \lim_{k\to\infty} \frac{m + \sum_{j=m+1}^n \left(\frac{x_j}{x_D}\right)^{k+1}}{m + \sum_{j=m+1}^n \left(\frac{x_j}{x_D}\right)^k} = x_D \ ,$$

and

$$\lim_{k\to\infty} \sqrt[2k+1]{\frac{1}{m} S_{2k+1}} = \lim_{k\to\infty} \sqrt[2k+1]{\frac{1}{m} \sum_{j=1}^n x_j^{2k+1}} = \lim_{k\to\infty} \sqrt[2k+1]{x_D^{2k+1} + \frac{1}{m} \sum_{j=m+1}^n x_j^{2k+1}}$$

$$= x_D \lim_{k\to\infty} \sqrt[2k+1]{1 + \frac{1}{m} \sum_{j=m+1}^n \left(\frac{x_j}{x_D}\right)^{2k+1}} = x_D \ .$$

For a dominated root, we apply the first part of the Theorem to the set of numbers x_j^{-1}, $j = 1, 2, \ldots, n$. □

Remarks.1) The multiplicity m is necessary only in the formulas based on radicals. In these formulas we consider only odd indices, to avoid the calculus with complex numbers when the numbers x_j are real. The radical formulas are not suitable for calculating polynomial roots because require the calculation of high order radicals. However, to show how these formulas work, we use them in some of the examples below.
2) In this paper we apply the formulas given in Theorem 1 to calculate the extreme roots of a polynomial, namely the extreme numbers in the set of all non-null roots of the polynomial.

3 Approximate calculus of the matriceal eigenvalues

First we will calculate the eigenvalues $x_1, x_2, \ldots, x_n$, not necessary distinct, of a square matrix A of order n, namely the roots of its characteristic polynomial $P(x) = det(xI_n - A)$, without effective knowledge of this polynomial. Because the trace of matrix power A^k is $S_k = \sum_{j=1}^n x_j^k = Tr(A^k)$, in conformity with (2) and (3), the non-null extreme eigenvalues are given by the formulas

$$x_D = \lim_{k\to\infty} \frac{Tr(A^{k+1})}{Tr(A^k)} = \lim_{k\to\infty} \sqrt[2k+1]{Tr(A^{2k+1})/m}\,, \tag{4}$$

and

$$x_d = \lim_{k\to\infty} \frac{Tr(A^{-k})}{Tr(A^{-k-1})} = \frac{1}{\lim_{k\to\infty} \sqrt[2k+1]{Tr(A^{-2k-1})/m}}\,, \tag{5}$$

where m is the multiplicity of the extreme eigenvalue. This straightforward calculation is in contrast with that given in the paper [1] in which it is proposed to calculate the eigenvalues from the characteristic polynomial equation whose coefficients to be algebraic calculated by the Newton's identities, given in formula (9) below, the sums S_k occurring in these identities being computed as traces of matrix powers.

Example 1. Let the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ with the eigenvalues 8.7040, −2.0829 and 1.3789, all decimal being exact. Because
$A^7 = \begin{bmatrix} 614555 & 915508 & 1073314 \\ 853490 & 1270947 & 1490236 \\ 1087438 & 1619794 & 1899091 \end{bmatrix}$, $A^8 = \begin{bmatrix} 5349901 & 7967937 & 9342204 \\ 7427514 & 11063308 & 12971001 \\ 9465705 & 14098266 & 16529713 \end{bmatrix}$,
$Tr(A^7) = 3784593$ and $Tr(A^8) = 32942922$, the dominant eigenvalue is $x_D \approx \dfrac{Tr(A^8)}{Tr(A^7)} \approx 8.7045$, or $x_D \approx \sqrt[7]{Tr(A^7)} \approx 8.7040$. Also
$A^{-7} \approx \begin{bmatrix} 0.0379 & 0.0140 & -0.0324 \\ 0.0518 & 0.0109 & -0.0378 \\ -0.0659 & -0.0173 & 0.0508 \end{bmatrix}$, $A^{-8} \approx \begin{bmatrix} 0.0307 & 0.0076 & -0.0233 \\ 0.0327 & 0.0117 & -0.0277 \\ -0.0455 & -0.0143 & 0.0370 \end{bmatrix}$,
$Tr(A^{-7}) \approx 0.099$ and $Tr(A^{-8}) \approx 0.0794$, hence the dominated eigenvalue is $x_d \approx \dfrac{Tr(A^{-7})}{Tr(A^{-8})} \approx 1.2469$, or $x_d \approx 1/\sqrt[7]{Tr(A^{-7})} \approx 1.3915$. The third eigenvalue can be calculated by the relation $x_3 = Tr(A) - x_D - x_d$.

Remark. This example is taken from[12], where the dominant eigenvalue is computed only by the radical method. See also [13].

Example 2. Let the matrix $A = \begin{bmatrix} 6 & -1 & -1 \\ 5 & 2 & -9 \\ -1 & 0 & 7 \end{bmatrix}$ with eigenvalues 6, 6 and 3. Then $x_D \approx \dfrac{Tr(A^7)}{Tr(A^6)} = \dfrac{562059}{94041} \approx 5.98$, $x_d \approx \dfrac{Tr(A^{-6})}{Tr(A^{-7})} \approx \dfrac{0.001414}{0.000464} \approx 3.04$, $x_3 = Tr(A) - x_D - x_d \approx 5.98$. Also $x_D \approx \sqrt[7]{Tr(A^7)/2} \approx 6.0033$ and $x_d \approx 1/\sqrt[7]{Tr(A^{-7})} \approx 2.99$.

Example 3. The method can be eventually used when the matrix has not a dominant eigenvalue (in modulus), but has a dominated one, as we can view in this example. Let the matrix $A = \begin{bmatrix} 57 & 153 & 144 \\ -30 & -84 & -84 \\ 9 & 27 & 30 \end{bmatrix}$ with the eigenvalues ± 6 and 3. If we try to determine a dominant eigenvalue, we obtain $\dfrac{Tr(A^3)}{Tr(A^2)} = \dfrac{27}{81} \approx 0.3$, $\dfrac{Tr(A^4)}{Tr(A^3)} = \dfrac{2673}{27} \approx 99$, $\dfrac{Tr(A^5)}{Tr(A^4)} = \dfrac{243}{2673} \approx 0.9$, $\dfrac{Tr(A^6)}{Tr(A^5)} = \dfrac{94041}{243} \approx 387$, $\dfrac{Tr(A^7)}{Tr(A^6)} = \dfrac{2187}{94041} \approx 0.023$ and so on. Therefore, the matrix has not a dominant eigenvalue. On the other hand, we have $x_d \approx \dfrac{Tr(A^{-6})}{Tr(A^{-7})} \approx \dfrac{0.0014}{0.0005} = 2.8$. Because $x_1 + x_2 + x_d = Tr(A) = 3$ and $x_1 x_2 x_d = det(A) = -108$, we have $x_1 + x_2 \approx 0.2$ and $x_1 x_2 \approx -38.57$, hence the other eigenvalues are approximately given by the quadratic equation $x^2 - 0.2x - 38.57 = 0$, hence are $x_1 \approx 6.31$ and $x_2 \approx -6.11$.

4 Approximate calculus of the polynomial roots by the companion matrix method

We consider the polynomial equation of complex variable x,

$$P(x) = \sum_{k=0}^{n} a_k x^{n-k} = \sum_{k=0}^{n} a_{n-k} x^k = 0\ , \tag{6}$$

with complex coefficients $a_0 \neq 0$, $a_1, \ldots, a_n$ and the complex roots $x_1, x_2, \ldots, x_n$, not necessarily distinct. If $P(x)$ is monic, hence $a_0 = 1$, we consider its *companion matrix* C, that is a square matrix of order n, given by the formula

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & -a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix}$$

for which $P(x)$ is the characteristic polynomial and its roots are the eigenvalues of the matrix C. Therefore we have $S_k = \sum_{j=1}^{n} x_j^k = Tr(C^k)$, $k = 0, \pm 1, \pm 2, \ldots$. Determining the roots of the polynomial is then reduced to calculation of the eigenvalues of the matrix C by the method given in the previous Section 3.

Example 1. Let be the polynomial equation $x^3 - 5x^2 + 6x - 1 = 0$, with the roots $3.2469, 1.5549$ and 0.19806, having all decimal exact. The companion matrix and its powers, $C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -6 \\ 0 & 1 & 5 \end{bmatrix}$, $C^5 = \begin{bmatrix} 19 & 66 & 221 \\ -109 & -377 & -1260 \\ 66 & 221 & 728 \end{bmatrix}$ and $C^6 = \begin{bmatrix} 66 & 221 & 728 \\ -377 & -1260 & -4147 \\ 229 & 728 & 2380 \end{bmatrix}$, give the dominant root $x_D \approx \dfrac{Tr(C^6)}{Tr(C^5)} = \dfrac{1186}{370} \approx 3.2054$, or $x_D \approx \sqrt[5]{Tr(C^5)} = \sqrt[5]{370} \approx 3.263$. Also $C^{-5} = \begin{bmatrix} 4004 & 793 & 157 \\ -3808 & -754 & -149 \\ 793 & 157 & 31 \end{bmatrix}$ and $C^{-6} = \begin{bmatrix} 20216 & 4004 & 793 \\ -19227 & -3808 & -754 \\ 4004 & 793 & 157 \end{bmatrix}$, the dominated root being $x_d \approx \dfrac{Tr(C^{-5})}{Tr(C^{-6})} = \dfrac{3281}{16565} \approx 0.198$, or $x_d \approx 1/\sqrt[5]{Tr(C^{-5})} = 1/\sqrt[5]{3281} \approx 0.198$.

Example 2. For the polynomial $x^5 - 3x^4 + x^3 + 3x^2 - x - 1 = 0$, with the simple root 1 and double roots 1.618 and -0.618, having all decimals exact, the companion matrix is $C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$. The extreme roots are $x_D \approx \dfrac{Tr(C^7)}{TrC^6)} = \dfrac{59}{37} \approx 1.5946$ and $x_d \approx \dfrac{Tr(C^{-6})}{Tr(C^{-7})} = \dfrac{37}{-57} \approx -0.62$. Alternatively, $x_D \approx \sqrt[7]{Tr(C^7)/2} = \sqrt[7]{59/2} \approx 1.62$ and $x_d \approx 1/\sqrt[7]{Tr(C^{-7})/2} = 1/\sqrt[7]{-57/2} \approx -0.62$.

5 Approximate calculus of the polynomial roots by deconvolution method

The sums of the power of roots used in Theorem 1, can be computed by some deconvolution formulas, deduced from the Newton's identities. To present this method, we first make some preparations.

5.1 Discrete convolution and deconvolution

We say, see [10], that the sequence $c = a \star b = (c_0, c_1, \ldots, c_k, \ldots)$ is the *discrete convolution* or *Cauchy product* of the sequences of complex numbers $a = (a_0, a_1, \ldots, a_k, \ldots)$ and $b = (b_0, b_1, \ldots, b_k, \ldots)$, if it has the components

$$c_k = \sum_{j=0}^{k} a_{k-j} b_j = \sum_{j=0}^{k} a_j b_{k-j}\ ,\ k = 0, 1, 2, \ldots\ , \tag{7}$$

also given by the multiplication algorithm

$$\begin{array}{cccc}
a_0 & a_1 & a_2 & \ldots \\
b_0 & b_1 & b_2 & \ldots \\
\hline
a_0 b_0 & a_1 b_0 & a_2 b_0 & \ldots \\
 & a_0 b_1 & a_1 b_1 & \ldots \\
 & & a_0 b_2 & \ldots \\
\hline
c_0 = a_0 b_0 & c_1 = a_1 b_0 + a_0 b_1 & c_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 & \ldots
\end{array}$$

The convolution is commutative, associative and distributive relative to addition of sequences and has $\delta = (1, 0, 0, \ldots)$ as unity. If $a_0 \neq 0$, we can perform the inverse operation $b = c/a$, named the *deconvolution* of the sequence c by a, such that $c = a \star b$. It has the components

$$b_0 = \frac{c_0}{a_0}, \quad b_k = \frac{1}{a_0}\left(c_k - \sum_{j=0}^{k-1} a_{k-j} b_j\right),\ k = 1, 2, \ldots, \tag{8}$$

also given by the division algorithm

c_0	c_1	$\ldots$	a_0	a_1	$\ldots$
c_0	$a_1 b_0$	$\ldots$	$b_0 = \dfrac{c_0}{a_0}$	$b_1 = \dfrac{c_1 - a_1 b_0}{a_0}$	$\ldots$
/	$c_1 - a_1 b_0$	$\ldots$			
	$c_1 - a_1 b_0$	$\ldots$			
	/	$\ldots$			

Remarks. 1) In MATLAB, the discrete convolution and deconvolution can be calculated by the instructions conv and deconv.

2) In the example in Section 5.4.1 below will show a concrete calculation of deconvolution, both by its algorithm and by MATLAB. In the other examples of the paper the deconvolution is similar calculated, without giving concrete details.

5.2 Newton's identities

If $P(x)$ is the polynomial given by (6), with complex coefficients $a_0 \neq 0$, $a_1, \ldots, a_n$, then the sums of integer powers of its roots, given by (1), satisfy the Newton's identities, see [2],

$$\sum_{j=0}^{k-1} a_j S_{k-j} = -k a_k \ , \ k = 1, 2, \ldots, n, \qquad \sum_{j=0}^{n} a_j S_{k-j} = 0 \ , \ k > n \ . \tag{9}$$

5.3 Automatic calculation of the sums of integer powers of polynomial roots

Theorem 5.1. *The sums (1) of integer powers of the roots of the polynomial (6), with the complex coefficients* $a_0 \neq 0, a_1, \ldots, a_{n-1}, a_n \neq 0$, *are given by the deconvolution formula*

$$(S_k \ : \ k = 1, 2, \ldots) = -(a_1, 2a_2, \ldots, na_n, 0, \ldots)/(a_0, a_1, \ldots, a_n, 0, \ldots), \tag{10}$$

respective

$$(S_{-k} \ : \ k = 1, 2, \ldots) = -(a_{n-1}, 2a_{n-2}, \ldots, na_0, 0, \ldots)/(a_n, a_{n-1}, \ldots, a_0, 0, \ldots) \ . \tag{11}$$

Proof. Using the definition (7) of the convolution and Newton's identities (9), we obtain

$$(S_k \ : \ k = 1, 2, \ldots) \star (a_0, a_1, \ldots, a_n, 0, 0, \ldots) =$$

$$= \sum_{j=1}^{n} a_j S_{k-j} \ : \ k = 1, 2, \ldots, n \, ; \sum_{j=0}^{n} a_j S_{k-j} \ : \ k > n) =$$

$$= -(a_1, 2a_2, \ldots, na_n, 0, 0, \ldots) \ ,$$

from which it results formula (10). To determine the sums of the negative powers of the roots, we make the change of variables $x = y^{-1}$. Then the polynomial equation (6) takes the form $\sum_{j=0}^{n} a_j y^j = 0$, with the roots $y_j = x_j^{-1}$, $j = 1, 2, \ldots, n$. Applying formula (10), the sums $S_{-k} = \sum_{j=1}^{n} x_j^{-k} = \sum_{j=1}^{n} y_j^k$, $k = 1, 2, \ldots$, are given by the formula (11). □

Example. Consider the polynomial equation

$$P(x) = x^7 + x^6 - 2x^5 - 4x^4 - x^3 + 3x^2 + 3x + 1 = 0.$$

Applying the formulas (10) and (11), we get

$$(S_k \ : \ k = 1, 2, \ldots) = -(1, -4, -12, -4, 15, 18, 7, 0, 0, \ldots)/(1, 1, -2, -4,$$

$$-1, 3, 3, 1, 0, 0, \ldots) = (-1, 5, 5, 5, 11, 13, 21, 23, 35, 43, 59, \ldots),$$

$$(S_{-k} \ : \ k = 1, 2, \ldots) = -(3, 6, -3, -16, -10, 6, 7, 0, 0, \ldots)/(1, 3, 3, -1,$$

$$-4, -2, 1, 1, 0, 0, \ldots) = (-3, 3, 3, -5, 7, -3, -3, 11, -15, 13, -3, \ldots) \, .$$

5.4 Polynomial equations with extreme roots

Applying the Theorem 1 to the set of roots of the polynomial, we obtain two methods for approximate calculus of its extreme roots, the *ratio method* and the *radical method*. The sums that appear in the formulas of Theorem 1 can be automatic calculated by the deconvolution formulas given in Theorem 2. We give here some examples.

5.4.1 Distinct roots

We will solve again the polynomial equation $x^3 - 5x^2 + 6x - 1 = 0$, considered in Example 1 of Section 4. Using the formula (10), the sums of positive powers of the roots of the equation are

$$(S_k \ : \ k = 1, 2, \ldots) = (5, -12, 3, 0, 0, \ldots)/(1, -5, 6, -1, 0, 0, \ldots) =$$

$$= (5, 13, 38, 117, 370, 1186, 3827, 12389, 40169, 130338, \ldots)\ ,$$

obtained by deconvolution algorithm

5	-12	3	0	0	0	...	1	-5	6	-1	0	0	...
5	-25	30	-5	0	0	...	5	13	38	117	370	1186	...
/	13	-27	5	0	0	...							
	13	-65	78	-13	0	...							
	/	38	-73	13	0	...							
		38	-190	228	-38	...							
		/	117	-215	38	...							
			117	-585	702	...							
			/	370	-664	...							
				370	-1850	...							
				/	1186	...							

or by the MATLAB instruction

$a = [5 \quad -12 \quad 3 \quad zeros(1, 20)]$, $b = [1 \quad -5 \quad 6 \quad -1 \quad zeros(1, 10)]$, $c = deconv(a, b)$

Therefore, the dominant root is $x_D \approx S_6/S_5 = 130338/40169 \approx 3.2447$, or $x_D \approx \sqrt[9]{S_9} = \sqrt[9]{40169} \approx 3.2474.$. Using formula (11), we obtain

$$(S_{-k} \ : \ k = 1, 2, \ldots) = -(6, -10, 3, 0, 0, \ldots)/(-1, 6, -5, 1, 0, 0, \ldots) =$$

$$= (6, 26, 129, 650, 3281, 16565, \ldots)\ ,$$

hence the dominated root is $x_d \approx S_{-5}/S_{-6} = 3281/16565 \approx 0.198$ or $x_d \approx 1/\sqrt[5]{S_{-5}} = 1/\sqrt[5]{3281} \approx 0.198$.

5.4.2 Golden and Fibonacci type numbers

We consider the quadratic equation $x^2 - x - 1 = 0$, with the roots $x_1 = \frac{1+\sqrt{5}}{2} = \phi$, the *golden number (or ratio)*, and $x_2 = \frac{1-\sqrt{5}}{2} = 1 - \phi = \psi$. Denoting $S_k = x_1^k + x_2^k = \phi^k + \psi^k$, $k = 0, \pm 1, \pm 2, \ldots$, we have

$$(S_k \ : \ k = 1, 2, \ldots) = (1, 2, 0, 0, \ldots)/(1, -1, -1, 0, 0, \ldots) =$$

$$= (1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots),$$

these being the Fibonacci type numbers, because each number in sequence is the sum of the previous two, but the initial data are 1 and 3. We have $\phi \approx S_{10}/S_9 = 123/76 \approx 1.618$. Also

$$(S_{-k} \ : \ k = 1, 2, \ldots) = (1, -2, 0, \ldots)/(-1, -1, 1, 0, \ldots) =$$
$$= (-1, 3, -4, 7, -11, 18, -29, 47, -76, \ldots),$$

hence $\psi \approx S_{-8}/S_{-9} = -47/76 \approx -0.618$.

Remark. In connection with this example, to present a little history. The first that invented the ratio method for solving equations, was the German mathematician and astronomer Johannes Kepler. He approximately calculated the golden number ϕ as the ratio of two consecutive Fibonacci numbers $F_k = \frac{\phi^k - \psi^k}{\sqrt{5}}$. His method was later extended to all polynomial equations by Daniel Bernoulli.

5.4.3 Multiple roots

We consider again the polynomial equation $x^5 - 3x^4 + x^3 + 3x^2 - x - 1 = 0$, in Example 2, Section 4. Now

$$(S_k) = (3, -2, -9, 4, 5, 0, \ldots)/(1, -3, 1, 3, -1, -1, 0, \ldots) =$$

$$= (3, 7, 9, 15, 23, 37, 59, \ldots),$$

hence $x_D \approx S_7/S_6 = 59/37 \approx 1.5946$, or $x_D \approx \sqrt[7]{S_7/2} = \sqrt[7]{59/2} \approx 1.62$. Also

$$(S_{-k}) = (1, -6, -3, 12, -5, 0, \ldots)/(-1, -1, 3, 1, -3, 1, 0, \ldots) =$$

$$= (-1, 7, -7, 15, -21, 37, -57, \ldots),$$

hence $x_d \approx S_{-6}/S_{-7} = -37/57 \approx -0.62$, or $x_d \approx 1/\sqrt[7]{S_{-7}/2} = 1/\sqrt[7]{-57/2} \approx -0.62$.

5.4.4 Complex coefficients

Let the polynomial equation $x^3 + ix^2 + x + 1 = 0$, of roots $0.2464 - 1.6815i$, $0.3682 + 0.8478i$ and $-0.6245 - 0.1663i$, with exact decimals. Then

$$(S_k) = -(i, 2, 3, 0, \ldots)/(1, i, 1, 1, 0, \ldots) = (-i, -3, -3 + 4i, 7 + 4i,$$

$$10 - 11i, -15 - 18i, -35 + 22i, 27 + 64i, 114 - 31i, -23 - 200i, \ldots),$$

hence the dominant root is $x_D \approx (-23 - 200i)/(114 - 31i) \approx 0.2564 - 1.6847i$ and

$$(S_{-k}) = -(1, 2i, 2, 0, \ldots)/(1, 1, i, 1, 0, \ldots) = (-1, 1 - 2i, -4 + 3i, 3 - 4i,$$

$$-1 + 10i, 1 - 16i, 6 + 21i, -21 - 32i, 41 + 42i, -79 - 42i, \ldots),$$

hence the dominated root is $x_d \approx (41 + 42i)/(-79 - 42i) \approx -0.625 - 0.1994i$.

5.5 Polynomials without extreme roots

Appropriate changes of variables can transform polynomials without extreme roots in some who have such roots and therefore we can applay them our methods.

5.5.1 Complex roots of the polynomials with real coefficients

Because the complex roots of the polynomials with real coefficients occur only in pairs of complex conjugate roots with the same multiplicity, they are not extreme. However, a translation with an imaginary number, change the polynomial into one that has extreme roots.

Example. Let $x^2 - x + 1 = 0$ be a quadratic equation with the complex roots $x_{1,2} = \dfrac{1 \pm i\sqrt{3}}{2} \approx 0.5 \pm 0.8662i$, which are not extreme. Performing the change of variables $x = y - i$, we obtain the equation $y^2 - (1 + 2i)y + i = 0$. Then

$$(s_k) = (y_1^k + y_2^k) = (1 + 2i, -2i, 0, \ldots)/(1, -1 - 2i, i, 0, \ldots) =$$

$$= (1 + 2i, -3 + 2i, -5 - 5i, 7 - 12i, 26 + 7i, \ldots),$$

hence $y_D \approx \dfrac{s_5}{s_4} = \dfrac{26 + 7i}{7 - 12i} \approx 0.5 + 1.87i$, $x_1 = y_D - i \approx 0.5 + 0.87i$ and $x_2 = 1 - x_1 \approx 0.5 - 0.87i$.

5.5.2 Radicals by real numbers

If $r > 1$, the radical $x = \sqrt[n]{r}$ can be calculated as root of the polynomial equation $x^n - r = 0$, equation without extreme roots. A translation with a real number, for example $x = y - 1$, gives the equation $(y-1)^n - r = 0$, hence

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^k - r = 0, \tag{12}$$

with all roots distinct, its real root $y_1 > 1$ being dominant. Using (2), the radical is given by the formula

$$\sqrt[n]{r} = \lim_{k\to\infty}\frac{s_{k+1}}{s_k} - 1, \tag{13}$$

the sums $s_k = \sum_{j=1}^{n} y_j^k$, $k = 1, 2, \ldots$, being computed by formula (10) applied to the equation (12).

Example 1. To compute $\sqrt{2}$ we must solve the equation $x^2 = 2$, hence with the variables change $x = y - 1$, the equation $y^2 - 2y - 1 = 0$. For this equation we have

$$(s_k) = (y_1^k + y_2^k) = (2, 2, 0, \ldots)/(1, -2, -1, 0, \ldots) =$$
$$= (2, 6, 14, 34, 82, 198, 478, \ldots),$$

hence $\sqrt{2} \approx \dfrac{s_7}{s_6} - 1 = \dfrac{478}{198} - 1 \approx 1.414$.

Example 2. To compute $\sqrt[5]{2}$ we make the change of variables $x = y - 1$ in the equation $x^5 = 2$, obtaining the equation $y^5 - 5y^4 + 10y^3 - 10y^2 + 5y - 3 = 0$. For this equation we have

$$(s_k) = (5, -20, 30, -20, 15, 0, \ldots)/(1, -5, 10, -10, 5, -3, 0, \ldots) =$$
$$= (5, 5, 5, 5, 15, 65, 215, 565, 1265, 2545, 4845, 9245,$$
$$18595, 40045, 90135, 204485, 456285, 993485, 2118885, \ldots),$$

so $\sqrt[5]{2} \approx \dfrac{s_{19}}{s_{18}} - 1 = \dfrac{2118885}{993485} - 1 \approx 1.13$.

5.5.3 Radicals by complex numbers

To compute $x = \sqrt{1+i}$ we make the change of variables $x = y - 1$ in the equation $x^2 = 1 + i$, obtaining the equation $y^2 - 2y - i = 0$. If $s_k = y_1^k + y_2^k$, we have

$$(s_k) = (2, 2i, 0, \ldots)/(1, -2, -i, 0, \ldots) =$$
$$= (2, 4 + 2i, \ldots, -8050 + 5264i, -19290 + 7384i, \ldots),$$

so $x \approx \dfrac{-19290+7384i}{-8050+5264i} - 1 \approx 1.098 + 0.455i$.

6 Compared with Bernoulli's method

In the Bernoulli's method, the real dominant root of the polynomial $P(x)$ is calculated using the formula $x_D = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$, where u_k are solutions of the difference equation $\sum_{k=0}^{n} a_{n-k}u_{k+j} = 0$, $j = 0, 1, 2, \ldots$, which has $P(x)$ as characteristic polynomial. Starting with different initial values $u_0, u_1, \ldots, u_{n-1}$, some deconvolution formulas for u_k was obtained in [4]. For example, if $u_0 = 1$ and $u_1 = u_2 = \cdots = u_{n-1} = 0$, then

$$(u_k) = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)^{-1} = \delta/(a_0, a_1, \ldots, a_n, 0, 0, \ldots),$$

where $\delta = (1, 0, 0, \ldots)$. So, in the Example 1 of Section 4, considered also in Section 5.4.1, the dominant root is obtained by the sequence

$$(u_k \ : \ k = 0, 1, 2, \ldots) = (1, -5, 6, -1, 0, 0, \ldots)^{-1} =$$

$$= (1, 0, 0, \ldots)/(1, -5, 6, -1, 0, 0, \ldots) =$$

$$= (1, 5, 19, 66, 221, 728, 2380, 7753, 25213, 81927, 266110, \ldots),$$

being $x_D \approx u_{10}/u_9 = 266110/81927 \approx 3.248$. Our method is simpler than the Bernoulli's, because eliminates the use of the difference equation, replacing the sequence u_k with the sequence S_k of sums of powers of polynomial roots. However, this simplification reduces the possibilities to obtain various deconvolution formulas for calculating the polynomial roots. The dominant roots can be calculated in the present method only by the formulas (2) and (10).

7 Application to solving algebraic equations

In the paper [9] has shown that the variation $t_n = x_{n+1} - x_n$ of the approximate value x_n of the exact root of an algebraic equation $f(x) = 0$ is a root of the *resolving Taylor polynomial equation* $\sum_{k=0}^{m} \dfrac{f^{(k)}(x_n)}{k!} t^k = 0$ in the point x_n, by a considered order m. Starting with the initial values x_0 and x_1, hence with $t_0 = x_1 - x_0$, the approximate values x_n of the exact solution can be successively calculated by the formula $x_{n+1} = x_n + t_n$, where t_n is a convenient root of the resolving Taylor equation in x_n, preferably its dominated root.

Example. We consider the algebraic equation $e^x - 5x^2 = 0$, which has the roots $x_- \approx -0.3714$ and $x_+ \approx 0.6053$, with all decimal exact. We take $x_0 =$

$x_1 = t_0 = 0$. The resolving Taylor polynomial equation of order $m = 3$ is $\frac{t^3}{6} - \frac{9}{2}t^2 + t + 1 = 0$, hence $t^3 - 27t^2 + 6t + 6 = 0$. Its dominant root can be obtained by the sequence

$$(S_k) = (27, -12, -18, 0, 0, \ldots)/(1, -27, 6, 6, 0, 0, \ldots) =$$

$$= (27, 717, 19179, 513363, 13741587, \ldots),$$

so is $t_D \approx 13741587/513363 \approx 26.7675$. This value can not be taken as variation of the approximate solutions of the algebraic equation. The dominated root is given by the sequence

$$(S_{-k}) = -(6, -54, 3, 0, 0, \ldots)/(6, 6, -27, 1, 0, 0, \ldots) \approx$$

$$(-1, 10, -15, 60.1, -129.3, 402.6, -994.6, 2827.8, -7370.6, 20261.5, \ldots),$$

so is $t_d \approx -7370.6/20261.5 \approx -0.3638$. From the first Viète formula, the third root of the resolving Taylor equation is $t_m \approx 27 - 26.7675 + 0.3638 = 0.5963$. Therefore, the approximate values of the two roots of the algebraic equation are $\widetilde{x}_- = x_1 + t_d \approx -0.3638$ and $\widetilde{x}_+ = x_1 + t_m \approx 0.5963$. Continuing the process, more accurate values for the roots of algebraic equation can be obtained.

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